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ON SOME RELATIONS BETWEEN CURVATURE AND METRIC TENSORS IN RIEMANNIAN SPACES

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ABSTRACT. A theorem of G.G. Gadzhisalioglu and A.H. Amirov shows how to calculate a metric tensor in a semigeodesic coordinate system from its initial values on a hypersurface and some components of a curvature tensor in a domain.

In the present paper the above mentioned theorem is generalized, with a simplified proof based upon Picard's existence theorem for ordinary differential equations.

G.G. Gadzhisalioglu and A.H. Amirov [2], [3] studied the problem of the interrelations between the metric and curvature tensor in Riemannian spaces. In the present paper the results of [2] and [3] are generalized and shorter proofs are given.

By a Riemannian space V_n we mean in this paper a pseudo-Riemannian space regardless of the sign of the metric.

We will give more general results on finding metrics from values of the curvature tensor. The proof of our main theorem has a constructive character.

1. The definition and properties of the semigeodesic coordinate system are given in [6] and their physical interpretation in the theory of relativity in [4], [5].

In a Riemannian space a *semigeodesic coordinate system* $(x^1, x^2, \dots, x^n)$ is defined by the help of some nonisotropic hypersurface $\Omega_{n-1} \subset V_n$. The points $x = (x^1, x^2, \dots, x^n) \in \Omega_{n-1}$ have $x^1 \equiv 0$. The coordinate x^1 of the point $x \in V_n$ is the oriented length of the arc $\widehat{x, x^*}$ of the geodesic curve γ which goes through the point x and is orthogonal to the hypersurface Ω_{n-1}, where $x^* = \gamma \cap \Omega_{n-1}$.

A metric in the semigeodesic coordinate system is of the form

$$ds^2 = e\, {dx^1}^2 + g_{ab}(x^1, x^2, \dots, x^n) dx^a dx^b \quad (a, b = \overline{2, n}), \quad e = \pm 1.$$

The following relations are valid for the metric tensor g_{ij}, its inverse matrix (g^{ij}), the Christoffels symbols Γ_{ijk} and the Riemannian tensor R^h_{ijk} in any semigeodesic coordinates:

$$g_{1i} = g^{1i} = e\,\delta^1_i, \quad \Gamma_{11i} = \Gamma_{1i1} = 0, \quad R^1_{1jk} = 0, \qquad i, j, k = \overline{1, n}.$$

The basic identities valid for the Riemannian tensor imply

$$R_{1ij1} = eR^1_{ij1} = -eR^1_{i1j} = g_{ia}R^a_{11j} = -g_{ia}R^a_{1j1}.$$

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2. Hereafter we will deal only with the semigeodesic coordinate system in the domain D_n, where

$$D_n = \{x = (x^1, x^2, \dots, x^n) \in R^n : \quad 0 < x^i < 1,\ i = \overline{2,n}, \quad 0 < x^1 < \delta\}.$$

Points $(0, x^2, \dots, x^n)$ belong to the determining hypersurface Ω_{n-1}. We denote by

$$D_{n-1} = \{x^* = (x^2, \dots, x^n) \in R^{n-1} : \quad 0 < x^i < 1,\ i = \overline{2,n}\}.$$

Theorem 1. *Let D_n be a domain where the semigeodesic coordinate system is defined. Let*

$$a_{ij}(x) \in C^0(\overline{D_n}), \quad \overset{o}{g}{}^{ij}(x^*) \in C^2(\overline{D_{n-1}}), \quad \overset{1}{g}{}^{ij}(x^*) \in C^2(\overline{D_{n-1}}), \quad i,j = \overline{2,n}$$

be symmetric matrices, where $|\overset{o}{g}{}^{ij}(x^*)| \neq 0$.

Then there is exactly one metric tensor $g_{ij}(x) \in C^2(\overline{D_n})$ in the domain

$$D_n^* = \{x = (x^1, x^2, \dots, x^n) \in R^n : \quad 0 < x^i < 1, i = \overline{2,n}, \quad 0 < x^1 < \delta^* \leq \delta\},$$

where δ^* is some positive real number, such that

$$g_{ij}(0, x^*) = \overset{0}{g}{}^{ij}(x^*), \quad \frac{\partial}{\partial x^1} g_{ij}(0, x^*) = \overset{1}{g}{}^{ij}(x^*), \quad i,j = \overline{1,n},$$

hold for all $x^* \in \overline{D_{n-1}}$ and

$$R_{1ij1}(x) = a_{ij}(x)$$

for all $x \in \overline{D_n^*}$, $i,j = \overline{2,n}$.

Proof. It is known [6] that the Riemannian tensor R_{hijk} can be expressed by the following formulae:

$$R_{hijk} = 1/2(\partial_{ij} g_{hk} + \partial_{hk} g_{ij} - \partial_{ik} g_{hj} - \partial_{ik} g_{hj}) + g^{\alpha\beta}(\Gamma_{hk\alpha}\Gamma_{ij\beta} - \Gamma_{hj\alpha}\Gamma_{ik\beta}), \tag{1}$$

where $\Gamma_{ijk} = 1/2(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$ are the Christoffel symbols of V_n. The symbols g^{ij} are the components of the inverse matrix of the metric tensor, that means they can be expressed as rational functions of components of the metric tensor g_{ij}.

Putting $h = k = 1$ and using the properties of the semigeodesic coordinate system we obtain from (1)

$$R_{1ij1} = 1/2\partial_{11} g_{ij} - 1/4 g^{ab}\partial_1 g_{ia}\partial_1 g_{jb}. \tag{2}$$

Here, we can suppose that the indices $i, j, a, b > 1$.

If we set

$$G_{ij} = \partial_1 g_{ij}, \tag{3}$$

the formulas (2) have the following form:

$$R_{1ij1} = 1/2\partial_1 G_{ij} - 1/4 g^{ab} G_{ia} G_{jb}. \tag{4}$$

As $R_{1ij1} = a_{ij}(x)$, we get from (3) and (4):

$$\begin{array}{llcl} (a) & \partial_1 g_{ij} & = & G_{ij}, \\ (b) & \partial_1 G_{ij} & = & 1/2 g^{ab} G_{ia} G_{jb} + 2a_{ij}. \end{array} \tag{5}$$

Let us complete the formulas (5) by the initial conditions

$$\forall x^* \in D_{n-1}\colon g_{ij}(0, x^*) = \overset{0}{g}{}_{ij}(x^*),\ G_{ij}(0, x^*) = \overset{1}{g}{}_{ij}(x^*),\ i,j = \overline{2,n}. \tag{6}$$

Then (5) can be considered as a system of first-order ordinary differential equations for the unknown functions $g_{ij}(x)$ and $G_{ij}(x)$ with initial conditions (6), where x^1 is the variable and the coordinates $x^* \equiv (x^2, \dots, x^n) \in D_{n-1}$ are supposed to be parameters. The right sides in (5) satisfy the conditions of the existence and uniqueness theorem [1, p. 263] in the domain D_n^* and have continuous derivatives with respect to g_{ij} and G_{ij}. The initial value problem (5) and (6) has precisely one solution $g_{ij}(x)$.

The function $g_{ij}(x)$ in the domain D_n^* are components of the required metric tensor.

Comparing (5) and (4) we can see that $R_{1ij1}(x) = a_{ij}(x)$ in the domain D_n^*. This completes the proof.

Note. As g_{ij} and G_{ij} are symmetric matrices, the indices i and j in (11) may be assumed to satisfy the inequality $i \leq j$.

3. If the system (5) is written in the form

$$\begin{array}{llcl} (a) & \partial_1 g_{ij} & = & G_{ij}, \\ (b) & \partial_1 G_{ij} & = & 1/2 g^{ab} G_{ia} G_{jb} + 2R_{1ij1}. \end{array} \tag{7}$$

then the system (7) with the initial condition (6) can be considered to be the problem of finding metrics for given components $R_{1ij1}(x)$ of the Riemannian tensor in the domain D_n.

This means that in the semigeodesic coordinate system there exists just one metric for the initial value problem (7) and (6) and given components $R_{1ij1}(x)$ in the domain D_n.

In the equations (7a) we can substitute the components R_{1ij1} for their equivalents:

$$R_{1ij1} = eR^1_{ij1} = -eR^1_{i1j} = g_{ia}R^a_{11j} = -g_{ia}R^a_{1j1}.$$

This shows how to transfer given components $R_{1ij1}(x)$ to other components of a Riemannian tensor.

These results generalize the results in [2], [3]. It has been shown here that the proof of existence and uniqueness of solutions follows from the classical results in the theory of systems of first-order ordinary differential equations.

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