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ON QUASIJET BUNDLES

JIŘÍ TOMÁŠ

ABSTRACT. We discuss the Weil approach to the bundles of quasijets and describe the inclusion of the bundle of non-holonomic r-jets into the bundle of quasijets of order r. Applying this approach we rederive a result by Dekrét characterizing non-holonomic r-jets among quasijets of order r.

1. PRELIMINARIES

We start from the concept of non holonomic r-jet, introduced by Ehresmann, [2] and investigated in works of Pradines, Kolář, Dekrét, Kureš, Virsik and others, [8], [3], [1], [7].

We follow the results of Dekrét from [1], namely the definition of quasijets with their basic properties and essentially use the result of Kolář and Mikulski from [4], giving the description of bundle functors defined on the category $\mathcal{M}_m \times \mathcal{M}$ from the point of view of the theory of Weil bundles. We use the standard notation from [5].

In the very beginning, we remind the basic concepts of non-holonomic r-jet and quasijet of order r. We also recall their basic properties and present the relation between them. We define the associated concept of $(k, r)$-quasivelocities and introduce the bundle functor of quasijets on $\mathcal{M}_m \times \mathcal{M}$.

Let $M, N, P$ be manifolds. We recall that a non-holonomic r-jet is defined by induction as follows.

**Definition 1.** For $r = 1$, the set of non-holonomic 1-jets $\tilde{J}^1(M, N)$ is the set of 1-jets $J^1(M, N)$ with their standard composition.

By induction, let $\alpha : \tilde{J}^{r-1}(M, N) \rightarrow M$ denote the source projection and $\beta : \tilde{J}^{r-1}(M, N) \rightarrow N$ the target projection of $(r - 1)$-th order non-holonomic jets. Then

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$X$ is said to be a non-holonomic $r$-jet with the source $x \in M$ and the target $y \in N$, if there is a local section $\sigma : M \to \tilde{J}^{r-1}(M,N)$ such that $X = j^2_{\rho} \sigma$ and $\beta(\sigma(x)) = y$.

Let $Y = j^2_{\rho} \sigma$ for a local section $\rho : N \to \tilde{J}^{r-1}(N,P)$, $y = \beta(\sigma(x))$. The composition $Y \circ X$ of non-holonomic $r$-jets is defined by

$$Y \circ X = j^1_{\rho}(\rho(\beta(\sigma(u)))) \circ_{r-1} \sigma(u)$$

where $\circ_{r-1}$ denotes the composition of non-holonomic $(r-1)$-jets and $u$ is an element of $M$ from a neighbourhood of $x$.

Now we are going to remind the concept of quasijet. For a manifold $M$, consider the $r$-times iterated tangent bundle $T^r M$. It is well-known that there are $r$ structures of vector bundle on $T^r M$, namely $T^{r-k} p^k_M : T^r M \to T^{r-1} M$, where $p^k_M : T^i M \to T^{i-1} M$ denote the tangent bundle projection. The definition of quasijet of order $r$ reads as follows

**Definition 2.** Let $x \in M$ and $y \in N$. A map $\varphi : (T^r M)_x \to (T^r N)_y$ is said to be a quasijet of order $r$ with the source $x$ and the target $y$, if it is a vector bundle morphism with respect to all vector bundle structures $(T^{r-k} p^k_M)_x$ and $(T^{r-k} p^k_N)_y$, $k = 1, \ldots, r$. The set of all such quasijets is denoted by $QJ^r_x(M,N)_y$.

We need the coordinate description of quasijets. Let $x^i = x^i_0$ denote the coordinates on a manifold $M$ and $x^i_1 = dx^i_0$ the additional coordinates on $TM$. Define the coordinates on $T^r M$ by induction as follows. Let $x^i_{\epsilon_1, \ldots, \epsilon_{r-1}}$ denote the coordinates on $T^{r-1}$, $\epsilon_i \in \{0,1\} \forall i \in \{1, \ldots, r-1\}$. Then $x^i_{\epsilon_1, \ldots, \epsilon_{r-1}}$ denote the base coordinates on $T^r M$ with respect to the tangent bundle projection $p^k_M : T^r M \to T^{r-1} M$, while $x^i_{\epsilon_1, \ldots, \epsilon_{r-1}} = dx^i_{\epsilon_1, \ldots, \epsilon_{r-1}}$ denote the fiber ones.

By Dekrét, [1], every quasijet $\varphi \in QJ^r_x(M,N)_y$ is expressed in coordinates by $a_{i_1, \ldots, i_k}$ defined by the following equation

$$(1) \quad y^p_{\epsilon_1, \ldots, \epsilon_r} = \sum_{(\gamma_1, \ldots, \gamma_k)} a^p_{i_1, \ldots, i_k} x^{i_1}_{\gamma_1} \ldots x^{i_k}_{\gamma_k}$$

where the sum is taken over all multiindices $\gamma_1, \ldots, \gamma_k$ satisfying the following conditions

(i) $\gamma_1 + \ldots + \gamma_k = \epsilon = (\epsilon_1, \ldots, \epsilon_r)$

(ii) $\deg \gamma_1 < \deg \gamma_2 \ldots < \deg \gamma_k$, where $\deg \gamma$ denotes the number of the first unit component in $\gamma$.

(Here $\gamma_i$ denotes the $i$-th multiindex, while $\gamma_i$ denotes the $i$-th component in the multiindex $\gamma$).

In what follows, we interpret non-holonomic $r$-jets as quasijets of order $r$ and prove the compatibility of their compositions. Every non-holonomic $r$-jet $X \in \tilde{J}^r_x(M,N)_y$ determines a quasijet $\mu X \in QJ^r_x(M,N)_y$ as follows

Let $r = 1$ and $X = j^1_{\rho} f$. Then $\mu X$ is defined as $T_x f$. By induction, we define $\mu X : T^r_x M \to T^r_y N$ for $X \in \tilde{J}^r_x(M,N)_y$. Let $X = j^1_{\rho} \sigma$ for a local $\alpha$-section $\sigma : X \to \tilde{J}^r(M,N)$. Then $\sigma(u) \in \tilde{J}^r_u(M,N)$ and $\mu(\sigma(u)) : T^r_u M \to T^r_{\beta(\sigma(u))} N$. We put $\mu X = T_x \mu(\sigma(u))$. 

Proposition 3. For a non-holonomic r-jet $X \in \tilde{J}_x^r(M,N)_y$, $\mu X$ is a quasijet. If $Y \in \tilde{J}_y^r(N,P)$, then $\mu(Y \circ X) = \mu(Y) \circ \mu(X)$.

Proof. We prove the assertion by induction. Let $X = j_x^1 \sigma$ for a local $\alpha$-section $\sigma : M \to \tilde{J}^{r-1}(M,N)$. By induction, $\mu(\sigma(u)) : T_x^{r-1}M \to T_{\beta(\sigma(u))}^{r-1}N$ is a quasijet. Moreover, we have a map $\mu(\sigma) : T_x^{r-1}M \to T_{\beta(\sigma(u))}^{r-1}N$ defined by $\mu(\sigma)(z) = \mu(\sigma(p(z)))(z)$, where $p : T_x^{r-1}M \to M$ denotes the base projection. By the induction assumption, $\mu(\sigma) : T_x^{r-1}M \to T_{\beta(\sigma(u))}^{r-1}N$ is a vector bundle morphism with respect to all vector bundle structures $T_x^{r-1}p_M^i$ and $T_{\beta(\sigma(u))}^{r-1}p_N^i$. Then it is easy to see that $T\mu(\sigma) : p_M^i \to p_N^i$ is a vector bundle morphism as well as $T\mu(\sigma) : T_x^{r-1}M \to T_{\beta(\sigma(u))}^{r-1}N$ for $i = 1, \ldots, r - 1$. Thus $\mu(X) = T_x\mu(\sigma) : T_x^{r-1}p_M^i \to T_{\beta(\sigma(u))}^{r-1}p_N^i$ is a quasijet, which proves the first claim.

For the proof of the second assertion, consider local sections $\sigma : M \to \tilde{J}^{r-1}(M,N)$ and $\rho : N \to \tilde{J}^{r-1}(N,P)$ and define $\mu(\rho(\sigma)) : T_{\beta(\sigma(u))}^{r-1}N \to T^{r-1}P$ by

$$\mu(\rho(\sigma))(\mu(\sigma)(u)) = \mu(\rho \circ \beta(\sigma(u)))(\mu(\sigma(u))).$$

We prove that $\mu(\rho \circ \mu(\sigma)(u)) = \mu(\rho(\sigma))(\mu(\sigma(u)))$. It holds

$$\mu(\rho \circ \mu(\sigma)(u)) = \mu(\rho)(\mu(\sigma)(u)) = \mu(\rho)(\mu(\sigma(p(u)))(u))$$

$$= \mu(\rho \circ \beta(\sigma(u)))(\mu(\sigma)(u))$$

$$= \mu(\rho \circ \beta(\sigma(u)))(\mu(\sigma(u))).$$

By induction, we have $\mu(\rho(\sigma))(\mu(\sigma(u))) = \mu((\rho \circ \beta(\sigma(u))) \circ \sigma(u))$ which implies $\mu((\rho \circ \beta(\sigma(u)) \circ \sigma(u)) = \mu(\rho) \circ \mu(\sigma)(u)$. Let $X = j_x^1 \sigma, Y = j_y^1(\sigma(z))\rho$. Applying $T$ to both sides of the last equations yields $\mu(Y \circ X) = \mu(Y) \circ \mu(X)$. This proves our claim. □

By Dekrét, [1], there is a bundle structure $QJ^r(M,N) \to M \times N$ on quasijets. Analogously to $J^r$, [5], we can consider $QJ^r$ as the bundle functor on the category $\mathcal{M}/_m \times \mathcal{M} \to \mathcal{F}/\mathcal{M}/_m$, if we define $QJ^r(f,g)(X) = j_f^1(\sigma)g \circ X \circ j_f(\sigma(X))^{-1}$ for any local diffeomorphism $f : M \to \tilde{M}$ and any smooth map $g : N \to \tilde{N}$. The composition in the last expression denotes the composition of quasijets, where holonomic r-jets $j_f^1(\sigma)g$ and $j_f(\sigma(X))^{-1}$ are considered as quasijets.

Now we are going to define the bundle of $(m,r)$-quasivelocities. We put $QT^r_mN = QJ^r_0(\mathbb{R}^m, N)$ for a manifold $N$ and $QT^r_mf = QJ^r_0(\text{id}_{\mathbb{R}^m}, f)$ for a smooth map $f : N \to P$. Thus we have the functor $QT^r_m : \mathcal{M}/_m \to \mathcal{F}/\mathcal{M}$. It can be easily verified that, that $QT^r_m$ is a product preserving functor and thus it is a Weil bundle $T_A$ for $A = QT^r_m\mathbb{R}$. The situation is analogous to that for non-holonomic r-jets and non-holonomic $(m,r)$-velocities. Denote by $Q^r_m$ the Weil algebra corresponding to the bundle of $(m,r)$-quasivelocities and $D^r_m$ the Weil algebra corresponding to the bundle of non-holonomic $(m,r)$-velocities.
2. WEIL APPROACH TO QUASIJET BUNDLES

We start this section from an important result of Kolář and Mikulski, [4], from which we gradually deduce the description of quasijet bundles from the point of view of the theory of Weil bundles. Applying this approach, we also describe the inclusion of non-holonomic jets into the bundle of quasivelocities.

Let $F$ be a bundle functor defined on the product category $\mathcal{M}f_m \times \mathcal{M}f$. For a couple of manifolds $(M, N) \in \mathcal{M}f_m \times \mathcal{M}f$ we have two fibered manifold projections $a : F(M, N) \to M$ and $b : F(M, N) \to N$. For another couple of manifolds $(M, N) \in \mathcal{M}f_m \times \mathcal{M}f$, a local diffeomorphism $g : M \to M$ and a smooth map $f : N \to N$, we have a morphism $F(g, f) : F(M, N) \to F(M, N)$. Kolář and Mikulski in [4] defined the associated bundle functor $G^F$ on $\mathcal{M}f_m \times \mathcal{M}f$ by $G^F(N) = F_0(\mathbb{R}^m, N), G^F(f) = F_0(\text{id}_{\mathbb{R}^m}, f)$. Moreover, they defined the action $H^F$ of the jet group $G^r_m$ on $G^F$ by $H^F_N(j_0^r\varphi) = F_0(\varphi, \text{id}_N)$ in the case $F$ is a bundle functor of order $r$ in the first factor. For every $j_0^r\varphi \in G^r_m$, $H^F(j_0^r\varphi)$ is a natural equivalence on $G^F$ and thus $H^F : G^r_m \to \mathcal{N}E(G^F)$ is a group homomorphism of $G^r_m$ into the group of all natural equivalences $\mathcal{N}E(G^F)$ on $G^F$.

Conversely, let $G$ be a bundle functor defined on $\mathcal{M}f_m \times \mathcal{M}f$ and $H : G^r_m \to \mathcal{N}E(G)$ be a group homomorphism. We remind the bundle functor $(G, H)$ on $\mathcal{M}f_m \times \mathcal{M}f$ defined in [4]. We have $(G, H)(M, N) = P^r M[GN, H_N]$, the bundle associated to the frame bundle $P^r M$ with the standard fiber $GN$ and the action $H_N$ of $G^r_m$ on $GN$. For a local diffeomorphism $g : M \to M$ and a smooth map $f : N \to N$, we have $(G, H)(g, f) = P^r g[f]$. We have bundle projections $a : (G, H)(M, N) \to M$ and $b : (G, H)(M, N) \to N$.

Then the result of Kolář and Mikulski reads as follows

**Proposition 4.** (i) For every bundle functor $F$ defined on $\mathcal{M}f_m \times \mathcal{M}f$ of order $r$ in the first factor it holds $F = (G^F, H^F)$.

(ii) For another bundle functor $\bar{F}$ of this kind, natural transformations $t : F \to \bar{F}$ are in a bijection with natural transformations $\tau : G^F \to G^F$ satisfying the equivariance condition

$$H^F_N(j_0^r\varphi) \circ \tau_N = \tau_N \circ H^F_N(j_0^r\varphi)$$

for any $j_0^r\varphi \in G^r_m$.

(iii) A bundle functor $F$ on $\mathcal{M}f_m \times \mathcal{M}f$ of order $r$ in the first factor preserves products in the second factor if and only if $G^F = T^A$ for some Weil algebra $A$ and $H$ induces a homomorphism $G^r_m \to \text{Aut}(A)$ of Lie groups.

The well-known bundle functors satisfying the assumptions of Proposition 4 are the functors of holonomic jets $J^r$, non-holonomic jets $\tilde{J}^r$ and semiholonomic jets $\tilde{J}^r$. It is easy to verify that the functor of quasijets $QJ^r$ satisfies the assumptions of (iii) from Proposition 4 too. Then $G^QJ^r = QT_m^r = TQ_m^r$, for the Weil algebra $Q_m^r$. The action of $G^r_m$ on $QT_m^r$ is defined by $H_N(j_0^r\varphi)(X) = X \circ (j_0^r\varphi)^{-1}$ for $X \in QT_m^r N$ and $j_0^r\varphi \in G^r_m$. The situation is analogous to $J^r$, $\tilde{J}^r$ and $\tilde{J}^r$.

We are going to determine the Weil algebra $Q_m^r = QT_m^r \mathbb{R} = QJ^r(\mathbb{R}^m, \mathbb{R})$. We come out from the coordinate expression of quasijets given by (1), using $x^i$ for the
canonical coordinates on $\mathbb{R}^m$ and $y$ on $\mathbb{R}$. In what follows, we use multiindices $\gamma$ formed by zeros and units, the number of which not exceed $r$. Denote by $E_r$ the multiindex composed from $r$ units. A multiindex $\gamma$ is said to be contained in $\delta$ if $\gamma_j \leq \delta_j$ for any $j = 1, \ldots, \text{length}(\gamma)$. Let us assign a polynomial $a_1^{\gamma_1} \cdots a_k^{\gamma_k} \tau_{\gamma_1} \cdots \tau_{\gamma_k}$ with variables $\tau_{\gamma_1}, \ldots, \tau_{\gamma_k}$ to a $(m, r)$-quasivelocity determined by coordinates $a_1^{\gamma_1} \cdots a_k^{\gamma_k}$. Consider the Weil algebra $\mathcal{D}_k$ of polynomials of $k$ variables of degree at most $r$. Then it holds

**Proposition 5.** Let $\mathcal{Q}_m = \mathcal{D}_m / I$ be generated by $\tau_{\gamma}^{(i)}$ for $i \in \{1, \ldots, m\}$, $\gamma \subseteq E_r$. Then $\mathcal{Q}_m = \mathcal{D}_m / I$ is the Weil algebra associated to the bundle of $(m, r)$-quasivelocities, where the ideal $I$ is of the form $\langle \tau_{\gamma}^{(i)} \rangle$ with $\gamma \subseteq E_r > \emptyset$. The multiplication is defined as follows. For $a = a_1^{\gamma_1} \cdots a_k^{\gamma_k} \tau_{\gamma_1}^{(i_1)} \cdots \tau_{\gamma_k}^{(i_k)}$ and $b = b_1^{\delta_1} \cdots b_l^{\delta_l} \tau_{\delta_1}^{(j_1)} \cdots \tau_{\delta_l}^{(j_l)}$, the element $c = ab$ satisfies

\[
 c = a_1^{\epsilon_1} \cdots a_k^{\epsilon_k} b_1^{\phi_1} \cdots b_l^{\phi_l} \tau_{\epsilon_1}^{(i_1)} \cdots \tau_{\epsilon_k}^{(i_k)} \tau_{\phi_1}^{(j_1)} \cdots \tau_{\phi_l}^{(j_l)}
\]

where the sum on the right-hand side of (2) is taken over all subsets $\{i_1, \ldots, i_k\}$ including the empty one.

**Proof.** Let $a, b \in \mathcal{Q}_m = QT_m \mathbb{R}$ be any $(m, r)$-quasivelocities. Denote by $\mu : \mathbb{R}^2 \to \mathbb{R}$ the multiplication of reals. Then $ab = TQ_m \mu(a, b) = TQ_m \mu(a, b) = j^r(\beta(a), \beta(b))\mu(a, b).$ Since $a, b$ can be considered as maps $T_0^r \mathbb{R}^m \to T^r \mathbb{R}$, fixing an element $x \in T_0^r \mathbb{R}^m$, we can evaluate $a(x)$ and $b(x)$. In coordinates, we can express $x$ by $x_i^x$ for $i \in \{1, \ldots, m\}$ and $a(x)$ and $b(x)$ as follows

\[
 a(x) = \beta(a) + a_1^{\gamma_1} \cdots a_k^{\gamma_k} x_i^x \cdots x_i^x
\]
\[
 b(x) = \beta(b) + b_1^{\delta_1} \cdots b_l^{\delta_l} x_i^{\phi_1} \cdots x_i^{\phi_l}
\]

The element $j^r(\beta(a), \beta(b))\mu$ can be considered as a quasijet satisfying $\mu^1 = \beta(b), \mu^2 = \beta(a), \mu_{12}^1 = \mu_{12}^2 = 0$ for any multiindices $\epsilon, \delta \subseteq E_r$ and $\mu^1 \cdots \mu_l^l = 0$ for $l > 2$. Thus $TQ_m \mu(a, b)(x) = \beta(b)a(x) + \beta(a)b(x) + a_1^{\gamma_1} \cdots a_k^{\gamma_k} b_1^{\delta_1} \cdots b_l^{\delta_l} x_i^x \cdots x_i^x x_i^{\phi_1} \cdots x_i^{\phi_l}$, where $\deg \gamma < \cdots \deg \gamma_k$ and $\deg \delta < \cdots < \deg \delta_l$. Comparing the coefficients by $x_i^x \cdots x_i^x$, we obtain

\[
 c = a_1^{\epsilon_1} \cdots a_k^{\epsilon_k} b_1^{\phi_1} \cdots b_l^{\phi_l} \tau_{\epsilon_1}^{(i_1)} \cdots \tau_{\epsilon_k}^{(i_k)} \tau_{\phi_1}^{(j_1)} \cdots \tau_{\phi_l}^{(j_l)}
\]

where the sum is taken over all proper subsets $\{i_1, \ldots, i_k\} \subset \{1, \ldots, h\}$. The coincidence of (4) with (2) proves our claim. □

Thus the functor $QJ^r$ can be expressed as $(TQ_m^r, C)$, where $C : G^r_m \to \text{Aut} \mathcal{Q}_m$ is defined by $C(j_0^r \varphi)(a) = a \circ j_0^r \varphi^{-1}$ for any $j_0^r \varphi \in G^r_m$ and $a \in \mathcal{Q}_m$. 
Let us remind, that the Weil algebra $\mathbb{W}_m$ of non-holonomic $(m,r)$-velocities is identified with $\mathbb{W}_m^0 \ldots \mathbb{W}_m^m$, [3]. Elements of $\mathbb{W}_m$ are considered as polynomials

$$a_1 \ldots a_r t_1^{(i_1)} \ldots t_r^{(i_r)}$$

with variables $t_1^{(i_1)} \ldots t_r^{(i_r)}$ for $i_1 \in \{0,1,\ldots,m\}$ and $t_j^{(0)} = 1$ for $j \in \{1,\ldots,r\}$.

The following assertion describes the canonical inclusion $i : \mathbb{W}_m \rightarrow \mathbb{W}_m^r$, from which we can deduce the inclusion $\mathbb{J}^r \rightarrow \mathbb{Q}^r_J$ from Proposition 4. Moreover, it determines non-holonomic $(m,r)$-velocities among $(m,r)$-quasivelocities by the fact that $A_{j_1 \ldots j_k}^{\gamma_1 \ldots \gamma_k} t_1^{(i_1)} \ldots t_r^{(i_r)}$ represents a non-holonomic $(m,r)$-velocity if and only if $A_{j_1 \ldots j_k}^{\gamma_1 \ldots \gamma_k}$ depend on $\gamma_1, \ldots, \gamma_k$ only up to $\text{deg} \gamma_1, \ldots, \text{deg} \gamma_k$.

**Proposition 6.** Let $i : \mathbb{W}_m \rightarrow \mathbb{W}_m^r$ be a map defined by $i(a_1 \ldots a_r t_1^{(i_1)} \ldots t_r^{(i_r)}) = A_{j_1 \ldots j_k}^{\gamma_1 \ldots \gamma_k} t_1^{(j_1)} \ldots t_r^{(j_k)}$ satisfying

$$A_{j_1 \ldots j_k}^{\gamma_1 \ldots \gamma_k} = a_{j_1 \text{deg} \gamma_1 \ldots j_k \text{deg} \gamma_k}$$

Then $i$ is an injective algebra homomorphism.

**Proof.** Let $a = a_1 \ldots a_r t_1^{(i_1)} \ldots t_r^{(i_r)}$ and $b = b_1 \ldots b_r t_1^{(j_1)} \ldots t_r^{(j_r)} \in \mathbb{W}_m$. Then $c = ab$ satisfies $c = a_1 \ldots a_r b_1 \ldots b_r t_1^{(i_1+j_1)} \ldots t_r^{(i_r+j_r)}$, where $t_j^{(i)} = 0$ whenever $i_j > 1$.

Then $D = i(a)i(b)$ satisfies $D_{j_1 \ldots j_k}^{\alpha_1 \ldots \alpha_k} = \sum_D A_{j_1 \ldots j_k}^{\beta_1 \ldots \beta_k} B_{j_1 \ldots j_k}^{\gamma_1 \ldots \gamma_k}$, where $A = i(a)$, $B = i(b)$ and $D$ is the set of all decompositions of $\{\alpha_1, \ldots, \alpha_k\}$ onto $\{\beta_1, \ldots, \beta_k\}$ with complementary $\{\gamma_1, \ldots, \gamma_k\}$ and the bottom indices $i_1, \ldots, i_k$ as well as $j_1, \ldots, j_k$ correspond to the top multiindices. By the definition of $i$ we have

$$D_{j_1 \ldots j_k}^{\alpha_1 \ldots \alpha_k} = \sum_D a_{i_1 \text{deg} \alpha_1 \ldots i_r \text{deg} \alpha_r} b_{j_1 \text{deg} \gamma_1 \ldots j_k \text{deg} \gamma_k}$$

Further, $C = i(ab)$ satisfies

$$C_{i_1 \ldots i_k}^{\alpha_1 \ldots \alpha_k} = \sum_{(j_1, \ldots, j_r)} a_{i_1 \text{deg} \alpha_1 \ldots i_r \text{deg} \alpha_r} b_{j_1 \ldots j_r}$$

where $0 \leq j_1 \leq 4 \text{deg} \alpha_1, \ldots, 0 \leq j_r \leq 4 \text{deg} \alpha_r$.

The last equality follows from (2), the multiplication formula for $(m,r)$-quasivelocities. Obviously, (7) corresponds bijectively with decompositions $\{\alpha_1, \ldots, \alpha_k\}$ and $\{\gamma_1, \ldots, \gamma_k\}$ in (6). This completes the proof. □

**Proposition 7.** Let $\mu : \mathbb{J}^r \rightarrow \mathbb{Q}^r_J$ be the inclusion of non-holonomic $r$-jets into quasi jets of order $r$ from Proposition 3. Then the restriction $\overline{\mu}_r$ of $\mu$ to $\mathbb{T}_m^r \mathbb{R} = \mathbb{W}_m^r$ coincides with $i : \mathbb{W}_m^r \rightarrow \mathbb{W}_m$ defined in Proposition 6.

**Proof.** In general let $b_{i_1 \ldots i_r}$ denote the coordinates of non-holonomic $r$-jets from $\mathbb{J}^r(\mathbb{K}^m, \mathbb{R})$, created by induction according to the definition of non-holonomic jets.
Let $\sigma : \mathbb{R}^m \to \tilde{J}^r(\mathbb{R}^m, \mathbb{R})$ be a local section in the neighbourhood of $0 \in \mathbb{R}^m$. Then $\sigma(u)$ is expressed as $b_{i_1 \ldots i_r}(u)$. Put $a_{i_1 \ldots i_r} = b_{i_1 \ldots i_r}(0) = \sigma(0)$. Further, assume $\mu(\sigma(u))$ has the coordinates $B^1_{i_1 \ldots i_k}$ and put $A^1_{i_1 \ldots i_k}(u) = B^1_{i_1 \ldots i_k}(u)$. As the assumption hypothesis we assume $B^1_{i_1 \ldots i_k}(u)$ has the coordinates $B^1_{i_1 \ldots i_k}(0)$ and put $B^1_{i_1 \ldots i_k}(0) = a(0)$. Further, assume $\mu(\sigma(u))$ has the coordinates $B^1_{i_1 \ldots i_k}(u)$ and put $B^1_{i_1 \ldots i_k}(u) = B^1_{i_1 \ldots i_k}(u)$. Further, assume $\mu(\sigma(u))$ has the coordinates $B^1_{i_1 \ldots i_k}(u)$ and put $B^1_{i_1 \ldots i_k}(u) = B^1_{i_1 \ldots i_k}(u)$. Further, assume $\mu(\sigma(u))$ has the coordinates $B^1_{i_1 \ldots i_k}(u)$ and put $B^1_{i_1 \ldots i_k}(u) = B^1_{i_1 \ldots i_k}(u)$. Further, assume $\mu(\sigma(u))$ has the coordinates $B^1_{i_1 \ldots i_k}(u)$ and put $B^1_{i_1 \ldots i_k}(u) = B^1_{i_1 \ldots i_k}(u)$.

Then $T_u(\mathbb{R}^m) \to T_{\mu(\sigma(u))}(\mathbb{R}^m)$ is expressed by $y = B^1_{i_1 \ldots i_k}(u)u^1_{i_1} \ldots u^i_{i_k}$, where $e_{\alpha}^i$ denotes the multiindex with just one unit on the $(r + 1)$-st position. Setting $v = 0 \in \mathbb{R}^m$, comparing the components $x^1_{\alpha_1} \ldots x^i_{\alpha_i}$ for $\alpha^i_{\alpha} \in E_{\alpha}^{i+1}$ and taking into account Proposition 5 and Proposition 6, we prove our claim. □

**Corollary 8.** Let $\mu : \tilde{J}^r(M,N) \to QJ^r_r(M,N)$ be the inclusion from Proposition 3. The $\mu$ is the natural inclusion corresponding to $i : \mathbb{R}_m^r \to \mathbb{R}_m^r$.

**Proof.** By Proposition 4 (i), every $X \in \tilde{J}^r(M,N)$ is identified with $\{j_0^*t_\alpha(X), X \circ j_0^*t_\alpha(X)\} \subseteq P^rM[\tilde{J}^r_\alpha N, H^r_\alpha]$, where $t_u$ is the translation, mapping $0$ onto $u$. It follows from Proposition 4 (ii) and (iii) and Proposition 7 that $\{j_0^*t_\alpha(X), \mu(X \circ j_0^*t_\alpha(X))\} \subseteq \{j_0^*t_\alpha(X), \mu(X \circ j_0^*t_\alpha(X))\} \approx \{j_0^*t_\alpha(X), \mu(X \circ j_0^*t_\alpha(X))\} \approx \mu(X)$. □

**Remark.** We finish this section by a more geometrical description of the Weil algebra $Q^2_{\alpha}$ and $QJ^2_r(M,N)$. In general, let $A_1 = \mathbb{R} \times N_1$ and $A_2 = \mathbb{R} \times N_2$ be Weil algebras with nilpotent ideals $N_1$ and $N_2$. Their direct sum $A_1 \oplus A_2$ is defined as $\mathbb{R} \times N_1 \times N_2$, where we put $n_1n_2 = 0$ for $n_1 \in N_1$ and $n_2 \in N_2$. By a direct evaluation, using Proposition 5, we obtain $Q^2_{\alpha} = \mathbb{D}^2_{\alpha} = \mathbb{D}^2_{\alpha} \oplus \mathbb{D}^2_{\alpha} = (\mathbb{D}^2_{\alpha} \otimes \mathbb{D}^2_{\alpha}) \oplus \mathbb{D}^2_{\alpha}$. In this way we find $QJ^2_r(M,N) \subseteq J^2_r(M,N) \otimes J^2_r(M,N)$.

3. QUASIJETS AND NON-HOLONOMIC JETS

In this section, we are going to apply the approach from Section 2 to reeducate a result by Dekrét in [1], giving the criterion how to recognize non-holonomic $r$-jets among quasijets of order $r$.

Let us recall the concept of the kernel injection, [1]. For a vector bundle $q : E \to M$ we have two structures of vector bundle on $TE$, namely $p : TE \to E$ and $Tq : TE \to TM$. Denote by $HE \to M$ (the so called heart of a vector bundle $E \to M$, [8], [6]) the vector bundle $VP \cap VTq \to M$. The identification $VE \approx E \times_M E$ is well-known. The kernel injection $V_0^E : E \approx HE \to TE$ is expressed by $V_0^E(x^1, y^p) = (x^1, 0, 0, y^p)$, [6].

Let us consider a vector bundle $T^{k-1}p_M : T^kM \to T^{k-1}M$ from Section 1. Denote by $V_{0k}^M : T^{k-1}p_M \to T^{k-1}p_M$ the kernel injection on $T^kM$ with respect to the $i$-th vector bundle structure on $T^kM$. In Section 1, we defined the coordinates $x^i_{\epsilon_1 \ldots \epsilon_k}$ on
\( T^k M \). There is a Weil bundle structure on \( T^k M \), namely \( T^D^k M \) corresponding to the Weil algebra \( D^k = D \otimes \ldots \otimes D \), where \( D \) denotes the algebra of dual numbers. Thus every element of \( T^k M \) with coordinates \( x^p_{\ell_1} \ldots x^p_{\ell_k} \) can be represented by \( p \) polynomials of the form \( x^p_{\ell_1} \ldots x^p_{\ell_k} \). It can be easily verified that

\[
(9) \quad V^i_{0k}(x^p_{\ell_1} \ldots x^p_{\ell_k}) = (1 - \varepsilon_i)x^p_{\ell_1} \ldots x^p_{\ell_k} - \varepsilon_i(1 - \varepsilon_i)x^p_{\ell_1} \ldots x^p_{\ell_k}
\]

The last formula is equivalent to \( \tau_j \rightarrow (1 - \delta^j_1)\tau_j + \delta^j_1\tau_{k+1} \). By direct evaluation we obtain that \( V^i_{0k} : D^k \rightarrow D^{k+1} \) is a homomorphism of Weil algebras and consequently, \( V^M_{0k} : T^k M \rightarrow T^{k+1} M \) is a natural transformation. In the same way we obtain that \( T^k V^M_{0k} : T^{k+1} M \rightarrow T^{k+1+i} M \) is a natural transformation too. Denote by \( \kappa_i : Q^k \rightarrow Q^{k-1} \) the projection of quasijet bundles induced by the projection \( T^{k-1} P^k : T^k \rightarrow T^{k-1} \), [1]. Then the result by Dekrét reads

A quasijet \( X \in QJ^k_x(M,N)_y \) represents a non-holonomic \( r \)-jet if and only if the following conditions are satisfied

\[
(10) \quad \begin{align*}
& (T^{k-2} V^M_{01})^{-1} \circ X \circ (T^{k-2} V^M_{01}) = \kappa_2 X \\
& (T^{k-3} V^M_{02})^{-1} \circ X \circ (T^{k-3} V^M_{02}) = (T^{k-3} V^M_{02})^{-1} \circ X \circ (T^{k-3} V^M_{02}) = \kappa_3 X \\
& \vdots \\
& (V^M_{0k-1})^{-1} \circ X \circ (V^M_{0k-1}) = \ldots \ldots = (V^M_{0k-1})^{-1} \circ X \circ (V^M_{0k-1}) = \kappa_k X
\end{align*}
\]

To deduce the result by our approach, denote by \( (V^j_{0i} M,N)^* : Q^j(M,N) \rightarrow Q^j(M,N) \) a map defined by \( X \rightarrow (V^j_{0i} M,N)^* \circ X \circ V^j_{0i} M \) for \( X \in Q^j(M,N) \). Analogously denote by \( (T^i V^j_{0i} M,N)^* : Q^{j+1}(M,N) \rightarrow Q^{j+1}(M,N) \) a map defined by \( X \rightarrow (T^i V^j_{0i} M,N)^* \circ X \circ (T^i V^j_{0i} M,N) \) for \( X \in Q^{j+1}(M,N) \). 

**Proposition 9.** Let \( M,N \) be manifolds. Then \( (T^{k-1} V^M_{01})^* : Q^k(M,N) \rightarrow Q^{k-1}(M,N) \) is a natural transformation for \( i = 1, \ldots, k - 1 \) and \( j = 1, \ldots, i \).

**Proof.** By Proposition 4, it is sufficient to prove that \( (T^{k-1} V^M_{01})^* : Q^k \rightarrow Q^k \) is a homomorphism of Weil algebras equivalent in respect to the action of \( G^m \) on \( Q^m \). We prove this for \( (V^j_{0i} R^m,R)^* : Q^i \rightarrow Q^i \) which proves our claim for \( i = k - 1 \). We show, that this proof can be easily extended to other cases of \( i \).

Let \( Y_\varepsilon = Y_{\varepsilon_1} \ldots \varepsilon_0 \) be the coordinates on \( T^i R = D^i \rightarrow D^{i+1} \rightarrow T^{i+1} R \) and \( y_\delta = y_{\delta_1} \ldots \delta_{i+1} \) the coordinates on \( T^{i+1} R = D^{i+1} \). Further, let \( a^1_{i_1} \ldots a^k_{i_k} \) be the coordinates on \( QT^i_{m} R = Q^{i+1} \) and \( x^j_{i_j} \) be the coordinates on \( T^i_{m} R = (D^i)^m \rightarrow (D^{i+1})^m = T^{i+1} R \). Then the formula (9) implies

\[
(11) \quad Y_\varepsilon = (1 - \varepsilon_i)(1 - \varepsilon_j)y_\varepsilon + \varepsilon_j y_\varepsilon + \varepsilon_{i+1}
\]

and the map \( (V^j_{0i} R^m R)^* \) satisfies

\[
(12) \quad Y_\varepsilon = (1 - \varepsilon_i)(1 - \varepsilon_j)a^1_{i_1} \ldots a^k_{i_k} x^1_{i_1} \ldots x^k_{i_k} + \varepsilon_j \varepsilon_{j_1} \ldots \varepsilon_{j_{i+1}} x^1_{i_1} \ldots x^k_{i_k}
\]
for \( \gamma^1 + \cdots + \gamma^k = \varepsilon \), \( \deg \gamma^1 < \cdots < \deg \gamma^k \). Evaluating the coefficients by \( x_1^{i_1} \cdots x_k^{i_k} \), we obtain the coordinates \( A^a_{i_1 \ldots i_k} \) on \( \mathbb{Q}_m \) expressed by \( a^{\delta^1 \ldots \delta^h} \) as follows

\[
A^a_{i_1 \ldots i_k} = (1 - \gamma^1_{i+1}) \cdots (1 - \gamma^k_{i+1})((1 - \sum_{l=1}^{k} \gamma^l_{j})a^{\gamma^1_{i_1} \ldots \gamma^k_{i_k}} + \gamma^l_{j}a^{\gamma^1_{i_1} \ldots \gamma^k_{i_k} + \delta^1 \ldots \delta^h})
\]

Let \( a = a^{\tau^1_{i_1} \ldots \tau^k_{i_k}} \in \mathbb{Q}^{i+1}_m \), \( b = b^{\gamma^1_{j_1} \ldots \gamma^k_{j_k}} \in \mathbb{Q}^{j+1}_m \) and \( A = (V_{\delta i}^{R^m \times R})^*(a), B = (V_{\delta i}^{R^m \times R})^*(b) \). Further, let \( C = AB \) and \( D = (V_{\delta i}^{R^m \times R})^*(ab) \). Then we have \( C^{a^1 \ldots a^h} = \sum_D C^{1 \ldots h} B^{a^h_{i_1 \ldots i_k}} \), where \( D \) is the set of all decompositions of \( \{\alpha^1, \ldots, \alpha^k\} \) into \( \{\beta^1, \ldots, \beta^h\} \) with the complementary \( \{\gamma^1, \ldots, \gamma^{k-h}\} \) and the corresponding bottom indices. By (13) we have

\[
C^{a^1 \ldots a^k} = \sum_D [(1 - \beta^1_{i+1}) \cdots (1 - \beta^h_{i+1})(1 - \sum_{l=1}^{h} \gamma^l_{j})b^{\gamma^1_{i_1} \ldots \gamma^{k-h}_{i_k} + \delta^1 \ldots \delta^h}]
\]

On the other hand

\[
D^{a^1 \ldots a^k} = (1 - \alpha^1_{i+1}) \cdots (1 - \alpha^h_{i+1})(1 - \sum_{l=1}^{k} \alpha^l_{j})a^{\beta^1_{j_1} \ldots \beta^h_{j_k} \gamma^1_{i_1} \ldots \gamma^{k-h}_{i_k}} + \gamma^l_{j}a^{\beta^1_{j_1} \ldots \beta^h_{j_k} \gamma^1_{i_1} + \delta^1 \ldots \delta^h_{i_k}}
\]

It is easy to see that \( C^{a^1 \ldots a^k} = D^{a^1 \ldots a^k} \) which follows that \( (V_{\delta i}^{R^m \times R})^* : \mathbb{Q}_m \rightarrow \mathbb{Q}_m \) is a homomorphism. The fact that \( (T^{k-i-1}V_{\delta i}^{R^m \times R})^* : \mathbb{Q}_m \rightarrow \mathbb{Q}_m^k \) is a homomorphism follows from (10), (12) and (13) remaining unchanged if we replace \( (V_{\delta i}^{R^m \times R})^* \) by \( (T^{i}V_{\delta i}^{R^m \times R})^* \). The equivariance of \( (T^{k-i-1}V_{\delta i}^{R^m \times R})^* \) with respect to the action of \( G^k_{\delta} \) on \( \mathbb{Q}_m^k \) follows from the fact that \( T^{k-i-1}V_{\delta i}^{R^m \times R} : \mathbb{D}^k \rightarrow \mathbb{D}^k \) is a natural transformation. This completes the proof. \( \square \)

We state the following assertion, the proof of which is omitted since it is almost the same as that of Proposition 9, only technically easier.

**Proposition 10.** The quasijet projection \( \kappa_i : QJ^{k+1} \rightarrow QJ^k \) induced by the \( l \)-th vector bundle structure \( T^{k+1-i}p_l : T^{k+1} \rightarrow T^k \) is a natural transformation.

We shall need the coordinate expressions of homomorphisms \( \kappa_{i+1} : Q_{m}^{k+1} \rightarrow Q_{m}^{k} \). Let \( a \in Q_{m}^{k+1} \) and \( A = \kappa_{i+1}(a) \). Further, let \( a = a^{\gamma^1 \ldots \gamma^h} \tau^{(i_1)} \ldots \tau^{(i_k)} \) and \( A = A^{\delta^1 \ldots \delta^h} \tau^{(j_1)} \ldots \tau^{(j_k)} \). Then it holds

\[
A^{\gamma^1 \ldots \gamma^h} = (1 - \gamma^1_{i+1}) \cdots (1 - \gamma^h_{i+1})a^{\gamma^1 \ldots \gamma^h}_{i_1 \ldots i_k}
\]
If we compare (16) with (13), we have

\[
\alpha_{i_1 \ldots i_h} = (1 - \sum_{l=1}^{h} \gamma_l' \alpha_{i_1 \ldots i_h}) + \gamma_l' \alpha_{i_1 \ldots i_h} + e_{i+1} \ldots \gamma_h
\]

for all multiindices \(\gamma^1, \ldots, \gamma^h\) of order \(k + 1\) satisfying \(\gamma^l \subseteq E_{k+1}, \ i + 1 \not\subseteq \gamma^l\) for \(l = 1, \ldots, h\), \(\deg \gamma^l < \cdots < \deg \gamma^h\).

By Proposition 6, a \((m, r)\)-quasivelocity represents a non-holonomic \((m, r)\)-velocity if and only if all \(a_{i_1 \ldots i_h}^{\gamma_1 \ldots \gamma^h}\) depend on \(\gamma^1 \ldots \gamma^h\) only up to \(\deg \gamma^1, \ldots, \deg \gamma^h\). We prove the result of Dekrét if we show the equivalence of the last condition with (17).

Fix \(\gamma^1, \ldots, \gamma^h\) except of \(l\) and consider \(\tilde{\gamma}^l\) derived from \(\gamma^l\) by \(\tilde{\gamma}^l_{i+1} = 0\) and \(\tilde{\gamma}^l_1 = 1\). Further, denote by \(e_{\deg \gamma^l}\) the multiindex containing the only unit at the \((\deg \gamma^l)\)-th position. Clearly \(\deg \gamma^l = \deg \tilde{\gamma}^l\) and \(a_{i_1 \ldots i_h}^{\gamma_1 \ldots \gamma^h} = a_{i_1 \ldots i_h}^{\gamma_1 \ldots \tilde{\gamma}^l \ldots \gamma^h}\) is equivalent with (17), setting \(j = \deg \gamma^l\). The condition \(a_{i_1 \ldots i_h}^{\gamma_1 \ldots \gamma^h} = a_{i_1 \ldots i_h}^{\gamma_1 \ldots e_{\deg \gamma^l} \ldots \gamma^h}\) is equivalent with (17), which is obtained by setting \(j = \deg \gamma^l\) and iterating the last step for all \(i + 1\) corresponding to units in the multiindex \(\gamma^l\). This way we obtain the result by Dekrét.

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