W. Bajguz

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ON EMBEDDING CURVES IN SURFACES

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ABSTRACT. We define here a closed surface to be such a continuum, every point of which has a neighborhood homeomorphic to the Euclidean plane. In the following one proves that for every locally plane Peano continuum X there exists a closed surface such that X is embedded in that surface. Hence the class of locally plane Peano continua appears to be much more regular then it was supposed to be ([2]).

1. PRELIMINARIES

By a *Moore decomposition* of the Euclidean plane we mean any upper semicontinuous decomposition Σ of E^2 such that each element $A \in \Sigma$ is a continuum and every neighborhood of A contains a neighborhood of A homeomorphic to the plane E^2. Then one has a theorem of R. L. Moore [5]:

(1.1) *The decomposition space of a Moore decomposition of E^2 is homeomorphic to E^2.*

A Peano curve $X \subset E^2$ is said to be an *S-curve* if the boundary of each component of $E^2 \setminus X$ is a simple closed curve and no two boundaries of different components intersect. According to this definition the Sierpiński plane universal curve is an S-curve.

We have then the following result due to G. T. Whyburn [6]:

(1.2) *Any two S-curves are homeomorphic.*

In the case of the S-curve X the union of boundaries of all components of $E^2 \setminus X$ is called the *rational part* of X. One then observes that for any simple closed curve S included in the Sierpiński plane universal curve X, S does not disconnect X iff S is the boundary of some component of $E^2 \setminus X$ - i.e. if S is the component of the rational part of X. Hence by (1.2) this property belongs to each S-curve X:

(1.3) *Let S be any simple closed curve included in the S-curve X. Then S does not disconnect X if and only if S is included in the rational part of X.*

The proof of the following fact the reader can be found in [2]:

(1.4) *Let X be a locally connected subcontinuum of space M and Σ a collection of indices. Assume that each index $\sigma \in \Sigma$ corresponds an open subset G_σ of M and a locally connected continuum $F_\sigma \subset M$ satisfying the following conditions:*

- *for every $\varepsilon > 0$ the inequality* $\operatorname{diam} F_\sigma < \varepsilon$ *holds for almost all indices $\sigma \in \Sigma$,*
- *$\sigma \neq \overline{\sigma}$ implies $G_\sigma \cap G_{\overline{\sigma}} = \emptyset$,*

The paper is in final form and no version of it will be submitted elsewhere.

– $\emptyset \neq X \cap \operatorname{Bd} G_\sigma \subset F_\sigma$ *for every index* $\sigma \in \Sigma$.

Then the set $Y = \left(X \setminus \bigcup_{\sigma \in \Sigma} G_\sigma \right) \cup \bigcup_{\sigma \in \Sigma} F_\sigma$ *is a locally connected continuum.*

In the following one makes an extensive use of the following notational conveniences. Let X be Peano continuum and $A, B \subset X$, then:

- d denotes a metric of X. For nonempty sets A, B one defines d as follows: $\mathrm{d}(A, B) = \inf \{\mathrm{d}(a, b) : a \in A, b \in B\}$;
- $\mathrm{B}(A, r) = \{x : d(\{x\}, A) < r\}$ for $A \neq \emptyset$;
- $\operatorname{Cl} A$, $\operatorname{Int} A$, $\operatorname{Bd} A$ denote the closure, the interior and the boundary of A in the space X respectively;
- The disk is identified here with the topological image of the square $[0,1]^2$;
- Let D be homeomorphic to the $[0,1]^2$. Then $\overset{\circ}{D}$ and $\overset{\bullet}{D}$ denote correspondingly the interior and the boundary of the bounded manifold D, while if D is a point, then $\overset{\circ}{D} = \emptyset$ and $\overset{\bullet}{D} = D$;

2. Brick partitions

A metric space X with a metric d is said to be *uniformly locally connected* iff for every $\varepsilon > 0$ there exists $\delta > 0$, such that: if $\mathrm{d}(x, y) < \delta$, then x and y are contained in a connected open set of diameter less than ε.
In the case of the complete connected metric space this is equivalent to the existence of an arc from x to y of diameter less than ε.

In the following we assume X to be a locally connected continuum.
A *partition* of X is a finite collection $\mathcal{F}$ of closed subsets of X such that $\mathcal{F}$ covers X, for each $F \in \mathcal{F}$, $\operatorname{Int} F$ is connected and dense in F, and for any pair of elements F_1, F_2 of $\mathcal{F}$: $F_1 \neq F_2$ and $\operatorname{Int} F_1 \cap \operatorname{Int} F_2 = \emptyset$.

For $\mathcal{F}$ being a partition of X and $G \subset X$ one use the notation

- $\operatorname{Star}_{\mathcal{F}} G = \bigcup \{F \in \mathcal{F} : G \cap F \neq \emptyset\}$.

A partition $\mathcal{F}$ of X is said to be of *order n* iff the intersection of any $n+1$ elements of $\mathcal{F}$ is empty.
A partition $\mathcal{F}$ of X is said to be *a brick partition* iff every element of $\mathcal{F}$ has uniformly locally connected interior and the interior of union of any pair of elements of F is uniformly locally connected.
A partition $\mathcal{F}$ is said to *refine* a partition $\mathcal{H}$ iff $\forall F \in \mathcal{F}\ \exists H \in \mathcal{H} : F \subset H$.
Let $\mathcal{F}$ and $\mathcal{H}$ be partitions of X. A partition $\mathcal{F}$ is said to be an *amalgam* of a partition $\mathcal{H}$ iff every element of $\mathcal{F}$ is an union of subcollection of $\mathcal{H}$ ($\mathcal{F}$ is a partition and therefore each element of $\mathcal{F}$ is connected).

Observe that

(2.1) *Any amalgam of brick partition of Peano continuum X is the brick partition of X.*

Proof. Let $\mathcal{F}$ be the brick partition of X, $\mathcal{G}$ be any amalgam of $\mathcal{F}$ and $G \in \mathcal{G}$. Then $G = \bigcup \mathcal{A}$ for some $\mathcal{A} \subset \mathcal{F}$. Consider $\varepsilon > 0$. Then for every pair $A, B \in \mathcal{A}$ there is $\delta_{A,B} > 0$ such that for any points $x, y \in \operatorname{Int}(A \cup B)$ if $\mathrm{d}(x, y) < \delta$, then x and y are contained in a connected open set of diameter less than $\frac{\varepsilon}{3}$. Let $\delta =$

$\min\{\delta_{A,B}; A, B \in \mathcal{A}\}$ and let $x, y \in \operatorname{Int} G$ be such that $d(x,y) < \min\left\{\frac{\varepsilon}{3}, \frac{\delta}{3}\right\}$. Then $x \in A_x$ and $y \in A_y$ for some $A_x, A_y \in \mathcal{A}$. According to locally connecteness of X there are connected open neighborhoods $U_x \ni x$, $U_y \ni y$ of diameter less than $\min\left\{\frac{\varepsilon}{3}, \frac{\delta}{3}\right\}$ and both U_x, U_y are included in $\operatorname{Int} G$. Let $x' \in U_x$, $y' \in U_y$ be points such that $x' \in \operatorname{Int} A_x$, $y' \in \operatorname{Int} A_y$. Then

$$d(x',y') \leq d(x',x) + d(x,y) + d(x,y') \leq \operatorname{diam} U_x + d(x,y) + \operatorname{diam} U_y < 3\frac{\delta}{3} = \delta \leq \delta_{A_x,A_y} .$$

Therefore there is a connected open set $U \subset \operatorname{Int}(A_x \cup A_y) \subset \operatorname{Int} G$ of diameter less than $\frac{\varepsilon}{3}$ such that x', y' are contained in U.
Thus x, y are elements of connected open set $U_x \cup U \cup U_y \subset \operatorname{Int} G$ and

$$\operatorname{diam}(U_x \cup U \cup U_y) \leq \operatorname{diam}(U_x) + \operatorname{diam}(U) + \operatorname{diam}(U_y) < 3\frac{\varepsilon}{3} = \varepsilon .$$

This means that $\operatorname{Int} G$ is uniformly locally connected and hence $\mathcal{G}$ is the brick partition. □

Since any amalgam of order n partition $\mathcal{F}$ is of order n partition and according to (2.1) one obtains:

(2.2) *Any amalgam of order n brick partition of X is of order n brick partition of X.*
A sequence $\{\mathcal{F}_i\}_{i=1}^{+\infty}$ of partitions of X is said to be a *decreasing sequence* of partitions iff $\mathcal{F}_i$ refines $\mathcal{F}_{i-1}$ for all $i > 1$ and $\lim\limits_{i\to+\infty} \operatorname{mesh} \mathcal{F}_i = 0$.

In [1] R. H. Bing proved an important brick partitioning theorem:

(2.3) *Every Peano continuum admits a decreasing sequence of brick partitions.*

In [3] a stronger version of the Bing's brick partitioning theorem is proved for the case of one-dimensional Peano continuum:

(2.4) *Any one-dimensional Peano continuum admits a decreasing sequence of order two brick partitions with zero-dimensional boundaries.*

A partition $\mathcal{F}$ of Peano continuum X we call a *net partition with nodes $N(\mathcal{F})$ and threads $T(\mathcal{F})$* if $\mathcal{F}$ is of order two brick partition such that:

- $\mathcal{F} = N(\mathcal{F}) \cup T(\mathcal{F})$ and for any different F_1, F_2 being both elements of $N(\mathcal{F})$ or both elements of $T(\mathcal{F})$ holds $F_1 \cap F_2 = \emptyset$,
- for any T being element of $T(\mathcal{F})$ there exist two different elements N_1, N_2 of $N(\mathcal{F})$ such that $\operatorname{Star}_{\mathcal{F}} T = N_1 \cup T \cup N_2$.

Every of order two brick partitions of one-dimensional Peano continuum X is related to the net partition of this continuum, more precisely:

(2.5) *Let $\mathcal{F}$ be of order two brick partition of one-dimensional Peano continuum X. Then there exists a net partition $\mathcal{G}$ of X such that the number of nodes of $\mathcal{G}$ is equal to the number of elements $\mathcal{F}$ and for every $A \in \mathcal{F}$ there is a node $G \in N(\mathcal{G})$ such that $G \subset \operatorname{Int} A$.*

Proof. Since $\{A \cap B : A, B \in \mathcal{F}, A \neq B\}$ is a finite family of mutually disjoint closed subsets of X, then there exists $\delta > 0$ such that:

(**1**) for each different elements A, B, C of $\mathcal{F}$, if $A \cap B$, $A \cap C$ are nonempty sets then $\mathrm{d}(A \cap B, A \cap C) > 3\delta$ and

(**2**) for each A being an element of $\mathcal{F}$ there exists a point $x_A \in A$ such that $\mathrm{d}(x_A, \mathrm{Bd}\, A) > \delta$.

Let now A be any element of $\mathcal{F}$. $A \setminus \mathrm{B}\left(\mathrm{Bd}\, A, \frac{\delta}{2}\right)$ contains only finite number of components, which are not included in $\mathrm{B}(\mathrm{Bd}\, A, \delta)$. Let $H_1, H_2, ..., H_n$ be an order all of these components into a sequence. $\mathrm{Int}\, A$ is connected and uniformly locally connected since A is element of brick partition $\mathcal{F}$. Therefore there exists arcs $L_1, L_2, ..., L_n$ in $\mathrm{Int}\, A$ such that L_i connect the point x_A and the component H_i. Let δ_A be any positive number such that $\delta_A < \min\left\{\delta, \mathrm{d}\left(\mathrm{Bd}\, A, \bigcup_{i=1}^{n} L_i\right)\right\}$ (here d is not a metric, it denotes the infimum of distance between pairs of points) and let M_A be a component of $A \setminus \mathrm{B}(\mathrm{Bd}\, A, \delta_A)$ such that $x_A \in M_A$. Then

(**3**) $$M_A \supset A \setminus \mathrm{B}(\mathrm{Bd}\, A, \delta) .$$

Let now $\delta_1 = \min\{\delta_A : A \in \mathcal{F}\}$. Let $\{\mathcal{F}_i\}_{i=1}^{\infty}$ be a decreasing sequence of order two brick partitions of X obtained from (2.4). Consider n such that $\mathrm{mesh}\, \mathcal{F}_n < \frac{\delta_1}{2}$. Then for $A \in \mathcal{F}$ we have:

$$\mathrm{Star}_{\mathcal{F}_n} M_A \subset \mathrm{B}\left(M_A, \frac{\delta_1}{2}\right) .$$

Therefore $\mathrm{Star}_{\mathcal{F}_n} M_A \cap \mathrm{B}\left(\mathrm{Bd}\, A, \frac{\delta_1}{2}\right) \subset \mathrm{B}\left(M_A, \frac{\delta_1}{2}\right) \cap \mathrm{B}\left(\mathrm{Bd}\, A, \frac{\delta_1}{2}\right)$
and furthermore $\mathrm{B}\left(M_A, \frac{\delta_1}{2}\right) \cap \mathrm{B}\left(\mathrm{Bd}\, A, \frac{\delta_1}{2}\right) = \emptyset$, since

$$\delta_1 \le \delta_A \quad \text{and} \quad M_A \cap \mathrm{B}(\mathrm{Bd}\, A, \delta_A) = \emptyset.$$

Thus

(**4**) $$\mathrm{Star}_{\mathcal{F}_n} M_A \cap \mathrm{B}\left(\mathrm{Bd}\, A, \frac{\delta_1}{2}\right) = \emptyset .$$

Now we can define families of sets $N(\mathcal{G})$ and $T(\mathcal{G})$ as follows:

(**5**) let $T(\mathcal{G})$ be a set of those components of the set $\bigcup\{H \in \mathcal{F}_n : H \not\subseteq \bigcup \mathrm{Star}_{\mathcal{F}_n} M_A : A \in \mathcal{F}\}$ which intersect $\bigcup\{\mathrm{Bd}\, A : A \in \mathcal{F}\}$ and

(**6**) let $N(\mathcal{G})$ be a set of components of $X \setminus \mathrm{Int}(\bigcup T(\mathcal{G})) = \bigcup\{H \in \mathcal{F}_n : H \not\subseteq \bigcup T(\mathcal{G})\}$.

Then by (2.2) $\mathcal{G} = N(\mathcal{G}) \cup T(\mathcal{G})$ is an order two brick partition and by (5), (6) the families $N(\mathcal{G})$ and $T(\mathcal{G})$ are disjoint.

According to (4), (5) each element of $N(\mathcal{G})$ is included in the interior of some element of $\mathcal{F}$. But each $A \in \mathcal{F}$ is connected and therefore by (5) if C is a component of $A \setminus \mathrm{Int}(\bigcup T(\mathcal{G}))$, then C contains M_A. Therefore each $A \in \mathcal{F}$ contains only one element of $N(\mathcal{G})$.

Let now $T \in T(\mathcal{G})$. Let us observe then that

$$\begin{aligned} T \;&\subset\; \bigcup\left\{H \in \mathcal{F}_n : H \not\subseteq \bigcup\{\mathrm{Star}_{\mathcal{F}_n} M_F : F \in \mathcal{F}\}\right\} = \\ &=\; \bigcup\left\{H \in \mathcal{F}_n : H \cap \bigcup\{M_F : F \in \mathcal{F}\} = \emptyset\right\} \subset \\ &\subset\; \bigcup\left\{H \in \mathcal{F}_n : H \cap \bigcup\{F \setminus \mathrm{B}(\mathrm{Bd}\, F, \delta) : F \in \mathcal{F}\} = \emptyset\right\} = \\ &=\; \bigcup\left\{H \in \mathcal{F}_n : H \subset \mathrm{B}\left(\bigcup\{\mathrm{Bd}\, F : F \in \mathcal{F}\}, \delta\right)\right\} \subset \mathrm{B}\left(\bigcup\{\mathrm{Bd}\, F : F \in \mathcal{F}\}, \delta\right) . \end{aligned}$$

From this inclusion and according to (1) every element of $T(\mathcal{G})$ connects exactly two elements of $N(\mathcal{G})$ since every element of $\mathcal{F}$ contains only one element of $N(\mathcal{G})$.
Finally if any two different G_1, G_2 are both elements of $N(\mathcal{G})$ or both elements of $T(\mathcal{G})$ then $G_1 \cap G_2 = \emptyset$, since $N(\mathcal{G})$ and $T(\mathcal{G})$ was defined as sets of components. □

From (2.4) and (2.5) it follows that

(2.6) *For every one-dimensional Peano continuum X and $\varepsilon > 0$ there exists a net partition $\mathcal{F}$ of X such that* $\operatorname{mesh} \mathcal{F} < \varepsilon$.

3. Embedding of locally connected continua in surfaces

In 1966 K. Borsuk presented a construction of locally plane and locally connected curve which was supposed to be not embedded in any surface [2]. The Borsuk's example relied on a missconviction that the curve under construction stays to be locally plane after each step of the construction. However this is not the case. As a result the opposite might be true.
In this paper one proves that the curve which is simultaneously locally plane, locally connected and not embeddable in a surface does not exist, i.e. one proves:
(3.1) Theorem. *For each locally plane Peano curve X there exists a closed surface such that X is embeddable in this surface.*

By the theorem (3.1) one arrives at the following stronger statement i.e.:

(3.2) Theorem. *For each locally plane Peano continuum X there exists a closed surface, such that X is embedded in this surface.*

Proof of Theorem (3.2). Let X be a locally plane Peano continuum. Due to the theorem (3.1) we can assume that $\dim X = 2$. Let X' be the set of all points $x \in X$ such that x has a neighborhood homeomorphic to the Euclidean plane. Consider a family $\mathcal{D} = \{D_j : j = 1, 2, ...\}$ of mutually disjoint closed disks in X' with diameter approaching to zero, such that $X \setminus \bigcup \overset{\circ}{D}_i$ is 1-dimensional.
Thus $\overset{\circ}{D}_1, \overset{\circ}{D}_2, ...$ is the family of open subsets of X and $\overset{\bullet}{D}_1, \overset{\bullet}{D}_2, ...$ is the family of locally connected continua satisfying conditions of (1.4). Therefore

$$Y = X \setminus \bigcup \overset{\circ}{D}_j \quad \text{is a Peano curve.}$$

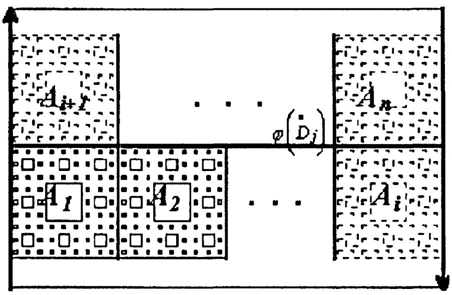
Consequently, according to the theorem (3.1), there exists a closed surface M and a homeomorphic embedding $\varphi : Y \longrightarrow M$.

In the next step we shall modify surface M in order to obtain a homeomorphic embedding of X into the new surface.
Consider disk D from family $\mathcal{D}$. D has an open neighborhood U in X homeomorphic to the Euclidean plane. Let $\sum$ be the decomposition of U such that every disk $D_j \subset U$ from family $\mathcal{D}$ is an element of $\sum$ and all other elements of $\sum$ are individual points. Then $\sum$ is the Moore decomposition and according to (1.1) the decomposition space $U/_{\sum}$ is homeomorphic to U.

Therefore every $\overset{\bullet}{D}_j$ has arbitrary small closed neighborhood in Y being an S-curve and is included in rational part of this S-curve.
For each simple closed curve C in M one has either (a) or (b):

(a) C has arbitrary small neighborhood in M homeomorphic to the annulus,
(b) C has arbitrary small neighborhood in M homeomorphic to the Möbius band.

Assume that the case (b) occurs for the simple closed curve $\varphi\left(\overset{\bullet}{D}_j\right)$. Let U be a sufficiently small neighborhood of $\varphi\left(\overset{\bullet}{D}_j\right)$ in M such that U can be represented as Cartesian product $[-1,1]\times(-1,1)$ with adequately identified $\{-1\}\times(-1,1)$ with $\{1\}\times(-1,1)$ and $\varphi\left(\overset{\bullet}{D}_j\right)$ represented as $[-1,1]\times\{0\}$. Let A be a sufficiently small closed neighborhood of $\varphi\left(\overset{\bullet}{D}_j\right)$ in $\varphi(Y)$ such that A is an S-curve included in U. A can be decomposed to a closed chain of S-curves $A_1, A_2, \ldots, A_n$ such that $A_i \cap A_{i+1}$ is an arc, only end-points of A_i are included in rational part of and one of its end-point is element of $\varphi\left(\overset{\bullet}{D}_j\right)$ (see picture below).



Then $\varphi\left(\overset{\bullet}{D}_j\right)$ does not included in the rational part of A and **hence case (b) is impossible.**

Let $A_j \subset M$ for each j be homeomorphic to the annulus such in a way that $\varphi\left(\overset{\bullet}{D}_j\right)$ is included in the boundary of A_j and $\varphi(Y)\cap A_j = \varphi\left(\overset{\bullet}{D}_j\right)$. Observe than, that only finite number of $\varphi\left(\overset{\bullet}{D}_j\right)$ does not disconnect M, since diameter of $\varphi\left(\overset{\bullet}{D}_j\right)$ approaches to zero and there exists $\varepsilon > 0$ such that any simple closed curve in M of diameter less than ε disconnects M.

Let $\tilde{M}$ be a component of $M\setminus\bigcup\left\{\operatorname{Int} A_j\left(\overset{\bullet}{D}_j\right) \text{ does not disconnect } M\right\}$. Then $\tilde{M}$ is a bounded surface. $\tilde{M}$ can be homeomorphically embedded into a closed surface N obtained from $\tilde{M}$ by adding disks to boundaries of $\tilde{M}$ – let γ be this embedding. In this way we obtain a homeomorphic embedding $\psi = \gamma\circ\varphi : Y \longrightarrow N$ such that every simple closed curve $\psi\left(\overset{\bullet}{D}_j\right)$ disconnects N.

As $\psi(Y)$ is contained in the closure of only one component of $N\setminus\psi\left(\overset{\bullet}{D}_j\right)$, we may choose F_j; $F_j \subset N$ as the closure of $N\setminus\psi\left(\overset{\bullet}{D}_j\right)$ component of such that $F_j \cap \psi(Y) = \psi\left(\overset{\bullet}{D}_j\right)$.

The diameter of $\psi\left(\overset{\bullet}{D}_j\right)$ is approaching to zero, therefore only finite number of F_j are not disks. We can replace these F_j by disks thus obtaining a closed surface $\tilde{N}$ and a homeomorphic embedding $\tilde{\psi} : Y \longrightarrow \tilde{N}$ such that each simple closed curve $\tilde{\psi}\left(\overset{\bullet}{D}_j\right)$ disconnects $\tilde{N}$ into two components. At the same time, the closure of component D_j of $\tilde{N}\backslash\tilde{\psi}\left(\overset{\bullet}{D}_j\right)$ – disjoined with $\tilde{\psi}(Y)$ – is the disk.

Finally we can extend $\tilde{\psi}$ into homeomorphic embedding $\Psi : X \longrightarrow \tilde{N}$ such that $\Psi|_Y = \psi$ and $\Psi(D_j) = F_j$ for each j. This proves the theorem (3.2).

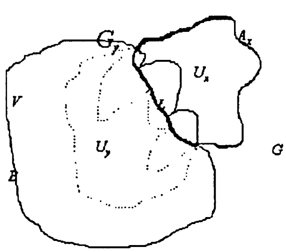
Now let X be Peano continuum included in the Euclidean plane E^2. One then proves the following two lemmas:

(3.3) Lemma. *Let $L \subset X$ be a point or an arc which irreducibly - with respect to subcontinua - disconnects X between points $x, y \in X^1$. Then there exists a component U of $E^2\backslash X$ and a simple closed curve S such that $S\backslash L \subset U$, $S \cap X = L$ and the points x, y belong to different components of $E^2\backslash S$.*

Proof. Let U_x, U_y be the components of $X\backslash L$ containing x, y respectively. Let G_y be a component of $E^2\backslash(U_x \cup L)$ containing U_y. If L is an arc – the ended points of L are elements of $\mathrm{Cl}\, U_x \cap \mathrm{Cl}\, U_y$ since L irreducibly disconnects X. Therefore $\overset{\bullet}{L}$ is included in the boundary $\mathrm{Bd}_{E^2} G_y$ of G_y in the Euclidean plane and hence $L \subset \mathrm{Bd}_{E^2} G_y$ and $A_x = \mathrm{Bd}_{E^2} G_y \backslash \overset{\circ}{L}$ is connected. Then $A_x \cup L$ disconnects E^2 such that G_y is included in one of components of $E^2\backslash(A_x \cup L)$ and U_x is included in the closure of the other of components of $E^2\backslash(A_x \cup L)$.

Observe that A_x is included in the boundary of one of components of $E^2\backslash X$, which in turn is included in G_y (since U_x is the component of $X\backslash L$). Denote this component as G.

In the boundary of G it can be founded (exactly one) a component V [2] of $X\backslash L$ such that for $B = \mathrm{Cl}_X V \cap \mathrm{Cl}_{E^2} G$ holds: $\overset{\bullet}{L} \subset B$ and B disconnects G_y such that U_y is included in the closure of one of components of $G_y\backslash B$ and A_x is included in the closure of the other one (see the picture below).



Now:

— when L is an arc we can find an arc $L' \subset \mathrm{Cl}_{E^2} G$ such that $\overset{\bullet}{L'} = \overset{\bullet}{L}$ and for which B and A_x are included in closures of different components of $G\backslash L'$. Hence the simple closed curve $S = L \cup L'$ fulfills the conditions of the above lemma;

— when L is a point we can find a simple closed curve $S \subset \mathrm{Cl}_{E^2} G$ such that $L \subset S$ and $S \backslash L \subset G$ for which B and A_x are included in closures of different components of $G \backslash S$ and hence this simple closed curve satisfies the conditions of lemma (3.3);

It completes the proof of lemma (3.3).

(3.4) Lemma. *Let $L \subset X$ be a continuum - a point or an arc, such that L irreducibly - with respect to subcontinua - disconnects X between points $x, y \in X$. Let S be a simple closed curve in E^2 such that $S \cap X = L$ and x, y are elements of different components of $E^2 \backslash S$. Let U_x, U_y be components of $X \backslash L$ containing x, y respectively. Let G be the union of components of $X \backslash L$ except for U_x contained in component $E^2 \backslash S$, which in turn contains the point x. Let δ - be any positive number and let $\varphi : U_x \cup L \cup U_y \longrightarrow E^2$ be a homeomorphic embedding.*
Then there exists a homeomorphic embedding $\psi : U_x \cup G \cup L \longrightarrow E^2$ and the disk $D \subset E^2$ such that:

1. $\psi|_{U_x \cup L} = \varphi|_{U_x \cup L}$,
2. $D \cap \psi(U_x \cup G \cup L) = \psi(L)$ *and* $\psi(L) \subset \overset{\bullet}{D}$,
3. $D \cup \psi(G) \subset \mathrm{B}(\psi(L), \delta)$.

Proof. Since L irreducibly disconnects X between points $x, y \in X$, then $\varphi(L)$ irreducibly disconnects $\varphi(U_x \cup L \cup U_y)$ between $\varphi(x)$ and $\varphi(y)$. Therefore and due to lemma (3.3) there exists a simple closed curve S' in E^2 such that both $S' \cap \varphi(U_x \cup L \cup U_y) = \varphi(L)$ and $\varphi(U_x), \varphi(U_y)$ are included in different components of $E^2 \backslash S'$. We can now find a disk D in the closure of component of $E^2 \backslash S'$ containing $\varphi(U_y)$, such that

$$D \subset \mathrm{B}\left(\varphi(L), \frac{\delta}{2}\right), D \cap \varphi(U_x \cup L) = \varphi(L) \quad \text{and} \quad \varphi(L) \subset \overset{\bullet}{D} .$$

Let $\{G_i\}_{i=1}^{\alpha}$, where $\alpha \leq +\infty$, be an appropriately ordered sequence of all subsets of G such that each G_i is an union of all components of G included in only one component of $E^2 \backslash (U_x \cup S)$ and let $\{U_i\}_{i=1}^{\alpha}$ be those components, i.e. $G_i \subset U_i$; (then $\{G_i\}_{i=1}^{\alpha}$ is the sequence of mutually disjoint sets).

Let B_i be a subcontinuum of S such that $\overset{\bullet}{B_i} \subset \mathrm{Cl}\, G_i \subset B_i \subset \mathrm{Cl}_{E^2} U_i$ [3]. Then $\overset{\circ}{B_i} \cap \overset{\circ}{B_j} = \emptyset$ for $i \neq j$ - since G_i, G_j are included in different components of $E^2 \backslash (U_x \cup S)$ and moreover $\overset{\circ}{B_i} \cap \mathrm{Cl}\, U_x = \emptyset$.

Let now A_i be a locally connected continuum in $U_i \cup B_i$ such that $A_i \backslash B_i$ is connected and $G_i \cup A_i \backslash B_i$ is connected too[4]. Let x_i be any point of $G_i \cup A_i$. B_i irreducibly - with respect to subcontinua - disconnects continuum $U_x \cup S \cup G_i \cup A_i$ between points x_i and x. Then according to the lemma 3.3, let S_i be such a simple closed curve in $(E^2 \backslash (U_x \cup S \cup G_i \cup A_i)) \cup B_i$ that $S_i \cap (U_x \cup S \cup G_i \cup A_i) = B_i$ and the points x_i, x belong to different components of $E^2 \backslash S_i$. Evidently S_i is included in $U_i \cup B_i$. Let D_i denote the disk such that $\overset{\bullet}{D_i} = S_i$ and $G_i \cup A_i \subset D_i$.

Then we can define inductively a sequence $\{\varphi_i\}_{i=0}^{\alpha}$ of homeomorphic embeddings such that:

[3] B_i is a point or an arc.

[4] A_i can be constructed as an union of arcs.

(a) $\varphi_i : U_x \cup S \cup \bigcup_{j=1}^{i} (G_j \cup A_j) \longrightarrow E^2$,

(b) $\varphi_0|_{U_x \cup L} = \varphi|_{U_x \cup L}$ and $\varphi_{i+1}|_{U_x \cup S \cup \bigcup_{j=1}^{i} (G_j \cup A_j)} = \varphi_i$,

(c) $\varphi_i (G_i \cup A_i)$ is included in $E^2 \setminus D$,

(d) $\varphi_i (G_i \cup A_i) \subset \mathrm{B}\left(\varphi_i (B_i), \frac{\delta}{2^i}\right)$ for $0 < i \leq \alpha$.

Let now $\varphi_0 : U_x \cup S \longrightarrow E^2$ be a homeomorphic embedding such that $\varphi_0|_{U_x \cup L} = \varphi|_{U_x \cup L}$ and $\varphi_0 (S) = \overset{\bullet}{D}$.
Then φ_0 fulfills conditions (a)-(d).
Assume now that φ_i for some $i < \alpha$ is defined. B_{i+1} is a point or an arc.

Suppose B_{i+1} consists just of one point and denote this point as z.
Then $\{z\}$ locally disconnects $U_x \cup S \cup \bigcup_{j=1}^{i} (G_j \cup A_j)$ and therefore $\varphi_i (\{z\})$ locally disconnects $\varphi_i \left(U_x \cup S \cup \bigcup_{j=1}^{i} (G_j \cup A_j)\right)$.
This implies that there exists a component K of $E^2 \setminus \left(D \cup \varphi_i \left(U_x \cup S \cup \bigcup_{j=1}^{i} (G_j \cup A_j)\right)\right)$ such that $\varphi_i (z)$ is an element of a closure of K in E^2.
Let K_{i+1} be a disk such that

$$\varphi_i (B_{i+1}) \subset \overset{\bullet}{K}_{i+1} \subset K_{i+1} \subset K \cup \varphi_i (B_{i+1}) \quad \text{and} \quad K_{i+1} \subset \mathrm{B}\left(\varphi_i (B_{i+1}), \frac{\delta}{2^{i+1}}\right)$$

and let $h_{i+1} : D_{i+1} \longrightarrow K_{i+1}$ be a homeomorphism such that $h_{i+1} (B_{i+1}) = \varphi_i (B_{i+1})$.
Then

$$\varphi_{i+1} = \varphi_i \cup h_{i+1}|_{G_{i+1} \cup A_{i+1} \cup B_{i+1}} : U_x \cup S \cup \bigcup_{j=1}^{i+1} (G_j \cup A_j) \longrightarrow E^2$$

is a homeomorphic embedding and evidently conditions (a)-(d) are fulfilled.
Now suppose B_{i+1} is an arc. $\overset{\circ}{B}_{i+1} \cap \mathrm{Bd}(G \setminus G_{i+1}) = \emptyset$ (this is possible due to $\overset{\circ}{B}_r \cap \overset{\circ}{B}_s = \emptyset$, $r \neq s$ and $\mathrm{Bd}\, G_r \subset B_r$). Then we have $\overset{\circ}{B}_{i+1} \cap \mathrm{Cl}\, U_x = \emptyset$. This implies that there exists a component of $E^2 \setminus \left(D \cup \varphi_i \left(U_x \cup S \cup \bigcup_{j=1}^{i} (G_j \cup A_j)\right)\right)$ and a disk K_{i+1} (with $\overset{\circ}{K}_{i+1}$ included in this component) such that
$\varphi_i (B_{i+1}) \subset \overset{\bullet}{K}_{i+1} \subset K_{i+1} \subset K \cup \varphi_i (B_{i+1})$ and $K_{i+1} \subset \mathrm{B}\left(\varphi_i (B_{i+1}), \frac{\delta}{2^{i+1}}\right)$. Let $h_{i+1} : D_{i+1} \longrightarrow K_{i+1}$ be a homeomorphism such that $h_{i+1} (B_{i+1}) = \varphi_i (B_{i+1})$.
Then $\varphi_{i+1} = \varphi_i \cup h_{i+1}|_{G_{i+1} \cup A_{i+1} \cup B_{i+1}} : U_x \cup S \cup \bigcup_{j=1}^{i+1} (G_j \cup A_j) \longrightarrow E^2$ is a homeomorphic embedding and conditions (a)-(d) are satisfied. In this way our construction of sequence $\{\varphi_i\}_{i=0}^{\alpha}$ is accomplished.

Observe now that if $\alpha = +\infty$ then $\operatorname{diam} \varphi_i (G_i \cup A_i) \xrightarrow{i \to +\infty} 0$ since $\operatorname{diam} \varphi_i (B_i) \xrightarrow{i \to +\infty} 0$ and due to (d). One concludes from the above that

$$\hat{\psi} = \bigcup_{i=0}^{\alpha} \varphi_i : U_x \cup S \cup G \longrightarrow E^2$$

is a homeomorphic embedding. (In the case of $\alpha < +\infty$; $\hat{\psi} = \varphi_\alpha$.)

Finally let $\psi : U_x \cup G \cup L \longrightarrow E^2$, $\psi =$. Then ψ is a homeomorphic embedding such that

1. $\psi|_{U_x \cup L} = \varphi_0|_{U_x \cup L} = \varphi|_{U_x \cup L}$;
2. $D \cap \psi(U_x \cup G \cup L) = \psi(L)$ since $D \cap \varphi(U_x \cup L) = \varphi(L)$ and $\varphi_i(G_i)$ is included in $E^2 \setminus D$. Moreover $\psi(L) = \varphi(L) \subset \overset{\bullet}{D}$.
3. $D \cup \psi(G) \subset \mathrm{B}(\psi(L), \delta)$ since $D \subset \mathrm{B}(\psi(L), \delta)$ and for each G_i

$$\psi(G_i) = \varphi_i(G_i) \subset \mathrm{B}\left(\varphi_i(B_i), \frac{\delta}{2^i}\right) \subset \mathrm{B}\left(D, \frac{\delta}{2^i}\right) \subset \mathrm{B}\left(\varphi(L), \frac{\delta}{2^i}\right)$$
$$\subset \mathrm{B}(\varphi(L), \delta).$$

□

4. A Proof of the theorem (3.1)

Let X be a locally plane and locally connected curve. Let $\varepsilon > 0$ be a real number such that each subset of X diameter less than ε is plane. According to (2.6) let $\mathcal{G}$ be a net partition of X with nodes $N(\mathcal{G})$ and threads $T(\mathcal{G})$ such that $\operatorname{mesh} \mathcal{G} < \frac{\varepsilon}{5}$. Then for every node K $\operatorname{Star}_\mathcal{G} \operatorname{Star}_\mathcal{G} K$ is plane.

Let us order all elements of $N(\mathcal{G})$ into a sequence $K_1, K_2, ..., K_k$ and all elements of $T(\mathcal{G})$ into a sequence $T_1, T_2, ..., T_t$. Let $\varphi_i : \operatorname{Star}_\mathcal{G} \operatorname{Star}_\mathcal{G} K_i \longrightarrow E^2$ for $i = 1, 2, ..., k$ be a homeomorphic embedding into the Euclidean plane.

Now – using $\mathcal{G}$ and embeddings φ_i – we shall construct:

a) closed and connected sets $X_1, X_2, ..., X_k$ which cover X,
b) homeomorphic embeddings $\psi_i : X_i \longrightarrow E^2$,
c) $\mathcal{L} = \{L_1, L_2, ..., L_t\}$ – the family of arcs and points such that $L_j \subset \operatorname{Int} T_j$ and
d) the family $\mathcal{D}$ of $2 \cdot t$ disks in E^2

with the following properties:

for each i such that $1 \le i \le k$:

(P1) $X_i \cap \bigcup \{X_j : 1 \le j \le k,\ j \ne i\} = \bigcup \left\{L_{a_j} : T_{a_j} \subset \operatorname{Star}_\mathcal{G} K_i\right\}$; $\forall i : 1 \le i \le k$;

(P2) if $T_{a_1}, T_{a_2}, ..., T_{a_{r(i)}}$ is denotation of all elements of $T(\mathcal{G})$ which are included in $\operatorname{Star}_\mathcal{G} K_i$ then there exist mutually disjoint disks $D_{a_1}, D_{a_2}, ..., D_{a_{r(i)}}$ from family $\mathcal{D}$ such that

$$\psi_i\left(X_i \setminus \bigcup_{j=1}^{r(i)} L_{a_j}\right) \subset E^2 \setminus \bigcup_{j=1}^{r(i)} D_{a_j} \quad \text{and} \quad \psi_i\left(L_{a_j}\right) \subset \overset{\bullet}{D}_{a_j}$$
$$\text{for} \quad i = 1, 2, ..., k, \quad j = 1, 2, ..., r(i).$$

For $1 \le j \le t$ there exist indices a, b such that $1 \le a, b \le k$ and $\operatorname{Star}_\mathcal{G} T_j = T_j \cup K_a \cup K_b$, since $\mathcal{G}$ is the net partition and then T_j disconnects $\operatorname{Star}_\mathcal{G} T_j$ between K_a and K_b.

Let $\mathcal{P}$ be a net partition obtained from (2.5) for the brick partition $\mathcal{G}$ and let P_j be a node in $\mathcal{P}$ which is included in the interior of T_j. Then P_j disconnects $\operatorname{Star}_\mathcal{G} T_j$ between K_a and K_b.

Let now $\mathcal{R}$ be a brick partition with sufficiently small elements (given from decreasing sequence of partitions of X by (2.3)) such that for every index j $\operatorname{Star}_{\mathcal{R}} P_j \subset \operatorname{Int} T_j$. Since $\operatorname{Star}_{\mathcal{R}} P_j$ disconnects $\operatorname{Star}_{\mathcal{G}} T_j$ between K_a and K_b we can choose a chain with elements from $\{F \in \mathcal{R} : F \subset \operatorname{Star}_{\mathcal{R}} P_j\}$ which disconnects $\operatorname{Star}_{\mathcal{G}} T_j$ between K_a and K_b. (Since T_j is one-dimensional, we can find a chain which is not closed, i.e. the first and the last elements of the chain are mutually disjoint). Then we can choose such an arc in this chain which disconnects $\operatorname{Star}_{\mathcal{G}} T_j$ between K_a and K_b. This arc contains a continuum L_j which irreducibly disconnects $\operatorname{Star}_{\mathcal{G}} T_j$ between K_a and K_b. Evidently L_j is an arc or a point (L_j is to be equal to $X_a \cap X_b \cap T_j$ soon) and $L_j \subset \operatorname{Star}_{\mathcal{R}} P_j \subset \operatorname{Int} T_j$.
$\bigcup_{j=1}^{t} L_j$ disconnects X such that each K_i belongs to different component of $X \setminus \bigcup_{j=1}^{t} L_j$.
Let Y_i denote the component of $\operatorname{Star}_{\mathcal{G}}(K_i) \setminus \bigcup\{L_j : T_j \subset \operatorname{Star}_{\mathcal{G}} K_i\}$ containing K_i and let y_i be any point of Y_i for $i = 1, 2, ..., k$.

Now, for $i = 1, 2, ..., k$ we can define successively a set X_i, a homeomorphic embedding $\psi_i : X_i \longrightarrow E^2$ and the family $\mathcal{D}$ of disks.
Let then $T_{a_1}, T_{a_2}, ..., T_{a_{r(i)}}$ be an order of all elements of $T(\mathcal{G})$ such that each $T_{a_j} \subset \operatorname{Star}_{\mathcal{G}} K_i$ for a given i. Let K_{b_j} for each T_{a_j} denotes such element of $N(\mathcal{G})$ that $\operatorname{Star}_{\mathcal{G}} T_{a_j} = K_{b_j} \cup T_{a_j} \cup K_i$ (sets K_{b_m}, K_{b_n} may be the same for different T_{a_m}, T_{a_n}). Let

$$(4.1) \qquad \delta = \frac{\min\left\{d\left(\varphi_i\left(L_{a_j}\right), \varphi_i\left(\bigcup\{T_{a_n} : 1 \le n \le r(i), n \ne j\}\right)\right) : 1 \le j \le r(i)\right\}}{2}.$$

Now for succeeding $j = 1, 2, ..., r(i)$ and only in case when $i < b_j$ we shall define sets X_{i,a_j}, X_{b_j,a_j}. We define also the homeomorphic embedding $\psi_{i,a_j} : X_{i,a_j} \longrightarrow E^2$ and the disk D_{i,a_j} with the following properties:

(p1) $T_{a_j} \setminus L_{a_j} = T_{a_j} \cap \left(Y_i \cup Y_{b_j}\right) \cup X_{i,a_j} \cup X_{b_j,a_j}$ and sets $Y_i, Y_{b_j}, X_{i,a_j}, X_{b_j,a_j}$ are mutually disjoint,

(p2) $D_{i,a_j} \cup \psi_{i,a_j}\left(X_{i,a_j}\right) \subset \mathrm{B}\left(\varphi_i\left(L_{a_j}\right), \delta\right)$,

(p3) $D_{i,a_j} \cap \left(\varphi_i\left(\left(Y_i \cap \operatorname{Star}_{\mathcal{G}} T_{a_j}\right) \cup L_{a_j}\right) \cup \psi_{i,a_j}\left(X_{i,a_j}\right)\right) = \varphi_i\left(L_{a_j}\right)$ and $\varphi_i\left(L_{a_j}\right) \subset \dot{D}$,

(p4) the map $\varphi_i|_{Y_i \cap \operatorname{Star}_{\mathcal{G}} T_{a_j} \cup L_{a_j}} \cup \psi_{i,a_j} : \left(Y_i \cap \operatorname{Star}_{\mathcal{G}} T_{a_j}\right) \cup L_{a_j} \cup X_{i,a_j} \longrightarrow E^2$ is a homeomorphic embedding.

In order to proceed we shall investigate the two cases as follows:

(i) if $i < b_j$, then let S_{a_j} be the simple closed curve in E^2 obtained according to the lemma (3.3):
  - — for the case of a plane Peanian curve $\varphi_i\left(\operatorname{Star}_{\mathcal{G}} T_{a_j}\right)$,
  - — for the case of points $\varphi_i(y_i), \varphi_i\left(y_{b_j}\right)$ and
  - — for the case of an arc or a point $\varphi_i\left(L_{a_j}\right)$.

Let P_{i,a_j}, P_{b_j,a_j} be the unions of components of $\varphi_i\left(\left(\operatorname{Star}_{\mathcal{G}} T_{a_j}\right) \setminus \left(L_{a_j} \cup Y_i \cup Y_{b_j}\right)\right)$, which are included in component of $E^2 \setminus S_{a_j}$ containing points $\varphi_i(y_i)$, $\varphi_i\left(y_{b_j}\right)$ - respectively.
Let $X_{i,a_j} = \varphi_i^{-1}\left(P_{i,a_j}\right)$, $X_{b_j,a_j} = \varphi_i^{-1}\left(P_{b_j,a_j}\right)$. Then the condition (p1) is fulfilled.

One may now use the lemma (3.4) for the case of:

— a continuum $\varphi_i\left(\mathrm{Star}_{\mathcal{G}}\, T_{a_j}\right)$,

— $\varphi_i\left(L_{a_j}\right)$ disconnecting $\varphi_i\left(\mathrm{Star}_{\mathcal{G}}\, T_{a_j}\right)$ between points $\varphi_i\left(y_i\right), \varphi_i\left(y_{b_j}\right)$,

— a simple closed curve S_{a_j} ,

— components $\varphi_i\left(Y_i \cap \mathrm{Star}_{\mathcal{G}}\, T_{a_j}\right)$, $\varphi_i\left(Y_{b_j} \cap \mathrm{Star}_{\mathcal{G}}\, T_{a_j}\right)$ of $E^2 \backslash S_{a_j}$ and the union of components $\varphi_i\left(X_{i,a_j}\right)$,

— a number δ given by (4.1) and

— a homeomorphic embedding being identity on $\varphi_i\left(\left(Y_i \cup Y_{b_j}\right) \cap \left(\mathrm{Star}_{\mathcal{G}}\, T_{a_j}\right) \cup L_{a_j}\right)$.

In this way we obtain both the homeomorphic embedding

$$\psi : \varphi_i\left(\left(Y_i \cap \mathop{\mathrm{Star}}_{\mathcal{G}} T_{a_j}\right) \cup X_{i,a_j} \cup L_{a_j}\right) \longrightarrow E^2$$

and the disk $D \subset E^2$ such that:

(4.2.1) $\psi|_{\varphi_i\left(\left(Y_i \cap \mathrm{Star}_{\mathcal{G}} T_{a_j}\right) \cup L_{a_j}\right)} = \mathrm{Id}|_{\varphi_i\left(\left(Y_i \cap \mathrm{Star}_{\mathcal{G}} T_{a_j}\right) \cup L_{a_j}\right)}$,

(4.2.2) $D \cap \psi\left(\varphi_i\left(\left(Y_i \cap \mathrm{Star}_{\mathcal{G}}\, T_{a_j}\right) \cup X_{i,a_j} \cup L_{a_j}\right)\right) = \psi\left(\varphi_i\left(L_{a_j}\right)\right)$ and $\psi\left(\varphi_i\left(L_{a_j}\right)\right) \subset \dot{D}$,

(4.2.3) $D \cup \psi\left(\varphi_i\left(X_{i,a_j}\right)\right) \subset \mathrm{B}\left(\psi\left(\varphi_i\left(L_{a_j}\right)\right), \delta\right)$.

Let $\psi_{i,a_j} = \psi \circ \varphi_i|_{X_{i,a_j}}$ and $D_{i,a_j} = D$. Then the condition (p2) follows from (4.2.3), while the condition (p3) follows from (4.2.1) and (4.2.2). The condition (p4) is obtained due

$$\psi \circ \varphi_i|_{\left(Y_i \cap \mathrm{Star}_{\mathcal{G}} T_{a_j}\right) \cup X_{i,a_j} \cup L_{a_j}} = \varphi_i|_{\left(Y_i \cap \mathrm{Star}_{\mathcal{G}} T_{a_j}\right) \cup L_{a_j}} \cup \psi_{i,a_j} .$$

(ii) if $b_j < i$, then X_{i,a_j}, X_{b_j,a_j} are defined and the condition (p1) is satisfied. Moreover the simple closed curve S_{a_j} in E^2 is defined too (see (i)). Then we have:

— $\varphi_{b_j}\left(\mathrm{Star}_{\mathcal{G}}\, T_{a_j}\right)$ is locally connected continuum in E^2,

— $\varphi_{b_j}\left(L_{a_j}\right)$ is a point or an arc, which disconnects $\varphi_{b_j}\left(\mathrm{Star}_{\mathcal{G}}\, T_{a_j}\right)$ between points $\varphi_{b_j}\left(y_i\right)$, $\varphi_{b_j}\left(y_{b_j}\right)$,

— S_{a_j} is a simple closed curve such that $S_{a_j} \cap \varphi_{b_j}\left(\mathrm{Star}_{\mathcal{G}}\, T_{a_j}\right) = \varphi_{b_j}\left(L_{a_j}\right)$ and points $\varphi_{b_j}\left(y_i\right)$, $\varphi_{b_j}\left(y_{b_j}\right)$ are included in different components of $E^2 \backslash S_{a_j}$,

— $\varphi_{b_j}\left(Y_i \cap \mathrm{Star}_{\mathcal{G}}\, T_{a_j}\right)$, $\varphi_{b_j}\left(Y_{b_j} \cap \mathrm{Star}_{\mathcal{G}}\, T_{a_j}\right)$ are components of $\varphi_{b_j}\left(\mathrm{Star}_{\mathcal{G}}\, T_{a_j}\right) \backslash \varphi_{b_j}\left(L_{a_j}\right)$ containing points $\varphi_{b_j}\left(y_i\right)$, $\varphi_{b_j}\left(y_{b_j}\right)$ - respectively,

— $\varphi_{b_j}\left(X_{i,a_j}\right)$ is an union of components of $\varphi_{b_j}\left(\mathrm{Star}_{\mathcal{G}}\, T_{a_j}\right) \backslash \varphi_{b_j}\left(L_{a_j}\right)$ except for $\varphi_{b_j}\left(Y_i \cap \mathrm{Star}_{\mathcal{G}}\, T_{a_j}\right)$ contained in the component of $E^2 \backslash S_{a_j}$, which contains the point $\varphi_{b_j}\left(y_i\right)$,

— δ is positive number,

— $\varphi_i \circ \varphi_{b_j}^{-1}\Big|_{\varphi_{b_j}\left(Y_i \cap \mathrm{Star}_{\mathcal{G}} T_{a_j}\right) \cup \varphi_{b_j}\left(L_{a_j}\right) \cup \varphi_{b_j}\left(Y_{b_j} \cap \mathrm{Star}_{\mathcal{G}} T_{a_j}\right)}$ is homeomorphic embedding in the Euclidean plane.

Therefore, according to the lemma (3.4), there exist a homeomorphic embedding

$$\psi : \varphi_{b_j}\left(\left(Y_i \cap \mathrm{Star}_{\mathcal{G}}\, T_{a_j}\right) \cup X_{i,a_j} \cup L_{a_j}\right) \longrightarrow E^2$$

and a disk $D \subset E^2$ such that:

(4.3.1) $\psi|_{\varphi_{b_j}\left(\left(Y_i \cap \mathrm{Star}_{\mathcal{G}} T_{a_j}\right)\cup L_{a_j}\right)} = \varphi_i \circ \varphi_{b_j}^{-1}\Big|_{\varphi_i\left(\left(Y_i\cap \mathrm{Star}_{\mathcal{G}} T_{a_j}\right)\cup L_{a_j}\right)}$,

(4.3.2) $D \cap \psi\left(\varphi_{b_j}\left(\left(Y_i \cap \mathrm{Star}_{\mathcal{G}}\, T_{a_j}\right) \cup X_{i,a_j} \cup L_{a_j}\right)\right) = \psi\left(\varphi_{b_j}\left(L_{a_j}\right)\right)$ and $\psi\left(\varphi_{b_j}\left(L_{a_j}\right)\right) \subset \dot{D}$,

(4.3.3) $D \cup \psi\left(\varphi_{b_j}\left(X_{i,a_j}\right)\right) \subset \mathrm{B}\left(\psi\left(\varphi_{b_j}\left(L_{a_j}\right)\right), \delta\right)$.

Let now $\psi_{i,a_j} = \psi \circ \varphi_{b_j}\Big|_{X_{i,a_j}}$ and $D_{i,a_j} = D$. Then one obtains the condition (p2) from (4.8.3), while the condition (p3) is met due to (4.3.1) and (4.3.2).

The condition (p4) follows from the equation

$$\psi \circ \varphi_{b_j}\Big|_{\left(Y_i \cap \mathrm{Star}_{\mathcal{G}} T_{a_j}\right)\cup X_{i,a_j}\cup L_{a_j}} = \varphi_i|_{\left(Y_i\cap \mathrm{Star}_{\mathcal{G}} T_{a_j}\right)\cup L_{a_j}} \cup \psi_{i,a_j}\ .$$

Finally let $X_i = Y_i \cup \bigcup_{j=1}^{r(i)} L_{a_j} \cup \bigcup_{j=1}^{r(i)} X_{i,a_j}$ and $\psi_i = \varphi_i|_{Y_i \cup \bigcup_{j=1}^{r(i)} L_{a_j}} \cup \bigcup_{j=1}^{r(i)} \psi_{i,a_j}$, i.e.

$\psi_i|_{Y_i \cup \bigcup_{j=1}^{r(i)} L_{a_j}} = \varphi_i|_{Y_i \cup \bigcup_{j=1}^{r(i)} L_{a_j}}$ and for $1 \leq j \leq r(i)$ $\quad \psi_i|_{X_{i,a_j}} = \psi_{i,a_j}$.

It is easy now to show – using (p2) and (p4) – that is the homeomorphic embedding into E^2.

The condition (P1) follows from (p1) and the condition (P2) – follows from (p2), (p3).

Let $r(i)$ – for each i such that $1 \leq i \leq k$ – denote number of elements of $T(\mathcal{G})$ which are included in $\mathrm{Star}_{\mathcal{G}}\, K_i$ and let $T_{i,1}, T_{i,2}, ..., T_{i,r(i)}$ be the sequence of all these elements. Let $L_{i,j}$ be an element of the family $\mathcal{L}$ such that $L_{i,j} \subset T_{i,j}$ and let $D_{i,j}$ be a disk from the family $\mathcal{D}$ such that $\psi_i(L_{i,j}) \subset \dot{D}$ for $1 \leq i \leq k$, $1 \leq j \leq r(i)$. Then for each pair of indices (i,j) such that $1 \leq i \leq k$, $1 \leq j \leq r(i)$ there exists exactly one pair (i',j') such that $i \neq i'$ and $T_{i,j} = T_{i',j'}$. Let us use (κ_1, κ_2) to define this very equivalence of indices i.e.

$$(\kappa_1(i,j), \kappa_2(i,j)) = (i',j').$$

Then both $L_{i,j} = L_{\kappa_1(i,j),\kappa_2(i,j)}$ and the homeomorphism

$$\sigma_{i,\kappa_1(i,j)} = \psi_{\kappa_1(i,j)} \circ \psi_i^{-1}\Big|_{\psi_i(L_{i,j})} : \psi_i(L_{i,j}) \longrightarrow \psi_{\kappa_1(i,j)}(L_{i,j})$$

can be extended into homeomorphism of simple closed curves

$$\hat{\sigma}_{i,\kappa_1(i,j)} : \dot{D}_{i,j} \longrightarrow \dot{D}_{\kappa_1(i,j),\kappa_2(i,j)}\ .$$

Finally let M_i be a bounded surface homeomorphic to 2-dimensional sphere with $r(i)$ boundaries and let $\gamma_i : E^2 \setminus \bigcup_{j=1}^{r(i)} \mathring{D}_{i,j} \longrightarrow M_i$ be a homeomorphic embedding for

$i = 1, 2, ..., k$. Let M be the closed surface obtained from $\bigcup_{i=1}^{k} M_i$ by identification of boundaries $\gamma_i\left(\overset{\bullet}{D}_{i,j}\right)$ with $\gamma_{\kappa_1(i,j)}\left(\overset{\bullet}{D}_{\kappa_1(i,j),\kappa_2(i,j)}\right)$ via homeomorphism

$$\gamma_{\kappa_1(i,j)} \circ \hat{\sigma}_{i,\kappa_1(i,j)} \circ \gamma_i^{-1}\Big|_{\gamma_i\left(\overset{\bullet}{D}_{i,j}\right)} : \gamma_i\left(\overset{\bullet}{D}_{i,j}\right) \longrightarrow \gamma_{\kappa_1(i,j)}\left(\overset{\bullet}{D}_{\kappa_1(i,j),\kappa_2(i,j)}\right).$$

From the properties (P1), (P2) it then follows that there exists a homeomorphic embedding of X into M. □

5. Final remarks

In view of the theorems proved in this paper one may conclude that

The class of locally plane Peano continua appears to be much more regular then it was supposed to be.

As the result the only continua for which a homeomorphic embedding into a topological surface does not exist are those continua, which are not locally plane or which are not locally connected.

Finally, locally plane Peano continua which appeared to be regular (due to the theorems (3.1) and (3.2)) deserve to be investigated further in detail with the well established topological surfaces methods at hand.

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Institute of Informatics, Białystok University
Sosnowa 64, Pl 15-887 Białystok, POLAND
E-mail: wbajguz@ii.uwb.edu.pl