

Jana Bělohávková; Josef Mikeš; Olga Pokorná

On 4-planar mapping of special almost antiquaternionic spaces

In: Jan Slovák and Martin Čadek (eds.): Proceedings of the 20th Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 2001. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 66. pp. 97–103.

Persistent URL: <http://dml.cz/dmlcz/701670>

Terms of use:

© Circolo Matematico di Palermo, 2001

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

ON 4-PLANAR MAPPINGS OF SPECIAL ALMOST ANTIQUATERNIONIC SPACES

JANA BĚLOHLÁVKOVÁ, JOSEF MIKEŠ, OLGA POKORNÁ

ABSTRACT. In the paper special 4-planar mappings of almost antiquaternionic Hermitian spaces are studied. Fundamental equations of these mappings are expressed in linear Cauchy form.

The 4-quasiplanar mappings of an almost quaternionic space have been studied in [5], [9] and [14]. These mappings generalize the geodesic, quasigeodesic and holomorphically projective mappings of Riemannian and Kählerian spaces, see [4], [12], [13], [15], [17], [18]. Similar problems are studied on complex manifolds in [2]. Antiquaternionic spaces which were studied e.g. in [11], [16] have some properties similar to those of quaternions [1]. This fact can be used in the study of 4-planar mappings of almost antiquaternionic spaces.

I. N. Kurbatova studied a special kind of 4-planar mappings (called 4-quasiplanar, see [9]) from a Riemannian space V_n onto another Riemannian space \bar{V}_n where an almost quaternionic structure on V_n is Hermitian and it satisfies additional conditions so that V_n a \bar{V}_n are Apt spaces.

Analyzing the results of [9] (theorems 2 – 6) we noticed that the space \bar{V}_n is implicitly supposed to be Hermitian and this assumption is essential. The Hermitian structure of \bar{V}_n is more important than the Hermitian structure of V_n and, moreover, it simplifies fundamental equations of 4-planar mappings. In this paper we do not assume V_n to be Hermitian.

1. A well-known definition says that an *almost antiquaternionic* space is a differentiable manifold M_n with almost product structures $\overset{1}{F}$ and $\overset{2}{F}$ satisfying

$$(1) \quad \overset{1}{F}_\alpha^h \overset{1}{F}_i^\alpha = \delta_i^h; \quad \overset{2}{F}_\alpha^h \overset{2}{F}_i^\alpha = \delta_i^h; \quad \overset{1}{F}_\alpha^h \overset{2}{F}_i^\alpha + \overset{2}{F}_\alpha^h \overset{1}{F}_i^\alpha = 0,$$

where δ_i^h is the Kronecker symbol, see e.g. [3], [14], [16].

2000 *Mathematics Subject Classification*: 53B25.

Key words and phrases: 4-planar mappings, almost antiquaternionic spaces, Hermitian spaces.

Supported by grant No. 201/99/0265 of The Grant Agency of Czech Republic.

The paper is in final form and no version of it will be submitted elsewhere.

The tensor $\overset{3}{F}_i^h \equiv \overset{1}{F}_i^\alpha \overset{2}{F}_\alpha^h$ defines an almost complex structure. The relations among the tensors $\overset{1}{F}, \overset{2}{F}, \overset{3}{F}$ are the following

$$(2) \quad \overset{1}{F}_i^h = -\overset{2}{F}_i^\alpha \overset{3}{F}_\alpha^h = \overset{3}{F}_i^\alpha \overset{2}{F}_\alpha^h, \quad \overset{2}{F}_i^h = -\overset{3}{F}_i^\alpha \overset{1}{F}_\alpha^h = \overset{1}{F}_i^\alpha \overset{3}{F}_\alpha^h, \quad \overset{3}{F}_i^h = \overset{1}{F}_i^\alpha \overset{2}{F}_\alpha^h = -\overset{2}{F}_i^\alpha \overset{1}{F}_\alpha^h,$$

i.e. that the three structures $\overset{1}{F}, \overset{2}{F}, \overset{3}{F}$ define an *almost antiquaternionic structure*.

Let $A_n \equiv (M_n, \Gamma, \overset{1}{F}, \overset{2}{F}, \overset{3}{F})$ be an almost antiquaternionic space with a torsion-free affine connection Γ .

Definition 1. A curve $\ell: x^h = x^h(t)$ in A_n is called *4-planar*, if the tangent vector $\lambda^h = dx^h/dt$, being parallelly transported along this curve, remains in the linear 4-dimensional space generated by the tangent vector λ^h and the corresponding vectors $\overset{1}{F}_\alpha^h \lambda^\alpha, \overset{2}{F}_\alpha^h \lambda^\alpha$ and $\overset{3}{F}_\alpha^h \lambda^\alpha$.

A curve is 4-planar if and only if the equations

$$\frac{d\lambda^h}{dt} + \Gamma_{\alpha\beta}^h \lambda^\alpha \lambda^\beta = \sum_{s=0}^3 \rho_s^h \overset{s}{F}_\alpha^h \lambda^\alpha$$

hold, where $\overset{0}{F}_i^h \equiv \delta_i^h, \Gamma_{\alpha\beta}^h$ are components of the affine connection on A_n and $\rho_s^h = \rho_s^h(t)$ ($s = 0, \dots, 3$) denote functions of the parameter t .

Any geodesic curve is a special case of a 4-planar curve where $\rho_1 \equiv \rho_2 \equiv \rho_3 \equiv 0$.

Consider two spaces A_n and \bar{A}_n with the same underlying manifold M_n and the same almost antiquaternionic structure $(\overset{1}{F}, \overset{2}{F}, \overset{3}{F})$ but with two different torsion-free affine connections Γ and $\bar{\Gamma}$, respectively.

Definition 2. A diffeomorphism $f: A_n \rightarrow \bar{A}_n$ is called a *4-planar mapping*, if it maps any geodesic of A_n to a 4-planar curve of \bar{A}_n .

Remark. In the following we shall attach to each local map φ around a point $p \in A_n$ the local map $\varphi \circ f^{-1}$ around the point $f(p) \in \bar{A}_n$. This means that any point $x \in A_n$ and the corresponding point $f(x) \in \bar{A}_n$ will have the same local coordinates.

The following theorem holds [14]:

Theorem 1. A diffeomorphism of A_n onto \bar{A}_n is a 4-planar mapping if and only if in every local coordinate system $x = (x^1, x^2, \dots, x^n)$ the conditions

$$(3) \quad \bar{\Gamma}_{ij}^h(x) = \Gamma_{ij}^h(x) + \sum_{s=0}^3 \psi_{(i}^s \overset{s}{F}_{j)}^h$$

hold, where Γ_{ij}^h and $\bar{\Gamma}_{ij}^h$ are components of the affine connections Γ and $\bar{\Gamma}$, respectively, $\psi_{(i}^s(x), s = 0, \dots, 3$, are covectors, and (ij) denotes a symmetrization of indices.

Using Theorem 1 one can prove that all 4-planar curves of A_n are mapped onto 4-planar curves of \bar{A}_n .

Finally, we will consider a special case of \bar{A}_n , namely an almost antiquaternionic Riemannian space $\bar{V}_n \equiv (M_n, \bar{g}, \overset{1}{F}, \overset{2}{F}, \overset{3}{F})$ in which $\bar{\Gamma}$ denote the Levi-Civita connection of \bar{g} .

The following theorem holds (see [14]).

Theorem 2. *A diffeomorphism $f: A_n \rightarrow \bar{V}_n$ is a 4-planar mapping if and only if the metric tensor $\bar{g}_{ij}(x)$ satisfies the following equations:*

$$(4) \quad \bar{g}_{ij,k} = \sum_{s=0}^3 \left(\psi_k \bar{g}_{\alpha(i} \overset{s}{F}_{j)}^{\alpha} + \psi_s (i \bar{g}_j)_{\alpha} \overset{s}{F}_k^{\alpha} \right),$$

where comma denotes the covariant derivative in A_n .

Recall that the covariant derivative of \bar{g} in \bar{V}_n is zero.

The proof follows from the fact that formulas (3) and (4) are equivalent in our special case.

2. Now we shall prove the following two lemmas.

Consider the spaces A_n, \bar{A}_n and let " , " or " | " before an index denote a covariant derivative w.r. to the corresponding local variable on A_n and \bar{V}_n , respectively. Now and further we will suppose that the affinors $\overset{s}{F}_i^h$ defining the almost antiquaternionic structure are traceless, i.e. $\overset{s}{F}_{\alpha}^{\alpha} = 0, s = 1, 2, 3$.

Lemma 1. *Let a 4-planar mapping $A_n \rightarrow \bar{A}_n$ be given and let $\psi_s i$ denote the corresponding covectors from (3). Then*

$$(5) \quad \overset{s}{F}_{i,\alpha}^{\alpha} = \overset{s}{F}_{i|\alpha}^{\alpha}, \quad s = 1, 2, 3.$$

holds if and only if the covectors $\psi_s i$ are expressed by formulas

$$(6) \quad \psi_s i = \frac{n^2 + 2n}{n^2 - 2n + 8} \psi_{\alpha} \overset{s}{F}_i^{\alpha}, \quad s = 1, 2, \quad \psi_3 i = -\frac{n^2 - 6n}{n^2 - 2n + 8} \psi_{\alpha} \overset{3}{F}_i^{\alpha}, \quad \psi_i \equiv \psi_0 i.$$

The proof of the Lemma 1 is a consequence of (5) and the fundamental equations of 4-planar mappings (3). We use also the algebraic properties (1) and (2) of antiquaternionic structures.

A manifold with an affine connection Γ and an almost complex structure F is said to be an *Apt space* (see [4], [9], or *nearly Kählerian space* or *Tachibana space* [4], [6], [7], [8], [10], [19]), if its structure F satisfies $F_{i,\alpha}^{\alpha} = 0$; a space $A_n = (M_n, \Gamma, \overset{1}{F}, \overset{2}{F}, \overset{3}{F})$ to be an *almost antiquaternionic Apt space*, if

$$\overset{s}{F}_{i,\alpha}^{\alpha} = 0, \quad s = 1, 2, 3.$$

Lemma 1 implies that an Apt space A_n is 4-planarly mapped on an Apt space \bar{A}_n iff (6) holds. Obviously, antiquaternionic Kählerian spaces are Apt spaces.

Contracting (3) with respect to h and j we got the lemma

Lemma 2. *If for a 4-planar mapping $A_n \rightarrow \bar{A}_n$ the formulae (6) hold and the spaces A_n and \bar{A}_n are equiaffine, then the vector ψ_i is a gradient, i.e. there exists a function ψ such that $\psi_i = \psi_{,i}$.*

3. Now we shall show that if a 4-planar mapping from A_n onto a Riemannian space \bar{V}_n is given, then the formulae (3) and (4) are both equivalent to the following formula:

$$(7) \quad \bar{g}^{ij}_{,k} = - \sum_{s=0}^3 \left(\psi_{,k} \bar{g}^{\alpha(i} \bar{F}_\alpha^{j)} + \psi_{,s} \bar{g}^{\alpha(i} \bar{F}_k^{j)} \right),$$

where \bar{g}^{ij} is the inverse matrix of metric tensor \bar{g}_{ij} . In fact, (7) is a consequence of the identity $\bar{g}^{ij}_{,k} = -\bar{g}_{\alpha\beta,k} \bar{g}^{\alpha i} \bar{g}^{\beta j}$.

In what follows we shall assume an antiquaternionic structure on \bar{V}_n which is *Hermitian*, i.e. we have

$$(8) \quad \bar{g}_{i\alpha} \bar{F}_j^\alpha + \bar{g}_{j\alpha} \bar{F}_i^\alpha = 0, \quad s = 1, 2, 3.$$

(8) is equivalent with

$$(9) \quad \bar{g}^{i\alpha} \bar{F}_\alpha^j + \bar{g}^{j\alpha} \bar{F}_\alpha^i = 0, \quad s = 1, 2, 3,$$

or with

$$(10) \quad \bar{g}^{\alpha\beta} \bar{F}_\alpha^i \bar{F}_\beta^j = e_s \bar{g}^{ij}, \quad s = 1, 2, 3, \quad e_1 = e_2 = -1, \quad e_3 = 1.$$

Using (9), the equations of 4-planar mappings are simplified to

$$(11) \quad \bar{g}^{ij}_{,k} = -2\psi_{,k} \bar{g}^{ij} - \sum_{s=0}^3 \psi_{,s} \bar{g}^{\alpha(i} \bar{F}_k^{j)}.$$

Suppose now that the covector ψ_i is a gradient, i.e. $\psi_i \equiv \psi_{,i} \equiv \psi_{,i}$ where ψ is a function. We define the tensor

$$a^{ij} \equiv e^{2\psi} \bar{g}^{ij}.$$

Then (11) can be rewritten in the form

$$(12) \quad a^{ij}_{,k} = \sum_{s=0}^3 \lambda_s^i \bar{F}_k^{(j)},$$

where

$$(13) \quad \lambda_s^i \equiv -\psi_{,s} \bar{g}^{\alpha i}.$$

By the definition of the tensor a^{ij} (10) is equivalent with

$$(14) \quad a^{\alpha\beta} \bar{F}_\alpha^i \bar{F}_\beta^j = e_s a^{ij}, \quad s = 1, 2, 3.$$

Due to the fact that \bar{V}_n is Hermitian and using (13) we see that the formula (6) is equivalent with

$$(15) \quad \lambda_s^i = \frac{n^2 + 2n}{n^2 - 2n + 8} \lambda^\alpha \bar{F}_\alpha^i, \quad s = 1, 2, \quad \lambda_3^i = -\frac{n^2 - 6n}{n^2 - 2n + 8} \lambda^\alpha \bar{F}_\alpha^i, \quad \lambda^i \equiv \lambda_0^i.$$

Now we come back to the affine case. Let a space A_n be given as before and let the system of equations (12), (14) and (15) have a solution for a regular matrix

function a^{ij} and a vector function λ^i . Then one can prove that the inverse matrix $\|\tilde{g}_{ij}\| = \|a^{ij}\|^{-1}$ defines a Riemannian metric \tilde{g} on M_n and the covector $\lambda^\alpha \tilde{g}_{\alpha i}$ is a gradient $\text{grad } \psi$. By the conformal change $\tilde{g}_{ij} = e^{2\psi} \tilde{g}_{ij}$ we obtain a new metric \bar{g} for which A_n becomes a almost antiquaternionic Hermitian space \bar{V}_n . Moreover, there exists a 4-planar mapping $A_n \rightarrow \bar{V}_n$.

Now we can conclude the above results with

Theorem 3. *Under the condition (5) an equiaffine space A_n admits a 4-planar mapping on an antiquaternionic Hermitian space \bar{V}_n if and only if there exists a regular tensor a^{ij} on A_n satisfying (12), (14), and (15).*

This result coincides with the result by N.S. Sinyukov for geodesic mappings and the results by V.V. Domashev and J. Mikeš for holomorphically projective mappings of Kählerian spaces etc., see [12], [13], [18].

4. In the following we will analyze the equations (12), (14) and (15). We consider the covariant derivatives of (14) in A_n , i.e.

$$a^{\alpha\beta} \overset{r}{F}_{\alpha,k}^i \overset{r}{F}_{\beta}^j + a^{\alpha\beta} \overset{r}{F}_{\alpha,k}^i \overset{r}{F}_{\beta}^j + a^{\alpha\beta} \overset{r}{F}_{\alpha}^i \overset{r}{F}_{\beta,k}^j = e_r a^{ij}_{,k}, \quad r = 1, 2, 3.$$

Putting (12) into the above equation we get

$$(16) \quad \sum_{s=0}^3 \left(e_r \lambda^i \overset{s}{F}_k^j - \lambda^\alpha \overset{r}{F}_\alpha^i \overset{s}{F}_k^j \right) = a^{\alpha\beta} \overset{r}{F}_{\alpha,k}^i \overset{r}{F}_\beta^j.$$

For $r = 3$, using (1), (2) and (15) we have

$$(17) \quad \lambda^i \delta_k^j - \lambda^\alpha \overset{3}{F}_\alpha^i \overset{3}{F}_k^j = \frac{n^2 - 2n + 8}{4(n + 2)} a^{\alpha\beta} \overset{3}{F}_{\alpha,k}^i \overset{3}{F}_\beta^j$$

and contracting (17) with respect to j and k we have the following expression of the vector λ^i :

$$(18) \quad \lambda^i = \frac{n^2 - 2n + 8}{4(n + 2)^2} a^{\alpha\beta} \left(\overset{3}{F}_{\alpha,\gamma}^i \overset{3}{F}_\beta^\gamma + \overset{3}{F}_\alpha^i \overset{3}{F}_{\beta,\gamma}^\gamma \right).$$

It implies that λ^i can be expressed as linear functions in a^{ij} . It implies

Theorem 4. *Under the condition (5) an equiaffine space A_n admits a 4-planar mapping onto a Hermitian almost quaternionic space \bar{V}_n if and only if the following system of differential equations of Cauchy type is solvable with respect to the unknown functions a^{ij} :*

$$(19) \quad a^{ij}_{,k} = \sum_{s=0}^3 \lambda^i \overset{s}{F}_k^j,$$

where

$$(20) \quad \lambda^i \equiv \lambda_0^i, \quad \lambda_s^i = \frac{n^2 + 2n}{n^2 - 2n + 8} \lambda^\alpha \overset{s}{F}_\alpha^i, \quad s = 1, 2, \quad \lambda_3^i = -\frac{n^2 - 6n}{n^2 - 2n + 8} \lambda^\alpha \overset{3}{F}_\alpha^i,$$

$$\lambda^i = \frac{n^2 - 2n + 8}{4(n + 2)^2} a^{\alpha\beta} \left(\overset{3}{F}_{\alpha,\gamma}^i \overset{3}{F}_\beta^\gamma + \overset{3}{F}_\alpha^i \overset{3}{F}_{\beta,\gamma}^\gamma \right),$$

and the matrix (a^{ij}) satisfies the algebraic condition

$$(21) \quad |a^{ij}| \neq 0 \quad \text{and} \quad a^{\alpha\beta} \overset{s}{F}_\alpha^i \overset{s}{F}_\alpha^j = e_s a^{ij}, \quad s = 1, 2, 3.$$

The system (19) does not have more than one solution for the initial Cauchy conditions $a^{ij}(x_o) = a_o^{ij}$ under the conditions (21). Therefore the general solution of (19) does not depend on more than $N_o = (n/2)^2$ parameters. The question of existence of a solution of (19) leads to the studium of integrability conditions, which are linear equations w.r. to the unknowns $a^{ij}(x)$ with coefficients from the space A_n .

Remarks. If the antiquaternionic structure is covariantly constant in A_n , i.e. $\overset{s}{F}_{i,j}^h = 0$, $s = 1, 2, 3$, then under the conditions of the Theorem 4, the 4-planar mapping is only affine. This follows from the fact, that by (20) $\lambda^i = 0$, $\lambda_s^i = 0$ and by (19) $a^{ij,k} = 0$. This is equivalent to $\bar{g}_{ij,k} = 0$, which is the condition for the mapping $A_n \rightarrow \bar{V}_n$ to be affine.

REFERENCES

- [1] Alekseevsky D. V., Marchiafava S., *Transformation of a quaternionic Kähler manifold*, C.R. Acad. Sci., Paris, Ser. I, 320, No. 6 (1995), 703–708.
- [2] Bailey, T. N., Eastwood, M. G., *Complex Paraconformal Manifolds – their Differential Geometry and Twistor Theory*, Forum Math., 3 (1991), 61–103.
- [3] Bejancu A., *Geometry of CR-submanifolds*, Math. and Its Appl., 23, Kluwer Acad. Publ. Group. XII (1986), 169p.
- [4] Beklemishev D. V., *Differential geometry of spaces with almost complex structure*. Geometria. Itogi Nauki i Tekhn., All-Union Inst. for Sci. and Techn. Information (VINITI), Akad. Nauk SSSR, Moscow (1965), 165–212.
- [5] Bělohávková J., Mikeš J., Pokorná O., *4-planar mappings of almost quaternionic and almost antiquaternionic spaces*, Proc. of the Third Int. Workshop on Diff. Geometry and its Appl. and the First German – Romanian Seminar on Geometry, Sibiu – Romania, Sept. 18 – 23. 1997. General Mathematics, vol. 5 (1997), 101–108.
- [6] Gray A., *The structure of nearly Kaehler manifolds*, Math. Ann. 223 (1976), 233–248.
- [7] Gray A., Hervella L. M., *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl., IV. Ser. 123 (1980), 35–58.
- [8] Koto S., *Some theorems on almost Kaehlerian spaces*, J. Math. Soc. Japan 12 (1960), 422–433.
- [9] Kurbatova I. N., *4-quasi-planar mappings of almost quaternion manifolds*, Sov. Math. 30, 100–104 (1986); translation from Izv. Vyssh. Uchebn. Zaved., Mat., No. 1 (1986), 75–78.
- [10] Libermann P., *Sur la classification des structures presque hermitiennes*, Proc. IV. int. Colloq. Differential geometry, Santiago de Compostela 1978 (1979), 168–191.
- [11] Marchiafava S., Nagy P. T., *3-webs and pseudo hypercomplex structures*, (manuscript).
- [12] Mikeš J., *Geodesic mappings on affine-connected and Riemannian spaces*, J. Math. Sci., New York, 78, 3 (1996), 311–333.
- [13] Mikeš J., *Holomorphically-projective mappings and its generalizations*, J. Math. Sci., New York, 89, 3 (1998), 1334–1353.
- [14] Mikeš J., Němčková J., Pokorná O., *On the theory of the 4-quasiplanar mappings of almost quaternionic spaces*, Proc. of Winter School "Geometry and Physic", Srní, January 1997, Suppl. Rend. Circ. Mat. Palermo, II. Ser. 54 (1998), 75–81.
- [15] Mikeš J., Sinyukov N. S., *On quasiplanar mappings of space of affine connection*, Sov. Math. 27, 63–70 (1983); translation from Izv. Vyssh. Uchebn. Zaved., Mat., No. 1 (1983), 55–61.
- [16] Nagy P. T., *Invariant tensor field and the canonical connection of a 3-webs*, Aequationes Math., 35 (1988), 31–44.
- [17] Petrov A. Z., *Simulation of physical fields*, Grav. i teor. Otnos., Issues 4–5, Kazan State Univ. Press, Kazan (1968), 7–21. (1968).

- [18] Sinyukov N. S., *Geodesic mappings on Riemannian spaces*, Nauka, Moscow, 1979.
[19] Tachibana, S.-I., *On automorphisms of certain compact almost-Hermitian spaces*, Tohoku Math. J., II. Ser. 13 (1961), 179–185.

DEPARTMENT OF MATHEMATICS, TECHNICAL UNIVERSITY
17. LISTOPADU, 708 33 OSTRAVA, CZECH REPUBLIC
E-mail: Jana.Belohlavkova@vsb.cz

DEPARTMENT OF ALGEBRA AND GEOMETRY
FACULTY OF SCIENCE, PALACKY UNIVERSITY
TOMKOVA 40, 779 00 OLMOUC, CZECH REPUBLIC
E-mail: Mikes@matnw.upol.cz

DEPARTMENT OF MATHEMATICS, CZECH UNIVERSITY OF AGRICULTURE
KAMÝČKÁ 129, PRAHA 6, CZECH REPUBLIC
E-mail: Pokorna@tf.czu.cz