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SPECIAL KAEHLER MANIFOLDS: A SURVEY

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ABSTRACT. This is a survey of recent contributions to the area of special Kaehler geometry. It is based on lectures given at the 21st Winter School on Geometry and Physics held in Srni in January 2001.

1. REMARKABLE FEATURES OF SPECIAL KAEHLER MANIFOLDS

A *(pseudo-) Kaehler manifold* (M, J, g) is a differentiable manifold endowed with a complex structure J and a (pseudo-) Riemannian metric g such that
(i) J is orthogonal with respect to the metric g, i.e. $J^*g = g$ and
(ii) J is parallel with respect to the Levi Civita connection D, i.e. $DJ = 0$.
In the following we will always allow pseudo-Riemannian, i.e. possibly indefinite metrics. The prefix "pseudo" will be generally omitted. The following definition is by now standard, see [F].

Definition 1. A *special Kaehler manifold* (M, J, g, ∇) is a Kaehler manifold (M, J, g) together with a flat torsionfree connection ∇ such that
(i) $\nabla\omega = 0$, where $\omega = g(\cdot, J\cdot)$ is the Kaehler form and
(ii) ∇J is symmetric, i.e. $(\nabla_X J)Y = (\nabla_Y J)X$ for all vector fields X and Y.

More precisely, one should speak of *affine* special Kaehler manifolds since there is also a projective variant of special Kaehler manifolds. In fact, there is a class of (affine) special Kaehler manifolds M, which are called *conic* special Kaehler manifolds and which admit a certain $\mathbb{C}^*$-action. The quotient of M by that action can be considered as projectivisation of M and is called a *projective* special Kaehler manifold, see [ACD]. Originally [dWVP1], in the supergravity literature, by a special Kaehler manifold one understood a projective special Kaehler manifold. This terminology was mantained in the first mathematical papers on that subject [C1, C2, AC] and abandoned with the publication of [F].
Example 1. Let (M, J, g) be a flat Kaehler manifold, i.e. the Levi Civita connection D is flat. Then $(M, J, g, \nabla = D)$ is a special Kaehler manifold and $\nabla J = 0$. Conversely, any special Kaehler manifold (M, J, g, ∇) such that $\nabla J = 0$ satisfies $\nabla = D =$ Levi Civita connection of the flat Kaehler metric g. This is the trivial example of a special Kaehler manifold.

The paper is in final form and no version of it will be submitted elsewhere.

Before giving a general construction of special Kaehler manifolds, which yields plenty of non-flat examples, I would like to offer some motivation for that concept.

- The notion of special Kaehler manifold was introduced by the physicists de Wit and Van Proeyen [dWVP1] and has its origin in certain supersymmetric field theories. More precisely, *affine* special Kaehler manifolds are exactly the allowed targets for the scalars of the vector multiplets of field theories with $N = 2$ *rigid* supersymmetry on four-dimensional Minkowski spacetime. *Projective* special Kaehler manifolds correspond to such theories with *local* supersymmetry, which describe $N = 2$ *supergravity* coupled to vector multiplets. $N = 2$ supergravity theories occur as low energy limits of type II superstrings and play a prominent role in the study of moduli spaces of certain two-dimensional superconformal field theories [CFG]. The structure of these moduli spaces is desribed as the product of a projective special Kaehler manifold and a quaternionic Kaehler manifold. Besides these strong physical motivations there is also a number of rather mathematical reasons to study special Kaehler manifolds.
- Interesting *moduli spaces* carry the structure of a special Kaehler manifold, for example:
  - The (Kuranishi) moduli space M_X of gauged complex structures associated to a Calabi-Yau 3-fold X is a special Kaehler manifold of complex signature $(1, n)$, $n = h^{2,1}(X)$. This fact can be found in the physical literature, see e.g. [S] and references therein. M_X parametrises pairs $(J, \mathrm{vol}\,)$, where J is a complex structure and vol a J-holomorphic volume form on a given compact Calabi-Yau manifold of complex dimension 3. Let me recall that (from the Riemannian point of view) a *Calabi-Yau n-fold* is a Riemannian manifold with holonomy group SU(n). More generally, the affine cone over any abstract variation of polarized Hodge structure of weight 3 and with $h^{3,0} = 1$ is a (conic) special Kaehler manifold, see [C2]. Such cones can be considered as formal moduli spaces, i.e. the underlying variation of Hodge structure is not necessarily induced by the deformation of complex structure of some Kaehler manifold.
  - The moduli space of deformations of a compact complex Lagrangian submanifold Y in a hyper-Kaehler manifold X is a special Kaehler manifold with positive definite metric [H1]. A *hyper-Kaehler manifold* is a Riemannian manifold with holonomy group in Sp(n). Such a manifold X is automatically Kaehler of complex dimension $2n$ and carries a holomorphic symplectic structure Ω. A complex submanifold $Y \subset X$ of complex dimension n is called *Lagrangian* if $\iota^*\Omega = 0$, where $\iota : Y \to X$ is the inclusion map.
- The cotangent bundle of any special Kaehler manifold carries the structure of a hyper-Kaehler manifold. This corresponds to the dimensional reduction of $N = 2$ supersymmetric theories from four to three spacetime dimensions [CFG]. This construction, which is called the *c-map* in rigid supersymmetry, is discussed, applied and generalised in the mathematical literature [C2, F, H1, ACD]. For example, it is used in [C2] to obtain a hyper-Kaehler structure (of complex signature $(2, 2n)$) on the bundle $\mathcal{J} \to M_X$ of intermediate Jacobians over the above moduli space M_X associated to a Calabi-Yau 3-fold X. The fibre of the holomorphic bundle $\mathcal{J}$ over

$(J, \mathrm{vol}\,) \in M_X$ is the *intermediate Jacobian*

$$\frac{H^3(X,\mathbb{C})}{H^{3,0}(X,J)+H^{2,1}(X,J)+H^3(X,\mathbb{Z})}$$

of (X, J).

- There is also a c-map in local supersymmetry, i.e. in supergravity, which to any projective special Kaehler manifold of (real) dimension $2n$ associates a quaternionic Kaehler manifold of dimension $4n+4$ [FS]. It corresponds to the dimensional reduction of $N = 2$ supergravity coupled to vector multiplets from dimension four to three. For mathematical discussions of this deep construction, see [H3, K].
- The base of any algebraic completely integrable system is a special Kaehler manifold, see [DW, F]. An *algebraic completely integrable system* is a holomorphic submersion $\pi : X \to M$ from a complex symplectic manifold X to a complex manifold M with compact Lagrangian fibres and a smooth choice of polarisation on the fibres. This is essentially the inverse construction of the rigid c-map. There should also exist an inverse construction for the local c-map.
- It was shown in [ACD] that the notion of special Kaehler manifold has natural generalisations in the absence of a metric: "special complex" and "special symplectic" manifolds. The cotangent bundle of such manifolds carries interesting geometric structures which generalise the hyper-Kaehler structure on the cotangent bundle of a special Kaehler manifold. Special complex geometry (in the absence of a metric) may provide insight in physical theories for which no Lagrangian formulation (and for that reason no target metric) is available.
- There is a close relation between special Kaehler manifolds and affine differential geometry discovered in [BC1]. In fact, any simply connected special Kaehler manifold has a canonical realisation as a parabolic affine hypersphere. This will be explained in detail in section 3.
  Any projective special Kaehler manifold has a canonical (pseudo-) Sasakian circle bundle which is realised as a proper affine hypersphere [BC3].
  It was discovered in [BC2] that special Kaehler manifolds with a flat indefinite metric have a nontrivial moduli space, which is closely related to the moduli space of Abelian simply transitive affine groups of symplectic type.
- Homogeneous projective special Kaehler manifolds were classified, under various assumptions in [dWVP2, C1, AC]. Under the c-map they give rise to homogeneous quaternionic Kaehler manifolds. If one restricts attention to the homogeneous projective special Kaehler manifolds of *semi-simple* group, then one finds a list of Hermitian symmetric spaces of non-compact type which shows a remarkable coincidence with the list of irreducible special holonomy groups of torsionfree symplectic connections [MS], as was noticed in [AC]. Finally, the classification [AC] may lead to the generalisation of recent ideas of Hitchin about special features of geometry in six dimensions to other dimensions [H2].

2. The construction of special Kaehler manifolds

In this section we will see that the equations defining special Kaehler manifolds are completely integrable, in the sense that the general local solution can be obtained from a free holomorphic potential. The discussion follows [ACD] and is based on the extrinsic approach to special Kaehler manifolds developped in [C2]. For a similar discussion from the bi-Lagrangian point of view see [H1].

The ambient data in the extrinsic approach are the following: The complex symplectic vector space $V = T^*\mathbb{C}^n = \mathbb{C}^{2n}$ with canonical coordinates $(z^1, \dots, z^n, w_1, \dots, w_n)$. In these coordinates the symplectic form is

$$\Omega = \sum_{i=1}^{n} dz^i \wedge dw_i \,.$$

We denote by $\tau : V \to V$ the complex conjugation with respect to $V^\tau = T^*\mathbb{R}^n = \mathbb{R}^{2n}$. The algebraic data (V, Ω, τ) induce on V the Hermitian form

$$\gamma := \sqrt{-1}\Omega(\cdot, \tau\cdot)$$

of complex signature (n, n).

Let M be a connected complex manifold of complex dimension n. We denote its complex structure by J.

Definition 2. A holomorphic immersion $\phi : M \to V$ is called *Kaehlerian* if $\phi^*\gamma$ is nondegenerate and it is called *Lagrangian* if $\phi^*\Omega = 0$.

A Kaehlerian immersion $\phi : M \to V$ induces on M the pseudo-Riemannian metric $g = \mathrm{Re}\,\phi^*\gamma$ such that (M, J, g) is a Kaehler manifold.

Lemma 1. *Let $\phi : M \to V$ be a Kaehlerian Lagrangian immersion. Then the Kaehler form $\omega = g(\cdot, J\cdot)$ of the Kaehler manifold (M, J, g) is given by*

$$\omega = 2\sum_{i=1}^{n} d\tilde{x}^i \wedge d\tilde{y}_i \,,$$

where $\tilde{x}^i := \mathrm{Re}\,\phi^* z^i$ and $\tilde{y}_i := \mathrm{Re}\,\phi^* w_i$.

Proof. The metric $g_V := \mathrm{Re}\,\gamma$ is a flat Kaehler metric of (real) signature $(2n, 2n)$ on the complex vector space (V, J). Its Kaehler form is

$$\omega_V := \sum (dx^i \wedge dy_i + du^i \wedge dv_i) \,,$$

where $x^i := \mathrm{Re}\,z^i$, $y_i := \mathrm{Re}\,w_i$, $u^i := \mathrm{Im}\,z^i$ and $v_i := \mathrm{Im}\,w_i$. On the other hand, the two-form

$$\mathrm{Re}\,\Omega = \sum (dx^i \wedge dy_i - du^i \wedge dv_i)$$

vanishes on M. This shows that

$$\omega = \phi^*\omega_V = 2\sum \phi^*(dx^i \wedge dy_i) = 2\sum d\tilde{x}^i \wedge d\tilde{y}_i \,. \qquad \square$$

The lemma implies that the functions $\tilde{x}^1, \dots, \tilde{x}^n, \tilde{y}_1, \dots, \tilde{y}_n$ define local coordinates near each point of M. Therefore we can define a flat torsionfree connection ∇ on M by the condition $\nabla d\tilde{x}^i = \nabla d\tilde{y}_i = 0$, $i = 1, \dots, n$. Now we can formulate the following fundamental theorem.

Theorem 1. *Let $\phi : M \to V$ be a Kaehlerian Lagrangian immersion with induced geometric data (g, ∇). Then (M, J, g, ∇) is a special Kaehler manifold. Conversely, any simply connected special Kaehler manifold (M, J, g, ∇) admits a Kaehlerian Lagrangian* *immersion* $\phi : M \to V$ *inducing the data* (g, ∇) *on* M. *The Kaehlerian Lagrangian immersion* ϕ *is unique up to an affine transformation of* $V = \mathbb{C}^{2n}$ *with linear part in* $\mathrm{Sp}(\mathbb{R}^{2n})$.

For the proof of that result and its projective version see [ACD], where analogous extrinsic characterisations are obtained also for special complex and special symplectic manifolds.

The above theorem may be considered as an extrinsic reformulation of the intrinsic Definition 1. The important advantage of the extrinsic characterisation in terms of Kaehlerian Lagrangian immersions lies in the well known fact that Lagrangian immersions are locally defined by a generating function. More precisely, any holomorphic Lagrangian immersion into (V, Ω) is locally defined by a holomorphic function $F(z^1, \ldots, z^n)$, at least after suitable choice of canonical coordinates $(z^1, \ldots, z^n, w_1, \ldots,$
$w_n)$. In fact, such a function defines a Lagrangian local section $\phi = dF$ of $T^*\mathbb{C}^n = V$. It is a Kaehlerian Lagrangian immersion if it satisfies the nondegeneracy condition $\det \mathrm{Im}\, \partial^2 F \neq 0$. Similarly, projective special Kaehler manifolds are locally defined by a holomorphic function F satisfying a nondegeneracy condition and which in addition is homogeneous of degree 2.

3. Special Kaehler manifolds as affine hyperspheres

The main object of affine differential geometry are hypersurfaces in affine space $\mathbb{R}^{m+1}$ with its standard connection denoted by $\widetilde{\nabla}$ and parallel volume form vol. A hypersurface is given by an immersion $\varphi : M \to \mathbb{R}^{m+1}$ of an m-dimensional connected manifold. We assume that M admits a transversal vector field ξ and that $m > 1$. This induces on M the volume form $\nu = \mathrm{vol}(\xi, \ldots)$, a torsionfree connection ∇, a quadratic covariant tensor field g, an endomorphism field S (shape tensor) and a one-form θ such that

$$\begin{aligned} \widetilde{\nabla}_X Y &= \nabla_X Y + g(X,Y)\xi\,, \\ \widetilde{\nabla}_X \xi &= SX + \theta(X)\xi\,. \end{aligned}$$

We will assume that g is nondegenerate and, hence, is a pseudo-Riemannian metric on M. This condition does not depend on the choice of ξ. According to Blaschke [B], once the orientation of M is fixed, there is a unique choice of transversal vector field ξ such that ν coincides with the metric volume form vol^g and $\nabla\nu = 0$. This particular choice of transversal vector field is called the *affine normal* and the corresponding geometric data (g, ∇) are called the *Blaschke data*. Notice that, for the affine normal, $\theta = 0$ and S is computable from (g, ∇) (Gauß equations). Henceforth we use always the affine normal as transversal vector field.

Definition 3. The hypersurface $\varphi : M \to \mathbb{R}^{m+1}$ is called a *parabolic* (or improper) *hypersphere* if the affine normal is parallel, $\widetilde{\nabla}\xi = 0$. It is called a *proper hypersphere* if the lines generated by the affine normals intersect in a point $p \in \mathbb{R}^{m+1}$, which is called *the centre*. For parabolic hyperspheres the centre is at ∞.

Notice that $\widetilde{\nabla}\xi = 0 \Leftrightarrow S = 0 \Leftrightarrow \nabla$ is flat. For proper hyperspheres $S = \lambda \mathrm{Id}$, $\lambda \in \mathbb{R} - \{0\}$.

The main result of [BC1] is the following:

Theorem 2. *Let (M, J, g, ∇) be a simply connected special Kaehler manifold. Then there exists a parabolic hypersphere $\varphi : M \to \mathbb{R}^{m+1}$, $m = \dim_{\mathbb{R}} M = 2n$, with Blaschke data (g, ∇). The immersion φ is unique up to a unimodular affine transformation of $\mathbb{R}^{m+1}$.*

The proof of Theorem 2 makes use of the Fundamental Theorem of affine differential geometry [DNV], which is the generalisation of Radon's theorem [R] to higher dimensions:

Theorem 3. *Let (M, g, ∇) be a simply connected oriented pseudo-Riemannian manifold with a torsionfree connection ∇ such that the Riemannian volume form vol^g is ∇-parallel. Then there exists an immersion $\varphi : M \to \mathbb{R}^{m+1}$ with Blaschke data (g, ∇) if and only if the g-conjugate connection $\overline{\nabla}$ is torsionfree and projectively flat. The immersion is unique up to unimodular affine transformations of $\mathbb{R}^{m+1}$.*

Recall that the g-conjugate connection $\overline{\nabla}$ on M is defined by the equation:

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \overline{\nabla}_X Z) \quad \text{for all vector fields} \quad X, Y, Z\,.$$

Proof (of Theorem 2). Let (M, J, g, ∇) be a simply connected special Kaehler manifold. For any Kaehler manifold we have $\mathrm{vol}^g = \frac{\omega^n}{n!}$, $n = \dim_{\mathbb{C}} M$. This implies the first integrability condition $\nabla \mathrm{vol}^g = 0$, since $\nabla\omega = 0$. The conjugate connection is torsionfree and flat. This will follow from:

Lemma 2. [BC1] *Let (M, J, g, ∇) be a special Kaehler manifold. Then $\overline{\nabla} = \nabla^J$, where the connection ∇^J is defined by*

$$\nabla^J_X Y := J\nabla_X(J^{-1}Y) = -J\nabla_X(JY) = \nabla_X Y - J(\nabla_X J)Y$$

for all vector fields X and Y.

From the formula defining ∇^J we see that a vector field X is ∇-parallel if and only if JX is ∇^J-parallel. Therefore, if X_i, $i = 1, \ldots, 2n = \dim_{\mathbb{R}} M$, is a parallel local frame for the flat connection ∇ then JX_i, $i = 1, \ldots, 2n$, is a ∇^J-parallel local frame. This shows that $\overline{\nabla} = \nabla^J$ is flat. Similarly the torsionfreedom of $\overline{\nabla} = \nabla^J = \nabla - J\nabla J$ follows from that of ∇ and the symmetry of the tensor ∇J. So the assumptions of Theorem 3 are satisfied and we conclude the existence of a hypersurface $\varphi : M \to \mathbb{R}^{m+1}$, $m = 2n$. inducing on M the Blaschke data (g, ∇). Now the flatness of ∇ implies that φ is a parabolic hypersphere. □

As applications we obtain:

Corollary 1. *Any holomorphic function F on a simply connected open set $U \subset \mathbb{C}^n$ with $\det \mathrm{Im}\,\partial^2 F \neq 0$ defines a parabolic hypersphere of dimension $m = 2n$.*

This is a generalisation of a classical theorem of Blaschke about 2-dimensional parabolic spheres. An explicit representation formula for the parabolic hypersphere in terms of the holomorphic function F was given in [C3].

Corollary 2. *Let (M, ∇, g) be a special Kaehler manifold with (positive) definite metric g. If g is complete then ∇ is the Levi Civita connection and g is flat.*

Proof. This follows by combining Theorem 2 with the following classical theorem of Calabi and Pogorelov [Ca, P]. □

Theorem 4. *If the Blaschke metric g of a parabolic affine hypersphere (M, g, ∇) is definite and complete, then M is affinely congruent to the paraboloid $x^{m+1} = \sum_{i=1}^{m}(x^i)^2$ in $\mathbb{R}^{m+1}$. In particular, ∇ is the Levi Civita connection and g is flat.*

Lu [L] proved that any special Kaehler manifold (M, g, J, ∇) with a definite and complete metric g is flat without making use of Calabi and Pogorelov's Theorem. Special Kaehler manifolds (M, J, g, ∇) with a flat indefinite and geodesically complete metric g for which ∇ is complete and is *not* the Levi Civita connection were constructed in [BC2].

For projective special Kaehler manifolds $\overline{M}$ it has been established in [BC3] that a natural circle bundle $S \to \overline{M}$ can be canonically realised as a proper hypersphere. Moreover, the metric cone over S is a conic special Kaehler manifold M, which is in turn realised as a parabolic hypersphere in a compatible way. There are also projective analogues of Corollaries 1 and 2.

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