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EXPLICIT GEODESIC GRAPHS ON SOME H-TYPE GROUPS

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ABSTRACT. A *g.o. space* is a homogeneous Riemannian manifold $(M = G/H, g)$ on which every geodesic is an orbit of a one-parameter subgroup of the group G. (G acts transitively on M as a group of isometries.) Each g.o. space gives rise to certain rational maps called "geodesic graphs". We are particularly interested in the case when the geodesic graphs are of non-linear character.

H-type groups provide the examples of these spaces. In this article we study H-type groups with 2-dimensional and 3-dimensional center and we present geodesic graphs with respect to various groups of isometries.

1. INTRODUCTION

Let (M, g) be a connected Riemannian manifold, $p \in M$ a fixed point and let G be a connected group of isometries which acts transitively on M. Then M can be viewed as a homogeneous space $(G/H, g)$, where H is the isotropy subgroup at p. The Lie algebra of G, or H, respectively, will be denoted by $\mathfrak{g}$, or $\mathfrak{h}$, respectively.

Definition 1. A homogeneous space $(G/H, g)$ is called a *(Riemannian) g.o. space*, if each geodesic of $(G/H, g)$ (with respect to the Riemannian connection) is an orbit of a one-parameter subgroup $\{\exp(tZ)\}, Z \in \mathfrak{g}$, of the group of isometries G.

Definition 2. Let $(G/H, g)$ be a Riemannian g.o. space. A vector $X \in \mathfrak{g}\backslash\{0\}$ is called a *geodesic vector* if the curve $\exp(tX)(p)$ is a geodesic.

In a g.o. space we investigate those sets of geodesic vectors which generate all geodesics through a fixed point. These sets are called "geodesic graphs". Let us recall basic facts about geodesic graphs. (A comprehensive information can be found in [1].)

On the Lie algebra $\mathfrak{g}$ of the group G there exists an $\mathrm{Ad}(H)$-invariant decomposition (reductive decomposition) $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of the group H and $\mathfrak{m}$ is a vector space $\mathfrak{m} \subset \mathfrak{g}$. (Such a decomposition is not unique.) On the vector space $\mathfrak{m}$ there is a natural $\mathrm{Ad}(H)$-invariant scalar product. It comes from the identification of $\mathfrak{m} \subset T_eG$ with the tangent space T_pM via the projection $\pi : G \mapsto M$.

We define equivariant subalgebras $\mathfrak{q}_X \subset \mathfrak{h}$ for $X \in \mathfrak{m}$ in the following way

$$\mathfrak{q}_X = \{A \in \mathfrak{h} \mid [A, X] = 0\}$$

and we choose an invariant scalar product on $\mathfrak{h}$.

The paper is in final form and no version of it will be submitted elsewhere.

Definition 3. Let $(G/H, g)$ be a g.o. space and $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ an $\mathrm{Ad}(H)$-invariant decomposition of the Lie algebra $\mathfrak{g}$. The *canonical geodesic graph* is an $\mathrm{Ad}(H)$-equivariant map $\xi : \mathfrak{m} \mapsto \mathfrak{h}$ (defined on an open dense subset of $\mathfrak{m}$) such that $X + \xi(X)$ is a geodesic vector and $\xi(X) \perp \mathfrak{q}_X$ for each $X \in \mathfrak{m}\backslash\{0\}$.

For the existence of the canonical geodesic graph see [4], [2]. It is analytic on an open dense subset of $\mathfrak{m}$.

Definition 4. Let $(G/H, g)$ be a g.o. space and $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ an $\mathrm{Ad}(H)$-invariant decomposition of the Lie algebra $\mathfrak{g}$. A *general geodesic graph* is an $\mathrm{Ad}(H)$-equivariant map $\eta : \mathfrak{m} \mapsto \mathfrak{h}$ which is analytic on an open dense subset of $\mathfrak{m}$ and such that $X + \eta(X)$ is a geodesic vector for each $X \in \mathfrak{m}\backslash\{0\}$.

Remark. The subalgebras $\mathfrak{q}_X$ have the following property: If $X \in \mathfrak{m}$, $A \in \mathfrak{h}$ are the vectors such that $X + A$ is a geodesic vector then all geodesic vectors "based on X" are of the form $X + A + Q$, where $Q \in \mathfrak{q}_X$. If the algebra $\mathfrak{q}_X$ is nontrivial, this gives us the possibility to find more geodesic graphs than the canonical one. If the algebras $\mathfrak{q}_X$ are trivial, then only canonical geodesic graph exists.

An essential tool for constructing geodesic graphs is the following

Proposition 1 (cf. [2], Corollary 2.2). A vector $Z \in \mathfrak{g}\backslash\{0\}$ is geodesic if and only if

$$\langle [Z, Y]_{\mathfrak{m}}, Z_{\mathfrak{m}} \rangle = 0 \quad \text{for all} \quad Y \in \mathfrak{m}. \tag{1}$$

Here the subscript $\mathfrak{m}$ indicates the projection into $\mathfrak{m}$.

We replace the vector Z by a vector $X + \xi(X)$ expressed with respect to the bases $\{X_i\}$ of $\mathfrak{m}$ and $\{D_j\}$ of $\mathfrak{h}$ as

$$X = \sum_{i=1}^{\dim\mathfrak{m}} x_i X_i, \quad \xi(X) = \sum_{j=1}^{\dim\mathfrak{h}} \xi_j D_j$$

and for Y we substitute step by step all the elements X_i.

We obtain a system of linear equations for ξ_j with coefficients and right-hand sides depending on x_i. If this system doesn't have the unique solution ($\dim\mathfrak{q}_X = q > 0$ for generic $X \in \mathfrak{m}$) then we add q additional linear equations, which characterize the orthogonality $\xi(X) \perp \mathfrak{q}_X$ (see [1] for detailed construction).

This extended system has the unique solution and by using the Cramer's rule we obtain a vector $\xi(X)$, whose components with respect to the basis of $\mathfrak{h}$ are of the form $\xi_j = P_j/P$, where P_j and P are homogeneous polynomials in variables x_i and $\deg(P_j) = \deg(P) + 1$.

In the examples already known these polynomials have the common factor and the degree of the polynomials can be decreased. We define the *degree of a geodesic graph* as the degree of the denominator after cancelling the common factor out.

2. H-TYPE GROUPS

Definition 5. Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra with an inner product $\langle , \rangle$. Let $\mathfrak{z}$ be the center of $\mathfrak{n}$ and let $\mathfrak{v}$ be it's orthogonal complement. For each vector $Z \in \mathfrak{z}$

define the operator $J_Z : \mathfrak{v} \mapsto \mathfrak{v}$ by the relation

$$\langle J_Z X, Y\rangle = \langle Z, [X,Y]\rangle \quad \text{for all} \quad X, Y \in \mathfrak{v}\,. \tag{2}$$

The algebra $\mathfrak{n}$ is called a *generalized Heisenberg algebra (H-type algebra)* if, for each $Z \in \mathfrak{z}$, the operator J_Z satisfies the identity

$$J_Z{}^2 = -\langle Z, Z\rangle \mathrm{id}_{\mathfrak{v}}\,. \tag{3}$$

A connected, simply connected Lie group whose Lie algebra is an H-type algebra is diffeomorphic to $\mathbb{R}^n$ and it is called an *H-type group.* It is endowed with a left-invariant metric.

H-type algebras are completely classified (see [3]). For each dimension of the center $\mathfrak{z}$ there is a series of H-type algebras. Each algebra of the series contains the center $\mathfrak{z}$ and the complement $\mathfrak{v}$ which decomposes into irreducible $\mathfrak{z}$-modules (the operators J_Z make $\mathfrak{v}$ a $\mathfrak{z}$-module). Irreducible $\mathfrak{z}$-modules are all equivalent if $\dim\mathfrak{z} \neq 3 \pmod 4$, otherwise there exist two nonequivalent irreducible modules of the same dimension (called non-isotypic modules).

The H-type group is a g.o. space if and only if (see [7] or [3])
- $\dim\mathfrak{z} \in \{1,2,3\}$ or
- $\dim\mathfrak{z} \in \{5,6,7\}$ and $\dim\mathfrak{v} = 8$ or
- $\dim\mathfrak{z} = 7$ and $\dim\mathfrak{v} \in \{16,24\}$ and $\mathfrak{v}$ is decomposed into 8-dimensional modules of the same type.

Each H-type group with $\dim\mathfrak{z} = 1$ is a naturally reductive space. The geodesic graph for naturally reductive spaces is linear - of degree 0. H-type groups with $\dim\mathfrak{z} = 3$ are naturally reductive if and only if the complement $\mathfrak{v}$ is decomposed into equivalent modules. Other H-type groups which are g.o. spaces are not naturally reductive. In the following sections we will concentrate on H-type groups with $\dim\mathfrak{z} = 2$ or 3. The case $\dim\mathfrak{z} = 5$ is investigated in [5].

2.1 $\dim\mathfrak{z} = 2$

Let $\mathfrak{n}$ be a vector space of dimension $4n + 2$ equipped with a scalar product and let $\{E_1, \dots, E_{4n}, Z_1, Z_2\}$ form an orthonormal basis. We define the structure of a Lie algebra on $\mathfrak{n}$ by the following relations. For $p = 0, \dots, n-1$

$$\begin{aligned}
&[E_{4p+1}, E_{4p+2}] = 0\,,\\
&[E_{4p+1}, E_{4p+3}] = Z_1\,, \quad [E_{4p+2}, E_{4p+3}] = Z_2\,,\\
&[E_{4p+1}, E_{4p+4}] = Z_2\,, \quad [E_{4p+2}, E_{4p+4}] = -Z_1\,, \quad [E_{4p+3}, E_{4p+4}] = 0\,,
\end{aligned}$$

for other $k, l = 1, \dots, 4n$ we put $[E_k, E_l] = 0$, further $[Z_1, Z_2] = 0$, and for $k = 1, \dots, 4n$ and $l = 1, 2$ we put $[E_k, Z_l] = 0$.

The elements Z_1 and Z_2 span the center $\mathfrak{z}$ of the Lie algebra $\mathfrak{n}$ and one easily verifies the condition (3) for the operators J_Z, so this relations define an H-type algebra. Each quadruplet $\mathfrak{v}_p = \mathrm{span}(E_{4p+1}, \dots, E_{4p+4})$ for $0 \le p \le n-1$ is an irreducible $\mathfrak{z}$-module and these modules are equivalent to each other. Summarizing, we have

$$\mathfrak{n} = \mathfrak{z} + \mathfrak{v} = \mathfrak{z} + \sum_{p=0}^{n-1} \mathfrak{v}_p .$$

If $n = 1$ then we have the Lie algebra of the simplest (6-dimensional) H-type group with 2-dimensional center. It was the first example (by A. Kaplan) of a g.o. space which is not naturally reductive. Its geodesic graph was described in [2] and this section is a generalization to other H-type groups with $\dim\mathfrak{z} = 2$.

Let us express the H-type group N corresponding to $\mathfrak{n}$ as a homogeneous space G/H. For $p = 0,\dots,n-1$, the following operators acting on $\mathfrak{v}$ are skew-symmetric derivations of the Lie algebra $\mathfrak{n}$:

$$\begin{aligned} D_{3p+1} &= -A_{(4p+1,4p+2)} + A_{(4p+3,4p+4)} , \\ D_{3p+2} &= +A_{(4p+1,4p+3)} + A_{(4p+2,4p+4)} , \\ D_{3p+3} &= +A_{(4p+1,4p+4)} - A_{(4p+2,4p+3)} . \end{aligned}$$

Here $A_{(k,l)}$ are the elements of $\mathfrak{so}(\mathfrak{v})$ acting on $\mathfrak{v}$ by $A_{(k,l)}(E_i) = \delta_{ki}E_l - \delta_{li}E_k$. So each subalgebra $\mathfrak{h}_p = \operatorname{span}(D_{3p+1},\dots,D_{3p+3})$ acts effectively only on $\mathfrak{v}_p$.

We put

$$\mathfrak{h} = \operatorname{span}(D_1,\dots D_{3n}) = \bigoplus_{p=0}^{n-1} \mathfrak{h}_p \cong \bigoplus_{p=0}^{n-1} \mathfrak{su}(2) .$$

and consider the decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$. Obviously, $\mathfrak{g}$ is a well-defined Lie algebra. If we express the H-type group N corresponding to $\mathfrak{n}$ as a homogeneous space G/H then G can be considered as a transitive group of *isometries* of N.

Hence we have $N = G/H$, where $H \cong [\mathrm{SU}(2)]^n$ and $G = N \rtimes H$. Here the group G is not the full isometry group of N. But the group N is a g.o. space with respect to this group.

Now, we shall construct the canonical geodesic graph $\xi : \mathfrak{n} \mapsto \mathfrak{h}$. We put

$$X = \sum_{k=1}^{4n} x_k E_k + \sum_{l=1}^{2} z_l Z_l , \quad \xi(X) = \sum_{i=1}^{3n} \xi_i D_i .$$

We check easily that the subalgebras $\mathfrak{q}_X$ from the Introduction are trivial. From the equation (1) we obtain $4n+2$ linear equations for the components ξ_i ($i = 1,\dots,3n$) of the vector $\xi(X)$ depending on the variables x_k and z_l ($k = 1,\dots,4n$ and $l = 1,2$).

For each quadruplet of these equations corresponding to $Y = E_{4p+1},\dots,E_{4p+4}$ only three of them are linearly independent. Hence we omit the fourth equation from each quadruplet (corresponding to $Y = E_{4p+4}$ for $p = 0,\dots,n-1$). The last two equations (corresponding to $Y = Z_1$ and $Y = Z_2$) are trivial.

The matrix of this system of equations is equivalent to the block square matrix $\mathbf{A}$ (of rank $3n$) with nonzero 3×3 blocks just along the diagonal. These blocks are

$$\mathbf{A}_p = \begin{pmatrix} -x_{4p+2} & x_{4p+3} & x_{4p+4} \\ x_{4p+1} & x_{4p+4} & -x_{4p+3} \\ x_{4p+4} & -x_{4p+1} & x_{4p+2} \end{pmatrix} \quad \text{for} \quad p = 0,\dots,n-1 .$$

The right-hand side vector $\mathbf{b}$ (of $3n$ entries) can be written in block form as $\mathbf{b} = (\mathbf{b}_0,\dots,\mathbf{b}_{n-1})^t$, where

$$\mathbf{b}_p = \begin{pmatrix} x_{4p+3}z_1 + x_{4p+4}z_2 \\ -x_{4p+4}z_1 + x_{4p+3}z_2 \\ -x_{4p+1}z_1 - x_{4p+2}z_2 \end{pmatrix} \quad \text{for} \quad p = 0,\dots,n-1 .$$

Hence we solve the matrix equation $\mathbf{A}\xi = \mathbf{b}$, where $\xi = (\xi_1,\dots,\xi_{3n})^t$. Using the Cramer's rule we get explicitly

$$\xi_{3p+1} = \frac{-2\,(x_{4p+2}x_{4p+3} + x_{4p+4}x_{4p+1})\,z_1 - 2\,(x_{4p+2}x_{4p+4} - x_{4p+1}x_{4p+3})\,z_2}{{x_{4p+1}}^2 + {x_{4p+2}}^2 + {x_{4p+3}}^2 + {x_{4p+4}}^2},$$

$$\xi_{3p+2} = \frac{({x_{4p+1}}^2 - {x_{4p+2}}^2 + {x_{4p+3}}^2 - {x_{4p+4}}^2)\,z_1 + 2\,(x_{4p+3}x_{4p+4} + x_{4p+1}x_{4p+2})\,z_2}{{x_{4p+1}}^2 + {x_{4p+2}}^2 + {x_{4p+3}}^2 + {x_{4p+4}}^2},$$

$$\xi_{3p+3} = \frac{2\,(x_{4p+3}x_{4p+4} - x_{4p+1}x_{4p+2})\,z_1 + ({x_{4p+1}}^2 - {x_{4p+2}}^2 - {x_{4p+3}}^2 + {x_{4p+4}}^2)\,z_2}{{x_{4p+1}}^2 + {x_{4p+2}}^2 + {x_{4p+3}}^2 + {x_{4p+4}}^2},$$

$$\text{for} \quad 0 \le p \le n-1 .$$

Thus, there is a canonical geodesic graph of degree 2 for every H-type group with $\dim\mathfrak{z} = 2$. Our choice of the group G doesn't involve other geodesic graphs, because we have $\dim\mathfrak{q}_X = 0$ for generic X.

Now, let us express the group N in the new form G'/H', where G' is the full isometry group and look for geodesic graphs with respect to bigger groups of isometries. The 6-dimensional H-type group was treated in [1]. Here the full isometry group G' is one dimension bigger than G. In the decomposition $\mathfrak{g}' = \mathfrak{n} + \mathfrak{h}'$ we have $\mathfrak{h}' = \mathrm{span}(\mathfrak{h}, R)$. R is the operator

$$R = 2\,B_{(1,2)} + A_{(1,2)} + A_{(3,4)}.$$

(Here $B_{(1,2)}$ is the operator on $\mathfrak{z}$ acting by $B_{(1,2)}(Z_i) = \delta_{1i}Z_2 - \delta_{2i}Z_1$.) But the equation (1) implies that the component of the operator R in any geodesic graph is zero. In this case only canonical geodesic graph exists.

In the 10-dimensional case ($\mathfrak{n} = \mathfrak{z} + \sum_{p=0}^{1} \mathfrak{v}_p$) the algebra $\mathfrak{h}'$ in the decomposition $\mathfrak{g}' = \mathfrak{n} + \mathfrak{h}'$ is spanned by 11 skew-symmetric derivations on $\mathfrak{n}$. We denote them $D_1,\dots,D_6,P_1,\dots,P_5$. The new elements act on $\mathfrak{n}$ by

$$\begin{aligned} P_1 &= +A_{(1,5)} + A_{(2,6)} + A_{(3,7)} + A_{(4,8)}, \\ P_2 &= +A_{(1,6)} - A_{(2,5)} - A_{(3,8)} + A_{(4,7)}, \\ P_3 &= +A_{(1,7)} + A_{(2,8)} - A_{(3,5)} - A_{(4,6)}, \\ P_4 &= +A_{(1,8)} - A_{(2,7)} + A_{(3,6)} - A_{(4,5)}, \end{aligned}$$

$$P_5 = 2\,B_{(1,2)} + A_{(1,2)} + A_{(3,4)} + A_{(5,6)} + A_{(7,8)}\,.$$

Again, the equation (1) implies that the component of the operator P_5 in any geodesic graph is zero. We denote

$$\mathfrak{h}'' = \operatorname{span}(D_1,\dots,D_6,P_1,\dots,P_4) \cong \mathfrak{so}(5).$$

We are given the new expression for the group N as G''/H'', where $H'' = \mathrm{Spin}(5)$ and $G'' = N \rtimes H''$. In this case we have $\dim\mathfrak{q}_X = 3$ for generic $X \in \mathfrak{n}$. Hence general geodesic graphs do exist. Conjecture: there is no geodesic graph of degree 1.

2.2. $\dim\mathfrak{z} = 3$.

In this case we have a vector space $\mathfrak{n}$ of dimension $4(n+m)+3$ equipped with a scalar product and the elements $\{E_1,\dots,E_{4n},F_1,\dots,F_{4m},Z_1,\dots,Z_3\}$ form an orthonormal basis. The structure of a Lie algebra on $\mathfrak{n}$ is defined by the following relations. For $p = 0,\dots,n-1$

$$\begin{aligned}
&[E_{4p+1},E_{4p+2}] = Z_1\,,\\
&[E_{4p+1},E_{4p+3}] = Z_2\,,\quad [E_{4p+2},E_{4p+3}] = Z_3\,,\\
&[E_{4p+1},E_{4p+4}] = Z_3\,,\quad [E_{4p+2},E_{4p+4}] = -Z_2\,,\quad [E_{4p+3},E_{4p+4}] = Z_1\,,
\end{aligned}$$

for $q = 0,\dots,m-1$

$$\begin{aligned}
&[F_{4q+1},F_{4q+2}] = Z_1\,,\\
&[F_{4q+1},F_{4q+3}] = Z_2\,,\quad [F_{4q+2},F_{4q+3}] = -Z_3\,,\\
&[F_{4q+1},F_{4q+4}] = Z_3\,,\quad [F_{4q+2},F_{4q+4}] = Z_2\,,\quad [F_{4q+3},F_{4q+4}] = -Z_1\,.
\end{aligned}$$

For other $i,j = 1,\dots,4n$ and $k,l = 1,\dots,4m$ we put $[E_i,E_j] = 0$, $[F_k,F_l] = 0$, and for $i = 1,\dots,4n$, $j = 1,\dots,4m$ and $k,l = 1,\dots,3$ we put

$$[E_i,Z_k] = 0\,,\quad [F_j,Z_k] = 0\,,\quad [E_i,F_j] = 0\,,\quad [Z_k,Z_l] = 0\,.$$

We have $\mathfrak{z} = \operatorname{span}(Z_1,\dots,Z_3)$, $\mathfrak{v}_p = \operatorname{span}(E_{p+1},\dots,E_{p+4})$ for $0 \le p \le n-1$ and $\bar{\mathfrak{v}}_q = \operatorname{span}(F_{q+1},\dots,F_{q+4})$ for $0 \le q \le m-1$. The action of $\mathfrak{z}$ on $\mathfrak{v}_p$ (via the operators J_Z) can be viewed as multiplication of quaternions by imaginary quaternions on the left and the action on $\bar{\mathfrak{v}}_q$ as multiplication on the right. The modules $\mathfrak{v}_p$ and $\bar{\mathfrak{v}}_q$ are not equivalent.

We start with the simplest case $n = 1, m = 0$. It is the seven-dimensional algebra $\mathfrak{n}_{(1,0)} = \mathfrak{z} + \mathfrak{v}$ with $\mathfrak{v} = \mathfrak{v}_0$. (The double index at $\mathfrak{n}$ shows the number of modules of each type in the complement of $\mathfrak{z}$.) We will show geodesic graphs with respect to various groups of isometries and apply the results to the general case.

To express $N_{(1,0)} = G/H$ we put $\mathfrak{h} = \mathrm{Der}(\mathfrak{n}) \cap \mathfrak{so}(\mathfrak{n})$ in the decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$. We get the following operators on $\mathfrak{n}$

$$\begin{aligned}
D_1 &= -A_{(1,2)} + A_{(3,4)}\,, & D_4 &= 2\,B_{(2,3)} + A_{(1,2)} + A_{(3,4)}\,,\\
D_2 &= +A_{(1,3)} + A_{(2,4)}\,, & D_5 &= 2\,B_{(1,3)} - A_{(1,3)} + A_{(2,4)}\,,\\
D_3 &= +A_{(1,4)} - A_{(2,3)}\,, & D_6 &= 2\,B_{(1,2)} + A_{(1,4)} + A_{(2,3)}\,.
\end{aligned}$$

Again, $A_{(k,l)}$ are the elements of $\mathfrak{so}(\mathfrak{v})$ acting on $\mathfrak{v}$ by $A_{(k,l)}(E_i) = \delta_{ki}E_l - \delta_{li}E_k$ and $B_{(k,l)}$ are the elements of $\mathfrak{so}(\mathfrak{z})$ acting on $\mathfrak{z}$ by $B_{(k,l)}(Z_i) = \delta_{ki}Z_l - \delta_{li}Z_k$.

We have

$$\mathfrak{h} = \mathrm{span}(D_1, \dots, D_6) \cong \mathfrak{su}(2) \oplus \tilde{\mathfrak{su}}(2),$$

where $\tilde{\mathfrak{su}}(2)$ means another representation of $\mathfrak{su}(2)$ on $\mathfrak{n}$. The group G corresponding to the algebra $\mathfrak{g}$ is the maximal connected isometry group of N.

The system of equations obtained from the equation (1) in the same way as in **2.1.** is equivalent to the matrix equation $\mathbf{A}\xi = \mathbf{b}$ with

$$\mathbf{A} = \begin{pmatrix} -x_2 & x_3 & x_4 & x_2 & -x_3 & x_4 \\ x_1 & x_4 & -x_3 & -x_1 & x_4 & x_3 \\ x_4 & -x_1 & x_2 & x_4 & x_1 & -x_2 \\ 0 & 0 & 0 & 0 & 2\,z_3 & 2\,z_2 \\ 0 & 0 & 0 & 2\,z_3 & 0 & -2\,z_1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} x_2z_1 + x_3z_2 + x_4z_3 \\ -x_1z_1 - x_4z_2 + x_3z_3 \\ x_4z_1 - x_1z_2 - x_2z_3 \\ 0 \\ 0 \end{pmatrix}.$$

The solution of this system is not unique ($\dim \mathfrak{q}_X = 1$). We find the generator of the algebra $\mathfrak{q}_X$ as the solution of the homogeneous system $\mathbf{A} \cdot \mathbf{Q}(X) = 0$ (see [1]). The components $(Q_i)_{i=1}^6$ of the vector $\mathbf{Q}(X)$ may be chosen as corresponding maximal subdeterminants with the corresponding signs of the matrix $\mathbf{A}$. But all these determinants have the common factor $4\,x_4z_3$ and therefore we can cancel out by this common factor and get the simpler components

$$Q_1 = \left(-x_1{}^2 - x_2{}^2 + x_3{}^2 + x_4{}^2\right) z_1 - 2\,(x_3x_2 + x_1x_4)\,z_2 + 2\,(x_1x_3 - x_4x_2)\,z_3\,,$$
$$Q_2 = 2\,(x_3x_2 - x_1x_4)\,z_1 + \left(x_1{}^2 - x_2{}^2 + x_3{}^2 - x_4{}^2\right) z_2 + 2\,(x_1x_2 + x_4x_3)\,z_3\,,$$
$$Q_3 = 2\,(x_2x_4 + x_1x_3)\,z_1 + 2\,(x_4x_3 - x_1x_2)\,z_2 + \left(x_1{}^2 - x_2{}^2 - x_3{}^2 + x_4{}^2\right) z_3\,,$$
$$Q_4 = -\left(x_2{}^2 + x_1{}^2 + x_3{}^2 + x_4{}^2\right) z_1\,,$$
$$Q_5 = \left(x_2{}^2 + x_1{}^2 + x_3{}^2 + x_4{}^2\right) z_2\,,$$
$$Q_6 = -\left(x_2{}^2 + x_1{}^2 + x_3{}^2 + x_4{}^2\right) z_3\,.$$

We extend the matrix $\mathbf{A}$ by the row vector $\mathbf{Q}(X)^t$ and the vector $\mathbf{b}$ by the sixth component equal to 0. So we have added the condition $\mathbf{Q}(X) \perp \xi(X)$ (the invariant scalar product on $\mathfrak{h}$ is chosen so that $\{D_i\}_{i=1}^6$ form an orthonormal basis). The solution of the extended matrix equation (obtained by the Cramer's rule) is

$$\xi_1 = \frac{\left(-x_1{}^2 - x_2{}^2 + x_3{}^2 + x_4{}^2\right) z_1 - 2\,(x_1x_4 + x_3x_2)\,z_2 + 2\,(x_1x_3 - x_2x_4)\,z_3}{2\,(x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2)},$$

$$\xi_2 = \frac{2\,(x_2x_3 - x_1x_4)\,z_1 + (x_1{}^2 - x_2{}^2 + x_3{}^2 - x_4{}^2)\,z_2 + 2\,(x_1x_2 + x_4x_3)\,z_3}{2\,(x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2)},$$

$$\xi_3 = \frac{2\,(x_2x_4 + x_1x_3)\,z_1 + 2\,(x_3x_4 - x_1x_2)\,z_2 + (x_1{}^2 - x_2{}^2 - x_3{}^2 + x_4{}^2)\,z_3}{2\,(x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2)},$$

$$\xi_4 = 1/2\,z_1\,,$$
$$\xi_5 = -1/2\,z_2\,,$$
$$\xi_6 = 1/2\,z_3\,.$$

Hence the degree of the canonical geodesic graph in the full isometry group is equal to 2. But we may consider the map

$$\eta(X) = \xi(X) - \frac{1}{2\,(x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2)} \mathrm{Q}(X)\,.$$

We have $\mathrm{Ad}(h)\mathfrak{q}_X = \mathfrak{q}_{\mathrm{Ad}(h)X}$ and for the scalar product on $\mathfrak{h}$ such that $\{D_i\}_{i=1}^6$ form an orthonormal basis we have

$$\|Q(X), Q(X)\|^2 = 2\,\left(z_1{}^2 + z_2{}^2 + z_3{}^2\right)\left(x_2{}^2 + x_1{}^2 + x_3{}^2 + x_4{}^2\right)^2,$$

which is an invariant function with respect to the representation $\mathrm{Ad}(H)|_{\mathfrak{n}}$. The function $\frac{1}{2\,(x_1{}^2+x_2{}^2+x_3{}^2+x_4{}^2)}$ is invariant as well, so the map η is $\mathrm{Ad}(H)$-equivariant. It is a general geodesic graph and it is obvious, that this geodesic graph is linear. Indeed

$$\eta_1 = \eta_2 = \eta_3 = 0\,, \quad \eta_4 = z_1\,, \quad \eta_5 = -z_2\,, \quad \eta_6 = z_3\,.$$

The linear geodesic graph shows, that the space $N_{(1,0)}$ is naturally reductive. An interesting observation shows, that this linear geodesic graph can be obtained as the canonical geodesic graph in the smaller group of isometries. If we put $N_{(1,0)} = G'/H'$, where

$$\mathfrak{h}' = \mathrm{span}(D_4, \ldots, D_6) = \tilde{\mathfrak{su}}(2)\,,$$

we get the same map. We will use this idea in constructing linear geodesic graphs in a more general general case $N_{(n,0)}$ or $N_{(0,m)}$.

Let $\mathfrak{n}_{(n,0)} = \mathfrak{z} + \mathfrak{v}$ where $\mathfrak{v} = \sum_{p=0}^{n-1} \mathfrak{v}_p$ be an H-type algebra with $\dim\mathfrak{z} = 3$ and the complemetnt $\mathfrak{v}$ decomposed into n irreducible modules of the first type. We put $N_{(n,0)} = G/H$ where $\mathfrak{h} = \mathrm{span}(\tilde{D}_1, \ldots, \tilde{D}_3)$, acting by

$$\tilde{D}_1 = 2\,B_{(2,3)} + \sum_{p=0}^{n-1} \left(A_{(4p+1,4p+2)} + A_{(4p+3,4p+4)}\right)\,,$$

$$\tilde{D}_2 = 2\,B_{(1,3)} + \sum_{p=0}^{n-1} \left(-A_{(4p+1,4p+3)} + A_{(4p+2,4p+4)}\right)\,,$$

$$\tilde{D}_3 = 2\,B_{(1,2)} + \sum_{p=0}^{n-1} \left(A_{(4p+1,4p+4)} + A_{(4p+2,4p+3)}\right)\,.$$

We have $H \cong \mathrm{SU}(2)$, $G = N \rtimes H$. From (1) we get $4n+3$ linear equations with right-hand sides for 3 components $(\xi_i)_{i=1}^3$. This system splits into n quadruplets of equations corresponding to different values of p and one triplet of equations. In each quadruplet (for $0 \le p \le n-1$) only three equations are linearly independent and they can be expressed as a matrix equation $\widetilde{\mathbf{A}}_p\xi = \widetilde{\mathbf{b}}_p$ where

$$\widetilde{\mathbf{A}}_p = \begin{pmatrix} x_{4p+2} & -x_{4p+3} & x_{4p+4} \\ -x_{4p+1} & x_{4p+4} & x_{4p+3} \\ x_{4p+4} & x_{4p+1} & -x_{4p+2} \end{pmatrix}, \quad \widetilde{\mathbf{b}}_p = \begin{pmatrix} x_{4p+2}z_1 + x_{4p+3}z_2 + x_{4p+4}z_3 \\ -x_{4p+1}z_1 - x_{4p+4}z_2 + x_{4p+3}z_3 \\ x_{4p+4}z_1 - x_{4p+1}z_2 - x_{4p+2}z_3 \end{pmatrix}.$$

The components of the solution of each of these subsystems are

$$\xi_1 = z_1\,, \quad \xi_2 = -z_2\,, \quad \xi_3 = z_3$$

and one easily verifies, that it is the solution of the whole system.

Similarly, in the case of an H-type algebra with $\dim\mathfrak{z} = 3$ and the complemetnt $\bar{\mathfrak{v}}$ decomposed into m irreducible modules of the second type ($\mathfrak{n}_{(0,m)} = \mathfrak{z} + \bar{\mathfrak{v}}$ where $\bar{\mathfrak{v}} = \sum_{q=0}^{m-1} \bar{\mathfrak{v}}_q$) we take $\mathfrak{h} = \mathrm{span}(\widehat{D}_1, \ldots, \widehat{D}_3)$, acting by

$$\widehat{D}_1 = 2\,B_{(2,3)} + \sum_{q=0}^{m-1} \left(-\bar{A}_{(4q+1,4q+2)} + \bar{A}_{(4q+3,4q+4)}\right),$$
$$\widehat{D}_2 = 2\,B_{(1,3)} + \sum_{q=0}^{m-1} \left(\bar{A}_{(4q+1,4q+3)} + \bar{A}_{(4q+2,4q+4)}\right),$$
$$\widehat{D}_3 = 2\,B_{(1,2)} + \sum_{q=0}^{m-1} \left(-\bar{A}_{(4q+1,4q+4)} + \bar{A}_{(4q+2,4q+3)}\right).$$

(Here $\bar{A}_{(k,l)}$ acts on $\bar{\mathfrak{v}}$ in the same way as $A_{(k,l)}$ acts on $\mathfrak{v}$, namely $\bar{A}_{(k,l)}(F_i) = \delta_{ki}F_l - \delta_{li}F_k$.) We have $N_{(0,m)} = G/H$ where $H \cong \mathrm{SU}(2)$, $G = N \rtimes H$. We put

$$X = \sum_{k=1}^{4m} y_k F_k + \sum_{l=1}^{3} z_l Z_l\,, \quad \xi(X) = \sum_{i=1}^{3m} \xi_i \widehat{D}_i$$

and the equation (1) gives $4m + 3$ linear equations. For example the first quadruplet reduces to the matrix equation $\widehat{\mathbf{A}}_0\xi = \widehat{\mathbf{b}}_0$ for

$$\widehat{\mathbf{A}}_0 = \begin{pmatrix} -y_2 & y_3 & -y_4 \\ y_1 & y_4 & y_3 \\ y_4 & -y_1 & -y_2 \end{pmatrix}, \quad \widehat{\mathbf{b}}_0 = \begin{pmatrix} y_2z_1 + y_3z_2 + y_4z_3 \\ -y_1z_1 + y_4z_2 - y_3z_3 \\ -y_4z_1 - y_1z_2 + y_2z_3 \end{pmatrix}.$$

The solution, which solves other equations too, is again linear:

$$\xi_1 = -z_1\,, \quad \xi_2 = z_2\,, \quad \xi_3 = -z_3\,.$$

We see, that all the H-type groups mentioned in this section so far are naturally reductive spaces.

Finally, we shall consider the general case of an H-type algebra with $\dim\mathfrak{z} = 3$. We have $\mathfrak{n}_{(n,m)} = \mathfrak{z} + \mathfrak{v} + \bar{\mathfrak{v}}$ where $\mathfrak{v} = \sum_{p=0}^{n-1} \mathfrak{v}_p$ and $\bar{\mathfrak{v}} = \sum_{q=0}^{m-1} \bar{\mathfrak{v}}_q$. Now, we put

$$\mathfrak{h} = \mathrm{span}(D_1, \ldots, D_{3n}, \bar{D}_1, \ldots, \bar{D}_{3m}) \simeq \bigoplus_{p=0}^{n-1} \mathfrak{su}(2) \oplus \bigoplus_{q=0}^{m-1} \bar{\mathfrak{su}}(2)\,.$$

Each copy of $\mathfrak{su}(2)$ acts effectively only on unique $\mathfrak{v}_p$ and each copy of $\bar{\mathfrak{su}}(2)$ acts effectively on unique $\bar{\mathfrak{v}}_q$ by the following skew-symmetric derivations. For $p = 0, \ldots, n-1$

$$D_{3p+1} = -A_{(4p+1,4p+2)} + A_{(4p+3,4p+4)}\,,$$
$$D_{3p+2} = +A_{(4p+1,4p+3)} + A_{(4p+2,4p+4)}\,,$$
$$D_{3p+3} = +A_{(4p+1,4p+4)} - A_{(4p+2,4p+3)}$$

and for $q = 0, \dots, m-1$

$$\begin{aligned}
\bar{D}_{3q+1} &= +\bar{A}_{(4q+1,4q+2)} + \bar{A}_{(4q+3,4q+4)}\,, \\
\bar{D}_{3q+2} &= +\bar{A}_{(4q+1,4q+3)} - \bar{A}_{(4q+2,4q+4)}\,, \\
\bar{D}_{3q+3} &= +\bar{A}_{(4q+1,4q+4)} + \bar{A}_{(4q+2,4q+3)}\,.
\end{aligned}$$

There are other skew-symmetric derivations of $\mathfrak{n}_{(n,m)}$ involving the operators $B_{(k,l)}$ but they are not needed here.

Now we put

$$X = \sum_{i=1}^{4n} x_i E_i + \sum_{j=1}^{4m} y_j F_j + \sum_{k=1}^{3} z_k Z_k\,, \quad \xi(X) = \sum_{i=1}^{3n} \xi_i D_i + \sum_{j=1}^{3m} \bar{\xi}_j \bar{D}_j\,.$$

The equation (1) gives again the system of linear equations. It is equivalent to the matrix equation $\mathbf{A}\xi = \mathbf{b}$ for the block square matrix $\mathbf{A}$ (of rank $3(n+m)$) with nonzero 3×3 blocks along the diagonal. These blocks are

$$\mathbf{A}_p = \begin{pmatrix} -x_{4p+2} & x_{4p+3} & x_{4p+4} \\ x_{4p+1} & x_{4p+4} & -x_{4p+3} \\ x_{4p+4} & -x_{4p+1} & x_{4p+2} \end{pmatrix}, \quad \bar{\mathbf{A}}_q = \begin{pmatrix} y_{4q+2} & y_{4q+3} & y_{4q+4} \\ -y_{4q+1} & -y_{4q+4} & y_{4q+3} \\ y_{4q+4} & -y_{4q+1} & -y_{4q+2} \end{pmatrix}$$

for $p = 0, \dots, n-1$ and $q = 0, \dots, m-1$.

The right-hand side vector $\mathbf{b}$ (of $3(n+m)$ entries) can be written in block form as $\mathbf{b} = (\mathbf{b}_0, \dots, \mathbf{b}_{n-1}, \bar{\mathbf{b}}_0, \dots, \bar{\mathbf{b}}_{m-1})^t$, where

$$\mathbf{b}_p = \begin{pmatrix} x_{4p+2}z_1 + x_{4p+3}z_2 + x_{4p+4}z_3 \\ -x_{4p+1}z_1 - x_{4p+4}z_2 + x_{4p+3}z_3 \\ x_{4p+4}z_1 - x_{4p+1}z_2 - x_{4p+2}z_3 \end{pmatrix} \quad \text{for} \quad p = 0, \dots, n-1\,,$$

$$\bar{\mathbf{b}}_q = \begin{pmatrix} y_{4q+2}z_1 + y_{4q+3}z_2 + y_{4q+4}z_3 \\ -y_{4q+1}z_1 + y_{4q+4}z_2 - y_{4q+3}z_3 \\ -y_{4q+4}z_1 - y_{4q+1}z_2 + y_{4q+2}z_3 \end{pmatrix} \quad \text{for} \quad q = 0, \dots, m-1\,.$$

By using the Cramer's rule we get (after cancelling the common factors out) the components of the canonical geodesic graph of degree 2. The components ξ_i and $\bar{\xi}_j$

for $1 \leq i, j \leq 3$ are

$$\xi_1 = \frac{(-x_1{}^2 - x_2{}^2 + x_3{}^2 + x_4{}^2)\, z_1 - 2\,(x_3 x_2 + x_1 x_4)\, z_2 + 2\,(x_1 x_3 - x_2 x_4)\, z_3}{x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2},$$

$$\xi_2 = \frac{2\,(x_2 x_3 - x_1 x_4)\, z_1 + (x_1{}^2 - x_2{}^2 + x_3{}^2 - x_4{}^2)\, z_2 + 2\,(x_1 x_2 + x_3 x_4)\, z_3}{x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2},$$

$$\xi_3 = \frac{2\,(x_3 x_1 + x_4 x_2)\, z_1 + 2\,(x_3 x_4 - x_1 x_2)\, z_2 + (x_1{}^2 - x_2{}^2 - x_3{}^2 + x_4{}^2)\, z_3}{x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2},$$

$$\bar{\xi}_1 = \frac{(y_1{}^2 + y_2{}^2 - y_3{}^2 - y_4{}^2)\, z_1 + 2\,(y_2 y_3 - y_1 y_4)\, z_2 + 2\,(y_3 y_1 + y_4 y_2)\, z_3}{y_1{}^2 + y_2{}^2 + y_3{}^2 + y_4{}^2},$$

$$\bar{\xi}_2 = \frac{2\,(y_3 y_2 + y_4 y_1)\, z_1 + (y_1{}^2 - y_2{}^2 + y_3{}^2 - y_4{}^2)\, z_2 + 2\,(y_3 y_4 - y_1 y_2)\, z_3}{y_1{}^2 + y_2{}^2 + y_3{}^2 + y_4{}^2},$$

$$\bar{\xi}_3 = \frac{2\,(y_2 y_4 - y_1 y_3)\, z_1 + 2\,(y_1 y_2 + y_4 y_3)\, z_2 + (y_1{}^2 - y_2{}^2 - y_3{}^2 + y_4{}^2)\, z_3}{y_1{}^2 + y_2{}^2 + y_3{}^2 + y_4{}^2}$$

and the components ξ_{3p+i} and $\bar{\xi}_{3q+j}$ for $1 \leq p \leq n-1$ and $1 \leq q \leq m-1$ are obtained after replacing all x_k by the corresponding x_{4p+k} and all y_l by the corresponding y_{4q+l} $(k, l = 1, \ldots, 4)$.

In the general case $N_{(n,m)}$ we can't use the similar construction as in the case $N_{(1,0)}$ and construct linear geodesic graph. For example in $N_{(1,1)}$ we have $N_{(1,1)} = G/H$ where $\mathfrak{h} = \operatorname{span}(D_1, \ldots, D_3, \bar{D}_1, \ldots, \bar{D}_3)$. The group G may be enlarged to the full connected isometry group $\tilde{G}$ of N and we get $N_{(1,1)} = \tilde{G}/\tilde{H}$ with $\tilde{\mathfrak{h}} = \mathfrak{h} + \operatorname{span}(\tilde{D}_1, \ldots, \tilde{D}_3)$. The action of additional elements of $\tilde{\mathfrak{h}}$ on $\mathfrak{n}$ is given by

$$\tilde{D}_1 = 2\, B_{(2,3)} + A_{(1,2)} + A_{(3,4)} - \bar{A}_{(1,2)} + \bar{A}_{(3,4)},$$
$$\tilde{D}_2 = 2\, B_{(1,3)} - A_{(1,3)} + A_{(2,4)} + \bar{A}_{(1,3)} + \bar{A}_{(2,4)},$$
$$\tilde{D}_3 = 2\, B_{(1,2)} + A_{(1,4)} + A_{(2,3)} - \bar{A}_{(1,4)} + \bar{A}_{(2,3)}.$$

If we compute the canonical geodesic graph with respect to $\tilde{G}$ (here $\dim \mathfrak{q}_X = 1$) we get the same map as with respect to G, the components of $\{\tilde{D}_i\}_{i=1}^{3}$ are zero. It is not hard to show that the similar trick for decreasing the degree as in the case $N_{(1,0)}$ doesn't work. It corresponds to the fact, which is known from the general theory, namely that the H-type groups with $\dim \mathfrak{z} = 3$ and of general type are not naturally reductive.

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