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A SURVEY OF BOUNDARY VALUE PROBLEMS FOR BUNDLES OVER COMPLEX SPACES

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ABSTRACT. This article is the expanded version of a conference paper presented at the 21st Winter School on Geometry and Physics, Srni, Czech Republic, January 14th - 20th, 2001. The following is a summary of recent results of the author concerning solutions of the Cauchy-Riemann equation and Laplace equation around an isolated complex singularity, with applications to the problem of analytic continuation of Hermitian-holomorphic vector bundles.

1. THE CAUCHY-RIEMANN EQUATION ON A PUNCTURED NEIGHBOURHOOD

Let X be a reduced n-dimensional complex space, for which the set of singularities consists of finitely many points. If $X' \subseteq X$ denotes the set of smooth points, we will consider a holomorphic vector bundle $E \to X' \setminus A$, where A represents a closed analytic subset of complex codimension at least two. It will be assumed moreover that E comes equipped with a Hermitian metric h. The results surveyed here will provide criteria for holomorphic extension of E across A, or across the singular points of X if $A = \emptyset$. A similar survey was carried out in [6] assuming the existence of certain special connections on E, and assuming X to be everywhere smooth. The more fundamental approach taken here is via the metric h, and in particular via the L^2-theory of the Cauchy-Riemann equation for differential (p,q)-forms with coefficients in E (cf. e.g., Demailly [4]).

We begin with a brief review of the $\bar{\partial}$-Neumann problem for strongly pseudoconvex domains. Let $\mathcal{H}_1 \xrightarrow{S} \mathcal{H}_2 \xrightarrow{T} \mathcal{H}_3$ be a sequence of Hilbert spaces, exact with respect to unbounded linear operators S and T. If S^* and T^* denote the corresponding adjoints, recall that $im(S) = ker(T)$ if there exists a positive constant C for all $x \in Dom(T) \cap Dom(S^*)$ such that

$$(*) \qquad \|S^*x\|^2_{\mathcal{H}_1} + \|Tx\|^2_{\mathcal{H}_3} \geq C\|x\|^2_{\mathcal{H}_2}.$$

In particular, consider $\mathcal{H}_q := L^2(\Omega, \bigwedge^{p,q}(E))$, where Ω is a domain in $\mathbb{C}^n$ with C^2-boundary, and $\bigwedge^{p,q}(E)$ denotes the forms of type (p,q) with coefficients in a Hermitian-holomorphic vector bundle $E \to \Omega$. S and T consequently represent the Cauchy-Riemann operators $\bar{\partial}_E : \bigwedge^{p,q}(E) \to \bigwedge^{p,q+1}(E)$. In this context $\mathcal{H}_q$ may be regarded as the metric completion of compactly supported forms $C^\infty_c(\Omega, \bigwedge^{p,q}(E))$ with

The paper is in final form and no version of it will be submitted elsewhere.

respect to the L^2–inner product induced by h. While the formal adjoint of $\bar\partial_E$ on compactly supported forms is simply defined via integration by parts, the appearance of a boundary term obstructs the definition of $\bar\partial_E^*$ when applying Stokes' theorem over $\mathcal{H}_q$. Specifically $\alpha \in L^2(\Omega, \bigwedge^{p,q-1}(E))$, $\beta \in L^2(\Omega, \bigwedge^{p,q}(E))$ implies

$$(\bar\partial\alpha, \beta) = (\alpha, \bar\partial^*\beta) + \int_{\partial\Omega} \langle \nu^{0.1} \wedge \alpha, \beta\rangle$$

or equivalently $\alpha \in L^2(\Omega, \bigwedge^{p,q+1}(E))$ implies

$$(\bar\partial^*\alpha, \beta) = (\alpha, \bar\partial\beta) + \int_{\partial\Omega} \langle \nu^{0.1} \vee \alpha, \beta\rangle\,,$$

where $\nu^{0.1}$ denotes the (0.1)–component of the unit normal covector to $\partial\Omega$, and $\vee$ denotes the adjoint of exterior multiplication with respect to the pointwise inner product on forms. We therefore restrict the domain of definition of $\bar\partial_E^*$ to those forms α such that

$$\nu^{0.1} \vee \alpha\,|_{\partial\Omega} = \nu^{0.1} \vee \bar\partial\alpha\,|_{\partial\Omega} = 0\,.$$

We now introduce the self-adjoint "Laplace–Beltrami" operator $\Box = \bar\partial_E\bar\partial_E^* + \bar\partial_E^*\bar\partial_E$. Given $\eta \in C^\infty(\Omega, \bigwedge^{p,q}(E)) \cap C^0(\overline{\Omega}, \bigwedge^{p,q}(E))$, both existence and smoothness of a solution to the equation $\Box\alpha = \eta$ are then guaranteed by a stronger estimate than (*), of the form

$$(**) \qquad (\Box\varphi, \varphi) \geq C\|\varphi\|^2 + \int_{\partial\Omega} |\varphi|^2 + \|\nabla^{0.1}\varphi\|^2$$

for all φ in the restricted domain, where

$$\nabla^{0.1} : \bigwedge^{p,q}(E) \to (T^{0.1}X)^* \otimes \bigwedge^{p,q}(E)$$

denotes the natural connection on forms induced by complex conjugation of the Chern connection of E. The positive constant C was moreover shown by Hörmander to exist for all such φ when the boundary $\partial\Omega$ is strongly pseudoconvex [9]. With this fact we are now able to solve the "$\bar\partial$–Neumann problem" using the theory developed for this purpose by Kohn and Spencer [10]. Namely, if η is smoothly defined on a strongly pseudoconvex domain Ω then there exists

$$(\dagger) \qquad \varphi \in \{\alpha \in C^\infty_{L^2}(\Omega, \bigwedge^{p,q}(E)) \mid \nu^{0.1} \vee \alpha\,|_{\partial\Omega} = \nu^{0.1} \vee \bar\partial\alpha\,|_{\partial\Omega} = 0\}$$

such that $\Box\varphi = \eta$. Using the additional assumption that $\bar\partial\eta = 0$, we obtain an immediate solution of the Cauchy–Riemann equation $\bar\partial u = \eta$, since

$$0 = (\bar\partial\eta, \bar\partial\varphi) = (\bar\partial\bar\partial^*\bar\partial\eta, \bar\partial\varphi) = \|\bar\partial^*\bar\partial\varphi\|^2$$

(given the condition $\nu^{0.1} \vee \bar\partial\varphi\,|_{\partial\Omega} = 0$), and hence $u = \bar\partial^*\varphi$.

While the unit ball $B \subset \mathbb{C}^n$ is the prototype for all strictly pseudoconvex domains, it is perhaps surprising that almost nothing was known about solvability of the Cauchy–Riemann equation on the punctured ball $B \setminus \{0\}$, prior to the appearance of a paper of S. Bando in 1991 [1]. Consider a Hermitian-holomorphic vector bundle $(E, h) \to B \setminus \{0\} \subset \mathbb{C}^2$ such that the curvature form F_h, represented in any holomorphic frame of E by the $(1,1)$–form $\bar\partial(h^{-1}\partial h)$, belongs to $L^2(B \setminus \{0\})$. For η a $\bar\partial$–closed, compactly supported $(0,1)$–form on $B \setminus \{0\}$, Bando solved the equation $\bar\partial u = \eta$ in two steps. Given $B_{\varepsilon,1} := \{z \in \mathbb{C}^2 \mid 0 < \varepsilon < |z| < 1\}$, it was first shown that for all $\varepsilon > 0$ the equation $\Box\varphi_\varepsilon = \eta$ is solvable on $B_{\varepsilon,1}$, for φ_ε a smooth (0.1)–form satisfying the

$\bar{\partial}$-Neumann conditions (†) on $|z| = 1$, and the Dirichlet condition $\varphi_\varepsilon = 0$ on $|z| = \varepsilon$. The key idea of this part of Bando's argument is to manipulate Weitzenböck formulae expressing the Laplace-Beltrami operator in terms of the curvature form F_h, which is assumed to belong to $L^2(B \setminus \{0\})$. In this way he derives an estimate of the form (**) independently of ε, and obtains a smooth solution φ on $B \setminus \{0\}$ satisfying $\|\varphi\| \leq \|\eta\|$, taking the uniform limit as ε approaches zero.

The second part of the proof treats the vanishing of $\bar{\partial}^*\bar{\partial}\varphi$ - a much more delicate issue in the case of the punctured ball - for which Bando recalls a Moser iteration technique developed in earlier work with Kasue and Nakajima [2]. Solvability of the equation $\bar{\partial}u = \eta$ such that $\|u\| \leq \|\eta\|$ is then applied in the standard way to obtain a sufficiently large number N of global holomorphic sections generating E over $B \setminus \{0\}$, and hence an embedding $E \hookrightarrow \mathbb{C}^N \times B \setminus \{0\}$. It follows from Hartogs' theorem that the holomorphic structure of E extends uniquely across the origin as a coherent analytic sheaf, though this structure may only be assumed to be locally free when $B \setminus \{0\} \subset \mathbb{C}^2$.

In collaboration with Y. Tonegawa [7], the author sought to generalise Bando's removable singularities theorem by first solving the Cauchy- Riemann equation on $B \setminus \{0\} \subset \mathbb{C}^n$. With an appropriate adjustment of the Hölder and Sobolev exponents it is a relatively straightforward matter to apply Bando's analysis of the $\bar{\partial}$-Neumann/Dirichlet problem to solve the Cauchy-Riemann equation in higher dimension when the curvature F_h belongs to $L^n(B \setminus \{0\})$. In fact, the closed (0.1)-form η need not be compactly supported, but must also belong to $L^n(B \setminus \{0\})$ (cf. [7]). A possibly surprising observation in the case $n = 2$ is the consequent vanishing of the L^2 Dolbeault cohomology $H^{0.1}_{L^2}(B \setminus \{0\})$, as compared with the infinite dimensionality of $H^{0.1}(B \setminus \{0\})$.

2. Removable singularities for holomorphic vector bundles

Given $(E, h) \to B \setminus \{0\} \subset \mathbb{C}^n$, recall that the equation $\bar{\partial}u = F_h$ determines the cohomology obstruction to existence of a holomorphic connection on E (cf [3]). When F_h belongs to $L^n(B \setminus \{0\})$, solvability of the $\bar{\partial}$-Neumann/Dirichlet problem, together with previous work of N.P. Buchdahl and the author [3], consequently shows that E admits a unique locally free extension across the origin. For the more general case of a complex manifold X, containing an analytic subset A of complex codimension at least two, with Hermitian-holomorphic vector bundle $(E, h) \to X \setminus A$, a unique holomorphic extension of E across A is carried out by first stratifying A into smooth open subsets. At any point of this stratification, the restriction of E to a normal slice corresponds to the extension problem for the punctured ball. The main theorem of [7] may be stated as follows.

Theorem 1. *Let $(E, h) \to X \setminus A$ be a Hermitian-holomorphic vector bundle such that $F_h \in L^n(X \setminus A)$, then there exists a unique vector bundle $\hat{E} \to X$ such that $\hat{E}\,|_{X \setminus A} \cong E$.*

A priori, E will have L^n-curvature when restricted to *almost* every normal slice. The uniform extension of E across A therefore requires two fundamental lemmas.

Lemma 1. (cf., e.g., [13, Lemma 3.1], also [14, Proposition 6.8].) *Let Ω be a domain in $\mathbb{C}^{n-1}$, Δ a disc in $\mathbb{C}$, and $\Delta^* := \Delta \cap \mathbb{P}_1 \setminus \{0\}$. Given $g \in \mathcal{O}(\Omega \times \Delta^*, \mathbb{GL}(r, \mathbb{C}))$, suppose there exists $\omega \in \Omega$, $h^+ \in \mathcal{O}(\Delta, \mathbb{GL}(r, \mathbb{C}))$, and $h^- \in \mathcal{O}(\mathbb{P}_1 \setminus \{0\}, \mathbb{GL}(r, \mathbb{C}))$ such that*

$g(\omega,...) = h^+ \cdot h^-$. Then there exists a uniquely defined analytic hypersurface $\Gamma \subset \Omega$, and unique holomorphic matrix-valued functions $g^+ \in \mathcal{O}((\Omega\setminus\Gamma)\times\Delta, \mathbb{GL}(r,\mathbb{C}))$, $g^- \in \mathcal{O}((\Omega\setminus\Gamma)\times\mathbb{P}_1\setminus\{0\}, \mathbb{GL}(r,\mathbb{C}))$ such that
(i) $g = g^+ \cdot g^-$,
(ii) $g^-(...,\infty) \equiv \mathbf{1}$,
(iii) *g^+, g^- extend to meromorphic matrix-valued functions on $\Omega\times\Delta$ (resp. $\Omega\times\mathbb{P}_1\setminus\{0\}$) with polar locus $\Gamma\times\Delta$ (resp. $\Gamma\times\mathbb{P}_1\setminus\{0\}$).*

Lemma 2. (Rothstein, cf. [13], Theorem 1.12.) *Let Δ^{n-1} be a polydisc in $\mathbb{C}^{n-1}$, and $\Delta(r)$ a disc of radius r. Suppose f is a meromorphic function on $\Delta^{n-1}\times\Delta(r)$, such that $f_w := f(w,...)$ extends meromorphically to $\Delta(r+\varepsilon)$ for all $w \in \Delta^{n-1}$. Then f extends meromorphically to $\Delta^{n-1}\times\Delta(r+\varepsilon)$.*

The proof of extension via slicing follows from a straightforward induction, which is a simplified version of an argument used by Shevchishin for his "Thullen-type" extension theorem [13]. Due to an additional slicing technique of Siu [14], the statement of the main theorem may be generalised to include any closed set A of finite $2n-4$-dimensional Hausdorff measure. We recall moreover that sharpness of the main hypothesis follows from the example of $E \to \mathbb{C}^n\setminus\{0\}$ corresponding to the pullback of the holomorphic tangent bundle from $\mathbb{CP}_{n-1}$, $n \geq 3$. It is easily seen that E admits no locally free holomorphic extension across the origin (cf. [3]). For h corresponding to the pullback of the Fubini–Study metric, however, it can be seen by direct computation that $F_h \in L^p(B\setminus\{0\})$ for all $p < n$.

Now consider X a reduced and irreducible n-dimensional complex space with isolated singularity $x_0 \in X$. Let $\rho : X \to [0,\infty)$, $\rho(x_0) = 0$, be a strongly plurisubharmonic function, and $\Omega \subset X$ a neighbourhood of x_0 with smooth compact boundary $\Sigma = \{x \in X \mid \rho(x) = c < \infty\}$. It will be assumed $\Omega_0 := \Omega\setminus\{0\}$ is a complex manifold with Kähler form $\omega = i\bar\partial\partial\rho$ and associated metric g satisfying the following conditions:

$$(i) \qquad \int_{\Omega_0} |R_g|^n < \infty\,,$$

where R_g denotes the canonical curvature form associated with g. It will further be assumed that the Sobolev inequality holds with respect to this metric, i.e.,

$$(ii) \qquad \left(\int_{\Omega_0} |f|^{\frac{2n}{n-1}}\omega^n\right)^{\frac{n-1}{n}} \leq c(n)\int_{\Omega_0} |\nabla f|^2\omega^n$$

for smooth compactly supported functions f. In addition, (iii) let $\delta(x_0, x)$ denote the Riemannian metric distance function on Ω_0, and let $B_\delta(x_0, r)$ be the associated ball of radius r. For some sufficiently small $0 < c' < c$ it will be assumed that there exists a positive constant K such that

$$\int_{B_\delta(x_0,r)} \omega^n \leq Kr^{2n}\,, \quad \text{for all} \quad 0 < r \leq c'.$$

In fact, when $X \subset \mathbb{C}^m$ is an affine analytic variety, with ρ corresponding to the restriction of the Euclidean norm- squared function, conditions (ii) and (iii) hold automatically. Moreover, the curvature form $R_g = \beta\wedge\beta^*$, where β denotes the second fundamental form of the embedded variety, and it is an easy computation to show that

$$\int_{\Omega_0} |R_g|^n \leq \int_{\Omega_0} |\beta|^{2n} < \infty$$

when, for example, $X : z_{n+1}^k = f(z_1, ..., z_n)$ is an analytic hypersurface in $\mathbb{C}^{n+1}$ with isolated singularity at the origin, such that $2 \leq k < ord_f(0)$. On the other hand surfaces in $\mathbb{C}^3$ defined by an equation of the form $z^k = xy$, which constitute a special class of orbifold singularities (cf., e.g., [5]), do not satisfy condition (i) with respect to the restricted ambient metric. For any singular space X of this type, corresponding to the quotient of $\mathbb{C}^n$ by a finite subgroup of $\mathbb{S}\mathbb{U}(n)$, the most natural choice of ρ is that induced by $|z|^2$ on the Euclidean covering space, since the associated orbifold metric is flat.

Let $(E, h) \to X_0$ be a Hermitian–holomorphic vector bundle, with Kähler metric g on X_0 derived from $i\bar{\partial}\partial\rho$ as above, so that the curvature F_h belongs to $L^n(\Omega_0)$. If $\eta \in C^\infty(X_0, \Lambda^{0.1}(E))$ is a $\bar{\partial}$-closed (0.1)-form also belonging to $L^n(\Omega_0)$, then it is shown in [8], theorem 1, that the equation $\bar{\partial}u = \eta$ admits a smooth solution on Ω_0 such that $\|u\| \leq \|\eta\|$. Once again the proof is an appropriate modification of the method of Bando in solving the $\bar{\partial}$-Neumann/Dirichlet problem on $\Omega \setminus B_\delta(x_0, \varepsilon)$, before taking the limit as ε approaches zero. In obtaining the basic estimate $(**)$ for existence and regularity of solutions, the curvature of h and g must be taken into account from the Weitzenböck formula

$$(\Box\varphi, \varphi) = \|\bar{\partial}_E\varphi\|^2 + \|\bar{\partial}_E^*\varphi\|^2 = \|\nabla^{0.1}\varphi\|^2 + \int_\Sigma |\varphi|^2 +$$

$$\varphi_{\bar{a}}^\alpha g^{b\bar{a}} R_{bk\bar{l}}^k h_{\alpha\bar{\beta}} \varphi_{\bar{l}}^\beta + \varphi_{\bar{i}}^\alpha g^{k\bar{i}} R_{bk\bar{l}}^b h_{\alpha\bar{\beta}} \varphi_{\bar{l}}^\beta + \varphi_{\bar{i}}^\gamma g^{k\bar{i}} F_{\gamma k\bar{l}}^\alpha h_{\gamma\bar{\zeta}} \varphi_{\bar{l}}^\zeta .$$

Now by means of a standard application of L^2–theory of the Cauchy–Riemann equation, it is once more possible to embed $E\ |_{\Omega_0}$ in the trivial bundle $\mathbb{C}^N \times \Omega_0$, for N sufficiently large. We now obtain [8]

Theorem 2. *Let X^n be a reduced complex space with normal isolated singularity at $x_0 \in X$, and $\rho : X \to [0, \infty)$ a smooth, strongly plurisubharmonic exhaustion function centred at x_0 which satisfies the conditions $(i) - (iii)$ above. If $E \to X_0$ is a Hermitian–holomorphic vector bundle with L^n_{loc}–curvature, then there exists a reflexive sheaf $\mathcal{F} \to X$ such that $\mathcal{F}\ |_{X_0} \cong \mathcal{O}(E)$.*

A further consequence of the solubility of the $\bar{\partial}$ –Neumann problem on Ω_0 is the solvability of the equation $\bar{\partial}u = F_h$, which implies existence of a holomorphic connection on $E \to \Omega_0$ (cf. [7]). The extension argument of [3], theorem 2.2 will then automatically imply the following

Corollary 1. *Let X^n be a reduced analytic space with isolated singularity $x_0 \in X$ and strongly plurisubharmonic function $\rho : X \to [0, \infty)$ satisfying conditions (i)-(iii). Consider $\pi : Y \to X$ to be a surjective holomorphic map from a complex manifold Y such that $\pi^{-1}(x_0)$ has codimension greater than one. If $E \to X_0$ is a Hermitian–holomorphic vector bundle with L^n_{loc}–curvature, then there exists a unique holomorphic vector bundle $V \to Y$ such that $V\ |_{Y \setminus \pi^{-1}(x_0)} \cong \pi^* E$.*

A natural instance of this result occurs when π corresponds to a quotient of $\mathbb{C}^n$ under the action of a finite group $G \subset \mathbb{S}\mathbb{U}(n)$, i.e., X has an orbifold singularity at x_0. Another potential class of examples corresponds to isolated singularities with "small resolution". Explicit examples of such singularities, with X a hypersurface in $\mathbb{C}^4$ and $\pi^{-1}(x_0) \cong \mathbb{CP}_1$ were presented in [12]. At present an example which admits a strongly plurisubharmonic function ρ of the required type is not known to the author, however.

Another application concerns the problem of holomorphic extension from the strictly pseudoconvex CR-hypersurface Σ corresponding to $\rho = c$. Let $\sigma : \Sigma \to E$ be a section of $E \to X_0$, such that σ is closed with respect to the tangential Cauchy-Riemann operator on $E|_\Sigma$, i.e., $\bar{\partial}_b \sigma = 0$ (cf. [10]). Let $s : \Omega_0 \to E$ be a smooth extension of σ, with support in an arbitrarily small neighbourhood of Σ. From the standard theory of the tangential Cauchy-Riemann complex for Σ, we note that $\bar{\partial}_b \sigma = 0$ if and only if the $(n, n-1)$-form $\xi = \bar{\partial}^* \star \bar{s}$ satisfies the $\bar{\partial}$- Neumann conditions $(\dagger)$, where $\star$ denotes the Hodge duality operator on forms ([10], Proposition 5.2.2). The essential idea of the extension technique of Kohn and Rossi [16] is to obtain a solution to the equation $\bar{\partial}^* u = \xi$ such that u again satisfies the conditions $(\dagger)$, and this is done in a manner entirely analogous to the theory of the equation $\bar{\partial} u = \xi$ for a $(0,1)$-form ξ. Moreover, u satisfies $(a)\bar{\partial}(s - \star\bar{u}) = 0$ and $(b) \star \bar{u}|_\Sigma = 0$, hence $s - \star\bar{u}$ is a holomorphic extension of σ. For the case of Ω_0 corresponding to the punctured neighbourhood of an isolated singularity our adaptation of Bando's method is applied to this end. The argument here also is essentially a dualised version of the method of the previous sections, and goes through with only minor alterations.

Corollary 2. *Let X^n be a reduced analytic space with isolated singularity x_0, and let $\rho : X \to [0, \infty)$ be a strongly plurisubharmonic function satisfying the conditions (i)-(iii). If $E \to X_0$ is a Hermitian-holomorphic vector bundle with L^n_{loc}-curvature, and σ a $\bar{\partial}_b$-closed section of $E|_\Sigma$, then there exists a unique holomorphic extension of σ as a section of E on Ω_0, and hence as a section of the reflexive sheaf $\mathcal{F}$ on Ω.*

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