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NECESSARY CONDITIONS FOR LOCAL CONVEXIFIABILITY OF PSEUDOCONVEX DOMAINS IN $\mathbb{C}^2$

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ABSTRACT. In this paper we give explicit necessary conditions for local convexifiability of a weakly pseudoconvex domain at a point of finite type.

1. INTRODUCTION

Pseudoconvexity is in several ways analogous to ordinary geometric convexity. Beside analogy, there is a fundamental fact which gives a direct connection between the "strict" versions of pseudoconvexity and convexity.

Theorem 1 (Narasimhan). *Let $\Omega \subseteq \mathbb{C}^n$ be a smoothly bounded domain, which is strictly pseudoconvex in a neighbourhood of a boundary point p. Then there exist local holomorphic coordinates around p in which $b\Omega$ is strictly convex.*

This fact is essential for many geometric and analytic constructions on strictly pseudoconvex domains. In 1973 Kohn and Nirenberg discovered an example which shows that local convexifiability does not in general extend to weakly pseudoconvex domains ([KN]). This celebrated result disproved the popular conjecture that pseudoconvexity is nothing but local convexifiability. The example of Kohn-Nirenberg is a pseudoconvex domain in $\mathbb{C}^2$, which is of finite type 8 at the origin :

$$\Omega_0 = \{(z,w) \in \mathbb{C}^2 \mid Im\ w > |z|^8 + \frac{15}{7}|z|^2 Re\ z^6\}.$$

It is proved in [KN] that the zero set of any holomorphic function vanishing at zero intersects both the interior an exterior of Ω_0 in any neighbourhood of the origin. In particular, there is no supporting function at $p = 0$, and the domain is not locally convexifiable. A later result of Fornaess [F] shows that the same phenomenon

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exists for analogous domains of type 6. On the other hand, it is easy to show that pseudoconvex domains of type 4 in $\mathbb{C}^2$ always admit supporting functions and pseudoconvex model domains of type four are convexifiable (see e.g. [FS]).

While the knowledge about the existence of supporting functions and peak functions is quite satisfactory (see e.g. [BF]), much less is known about local convexifiability. It is still an open problem to find a verifiable characterization of locally convexifiable domains. A result in this direction was obtained in [Ko]. It gives sufficient conditions for the class of model domains in $\mathbb{C}^2$. The result is based on exact calculations for domains of Kohn-Nirenberg type. In this paper we complement this result by proving an optimal necessary condition for local convexifiability of a finite type domain in $\mathbb{C}^2$.

2. Domains of finite type

Let $\Omega \subseteq \mathbb{C}^2$ be a smoothly bounded domain and $p \in b\Omega$ be a boundary point. Let U be a neighbourhood of p and $r \in C^\infty(U)$ be a local defining function, i.e.,

$$\Omega \cap U = \{z \in U \mid r(z) < 0\},$$

where $\nabla r \neq 0$ in U. Recall that $b\Omega \cap U$ is pseudoconvex if for all $q \in b\Omega \cap U$ and $\zeta \in \mathbb{C}^2 \setminus \{0\}$ satisfying $\sum_{i=1}^2 \frac{\partial r}{\partial z_i}(q)\zeta_i = 0$ we have

$$\sum_{i,j=1}^{2} \frac{\partial^2 r}{\partial z_i \partial \bar{z}_j}(q)\zeta_i\bar{\zeta}_j \geq 0.$$

On the other hand, $b\Omega$ is convex in U if for all $q \in b\Omega \cap U$ and $\xi \in \mathbb{R}^4 \setminus \{0\}$ satisfying $\sum_{i=1}^4 \frac{\partial r}{\partial x_i}(q)\xi_i = 0$ we have

$$\sum_{i,j=1}^{4} \frac{\partial^2 r}{\partial x_i \partial x_j}(q)\xi_i\xi_j \geq 0.$$

Here we set $z_j = x_j + \sqrt{-1}x_{j+2}$. Strict pseudoconvexity and strict convexity are obtained by requiring that strict inequalities hold in these formulas.

Definition 1. Ω is called locally convexifiable at p if there exist local holomorphic coordinates in a full neighbourhood of p such that $b\Omega$ is convex with respect to the induced linear structure.

We recall a definition of finite type points for domains in $\mathbb{C}^2$. For a smooth function f defined in a neighbourhood of $0 \in \mathbb{C}$ let $\nu(f)$ denote the order of vanishing of f at 0.

Definition 2. Let $\Omega \subseteq \mathbb{C}^2$ be a domain with smooth boundary. A point $p \in b\Omega$ is a point of finite type, if there exists an integer m such that

$$\nu(r \circ \gamma) \leq m$$

for all holomorphic maps γ from a neighbourhood of $0 \in \mathbb{C}$ into $\mathbb{C}^2$, satisfying $\gamma(0) = p$ and $\gamma'(0) \neq 0$. The smallest such integer is called the type of p.

We will use the following standard description of finite type points on pseudoconvex domains (see e.g. [FS]).

Lemma 1. *Let $\Omega \subseteq \mathbb{C}^2$ be a pseudoconvex domain with smooth boundary. A point $p \in b\Omega$ is a point of finite type k if and only if there exist local holomorphic coordinates (z,w) around p in which the boundary is described by*

$$Im\ w = P(z,\bar{z}) + o(|z|^k, Re\ w),$$

where P is a real valued homogeneous polynomial of degree k

$$P(z,\bar{z}) = a_0|z|^k + \sum_{j=2,4,\ldots,k} |z|^{k-j} Re(a_j z^j) \tag{1}$$

for some $a_0 > 0$ and $a_j \in \mathbb{C}$.

For the original definition of finite type see [K]. A model domain at p is the domain

$$M = \{(z,w) \in \mathbb{C}^2 \mid Im\ w > P(z,\bar{z})\}. \tag{2}$$

Lemma 2. *The polynomial P in (1) is determined uniquely up to a transformation*

$$z^* = \alpha z, \qquad w^* = w + \beta z^k, \tag{3}$$

where $\alpha, \beta \in \mathbb{C}$.

Proof. Let (z,w), where $z = x+iy$, $w = u+iv$, be local holomorphic coordinates centered at p, such that the hyperplane $v = 0$ is tangent to $b\Omega$ at p and the positive v-axis points inside of Ω. Near p, the boundary of Ω is described by

$$v = F(z,\bar{z},u),$$

where F is a smooth function which vanishes together with its first order derivatives at the origin. Such coordinates can be obtained from the general case by an affine transformation. Consider holomorphic transformations of the form

$$\begin{aligned} z^* &= z + g(z,w) \\ w^* &= w + f(z,w), \end{aligned} \tag{4}$$

which preserve the above description of $b\Omega$ and satisfy the normalization condition $f_w(0) = 0$. Let F^* be the function describing $b\Omega$ in new coordinates. Substituting (4) into $v^* = F^*(z^*,\bar{z}^*,u^*)$ and restricting the variables z,w to $b\Omega$, we get a change of variables formula:

$$F^*(z+g,\bar{z}+\bar{g},u+Re\ f) = F(z,\bar{z},u) + Im\ f(z,u+iF(z,\bar{z},u)), \tag{5}$$

where g and $Re\ f$ are also evaluated at $(z,u+iF(z,\bar{z},u))$. We need to prove that if a transformation (4) preserves the form described in Lemma 1, then the new leading polynomial P^* of F^* can be obtained from P by a transformation (3). We will use weights for the variables in (4) and (5), namely weight 1 for $z,\bar{z}$ and weight k for u and w. Since F starts with terms of weight k, the terms of weight ν for $\nu \leq k$ in $Im\ f(z,u+iF(z,\bar{z},u))$ come from corresponding terms of weight ν in $f(z,w)$. It follows that f does not contain $z^2,\ldots,z^{k-1}$, since all other entries in (5) are of weight $\geq k$. Hence $Re\ f$ has also weight $\geq k$ and (5) takes from

$$P^*(z+g,\bar{z}+\bar{g}) + \cdots = P(z,\bar{z}) + Im\ f(z,u+iF(z,\bar{z},u)) + \ldots$$

where dots denote terms of weight $> k$. Comparing terms of weight k on both sides shows that P^* depends only on terms of weight 1 in g and weight k in f, in other words it is obtained from P by (3). □

Lemma 3. (i) *If Ω is convexifiable at p, then the model domain M at p is convexifiable.* (ii) *M is convexifiable if and only if there exists an $\alpha \in \mathbb{C}$ such that $P(z,\bar{z}) + Re(\alpha z^k)$ is a convex function.*

Proof. In order to prove (ii), consider coordinates (z, w) of the type described in the proof of lemma 2. By the argument used there, we have either

$$F(z,\bar{z},u) = Re\ \alpha z^j + O(|z|^{j+1}, u)\,,$$

where $2 \le j \le k-1$ and $\alpha \in \mathbb{C} \setminus \{0\}$, or

$$F(z,\bar{z},u) = \tilde{P}(z,\bar{z}) + O(|z|^{k+1}, u)\,,$$

where $\tilde{P}$ is obtained from P by (3). Applying this to M itself gives (ii). Part (i) follows by an obvious homogeneity argument. □

3. Conditions for convexifiability

The sufficient conditions for convexifiability of a model domain follow from exact conditions which are computable for domains of Kohn-Nirenberg type.

Definition 3. For two even integers k, l and a real number $a > 0$ we will call the domain

$$M_a^{k,l} = \{(z,w) \in \mathbb{C}^2 \mid Im\ w > P_a^{k,l}(z,\bar{z})\}\,,$$

where

$$P_a^{k,l}(z,\bar{z}) = |z|^k + a|z|^{k-l} Re\ z^l\,,$$

a Kohn-Nirenberg domain of degree (k, l).

The Kohn-Nirenberg domains can be viewed as the building blocks of a general model domain.

For the proof of the following two results see [Ko].

Proposition 1. *$M_a^{k,l}$ is convex if and only if $l^2 \ge 3k-2$ and*

$$a \le \frac{k}{l^2 - k}\,,$$

or $l^2 \le 3k-2$ and

$$a \le \sqrt{\frac{(4k - l^2 - 4)k^2}{(4k-4)(k^2 - l^2)}}\,.$$

Moreover, if l is not a divisor of k, then this condition is equivalent to convexifiability of $M_a^{k,l}$.

Let us denote $\gamma_{lk} = \frac{k}{l^2-k}$ if $l^2 \ge 3k-2$, and $\gamma_{lk} = \sqrt{\frac{(4k-l^2-4)k^2}{(4k-4)(k^2-l^2)}}$ if $l^2 \le 3k-2$.

Theorem 2. *If*

$$\sum_{j=2,4,\dots,k} \gamma_{jk}^{-1}|a_j| < a_0 ,$$

then M is convex.

The main purpose of this paper is to prove the following necessary condition for convexifiability of Ω at p.

Theorem 3. *Let the model domain at $p \in b\Omega$ be given by* (2). *If Ω is convexifiable at p, then*
(i) $\dfrac{|a_j|}{a_0} \le \gamma_{jk}$ *for all* $j > \frac{k}{2}$,
and
(ii) $\dfrac{|a_j|}{a_0} \le 2\gamma_{jk}$ *for all* $j \le \frac{k}{2}$.

Proof. By Lemma 3 it is enough to consider the model domain at p. Let us assume that M is convexifiable. Since the harmonic term is included in (1), by Lemma 3 we may assume that P is convex. Without any loss of generality we may assume that $a_0 = 1$. In polar coordinates we have

$$P(r,\theta) = r^k(1 + \sum_j |a_j| \cos j(\theta + \theta_j)) , \tag{6}$$

where $\theta_j = \arg a_j$. We will evaluate the real Hessian of P at a point $e^{i\theta}$ on the unit circle with respect to the rotated basis formed by the unit outer normal vector n and the unit tangent vector t. We will denote differentiation by subscripts. We have $P_{tt} = P_{\theta\theta} + P_r$. From (6) with $r = 1$ we get

$$\frac{1}{k}P_{tt} = 1 + \sum_j \frac{k - j^2}{k}|a_j| \cos j(\theta + \theta_j) .$$

Let $p(\theta)$ be this trigonometric polynomial. We will show that $p(\theta) \ge 0$ implies (i). Let $j > \frac{k}{2}$ be fixed. For any other index l in (1) we have $j \nmid l$, and so

$$\sum_{m=1}^{j} \cos l(\theta_0 + \frac{2\pi m}{j}) = 0 \tag{7}$$

for any θ_0. Assume $|a_j| > |\dfrac{k}{k - j^2}|$, and take $\theta_0 = \frac{\pi}{j} - \theta_j$. Then

$$1 + |a_j|\gamma_{jk}^{-1} \cos j(\theta_0 + \theta_j) < 0 . \tag{8}$$

From (7) and (8) we get

$$\sum_{m=1}^{j} p(\theta_0 + \frac{2\pi m}{j}) < 0 .$$

Hence there is a point at which $P_{tt} < 0$. In order to prove (ii), let us first assume that $j^2 \geq 3k-2$. Since $p(\theta) \geq 0$, we have

$$\int_0^{2\pi} p(\theta)\cos j(\theta+\theta_j) \leq |\int_0^{2\pi} p(\theta)\cos j(\theta+\theta_j)| \leq \int_0^{2\pi} p(\theta) = 2\pi\,.$$

On the other hand

$$\int_0^{2\pi} p(\theta)\cos j(\theta+\theta_j) = \int_0^{2\pi} \gamma_{jk}^{-1}|a_j|\cos^2 j(\theta+\theta_j) = \pi\gamma_{jk}^{-1}|a_j|\,.$$

Hence $|a_j| \leq 2\gamma_{jk}$. In order to prove (ii) when $j^2 < 3k-2$, consider the unit circle, parametrized by θ. Let $D^2 f(\theta,\xi)$ denote the value of the Hessian of a function f at $e^{i\theta}$ in the direction ξ, and let $P_j = P^{k,j}_{|a_j|}$. Let $\xi = \xi_1 n + \xi_2 t$. Evaluating the Hessian with respect to the above basis, using the relations $P_{nn} = P_{rr}$ and $P_{nt} = P_{r\theta} - P_\theta$, we obtain

$$D^2 P_j(\theta,\xi) = b + c\ cos\ j\theta + d\ sin\ j\theta\,,$$

where b, c, d are functions of ξ_1, ξ_2. Similarly, if $h_l = |z|^{k-l} Re\ z^l$, then

$$D^2 h_l(\theta,\xi) = e\ cos\ l\theta + f\ sin\ l\theta\,. \tag{9}$$

By Proposition 1, these functions are nonnegative for all values of ξ_1, ξ_2 if and only if $|a_j| \leq \gamma_{jk}$. Now we apply the above argument to the family of functions

$$p_\xi(\theta) = D^2 P(\theta,\xi)$$

in place of $p(\theta)$. That gives (ii). □

Proposition 1 shows that condition (i) in Theorem 3 is sharp for the Kohn-Nirenberg domains $M^{k,l}_a$. Condition (ii) is sharp in the following sense.

Lemma 4. *For any even integer j and arbitrary $\epsilon > 0$ there is a convexifiable model domain of finite type k given by (2) such that* $\dfrac{|a_j|}{a_0} > 2\gamma_{jk} - \epsilon$.

Proof. For a fixed j the numbers γ_{jk} form an increasing sequence which tends to 1 as $k \to \infty$. Hence it is enough to prove the statement with $2\gamma_{jk}$ replaced by 2. Consider the domain $\{(z,w) \in \mathbb{C}^2 \mid Im\ w > (Re\ z)^k\}$. We have $(Rez)^k = (\frac{z+\bar{z}}{2})^k$. Using binomial theorem we put the defining equation into form (1), and easily verify that for sufficiently large k this domain provides the example. □

The domain

$$\{(z,w) \in \mathbb{C}^2 \mid Im\ w > (Re\ z)^k\} \tag{10}$$

which appears in the proof of the previous lemma is exceptional also in another sense, as the following lemma shows.

Let the numbers a_j in (1) be fixed and assume that there exists an index j, $2 \le j < k$ for which a_j is nonzero. For a real number $A > 0$ let us consider the polynomial

$$P_A(z,\bar{z}) = A|z|^k + \sum_{j=2,4,\dots,k} |z|^{k-j} Re(a_j z^j).$$

Let M_A be the corresponding model domain. Using the definition and a standard homogeneity argument we verify that for sufficiently large values of A the domain M_A is pseudoconvex, while for sufficiently small values it is not. The same holds for convexifiability. Let now A_p denote the minimum value for which M_A is pseudoconvex and A_c denote the minimum value of A for which M_A is convexifiable. Clearly $A_c \ge A_p > 0$. We have the following necessary condition for convexifiability of M_{A_p}.

Lemma 5. *If $A_p = A_c$, then M_{A_p} is equivalent to $\{(z,w) \in \mathbb{C}^2 \mid Im\ w > (Re\ z)^k\}$. In particular, for every pair $(k,l) \ne (4,2)$ there exists a weakly pseudoconvex and nonconvexifiable Kohn-Nirenberg domain of degree (k,l).*

Proof. Let us assume that M_{A_p} is convexifiable. M_{A_p} contains a curve of weakly pseudoconvex points passing through 0 and lying in the hyperplane $\{u = 0\}$. Its projection to the z-plane is, by homogeneity, a line. By Lemma 3, there is an $\alpha \in \mathbb{C}$ such that $\tilde{P} = P + Re(\alpha z^k)$ is convex. By a rotation we achive that the weakly pseudoconvex curve projects to the y-axis. Since $\tilde{P} \ge 0$ and $\tilde{P}_{yy}(0,y) = \tilde{P}_{xx}(0,y) = 0$, we have also $\tilde{P}(0,y) = 0$. So

$$\tilde{P}(x,y) = \sum_{i=1}^{k} c_i x^i y^{k-i}.$$

Let c_{i_0} be the first nonzero coefficient. If $i_0 < k$, then clearly $c_{i_0} x^{i_0} y^{k-i_0}$ is not convex, but its Hessian dominates the rest of the Hessian of $\tilde{P}$ near the y-axis. It follows from homogeneity that $\tilde{P}$ is not convex in any neighbourhood of 0. Hence $i_0 = k$. □

Let us remark that the particular case of Lemma 5 should be contrasted with Lemma 2 of [Ko]. By this result, weakly pseudoconvex Kohn-Nirenberg domains of degree (k,l) always admit a supporting function if l is a divisor of k.

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