WSGP 21

Josef Mikeš; Lukáš Rachůnek

T-semisymmetric spaces and concircular vector fields

In: Jan Slovák and Martin Čadek (eds.): Proceedings of the 21st Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 2002. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 69. pp. [187]–193.

Persistent URL: http://dml.cz/dmlcz/701696

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 69 (2002), pp. 187-193

T-SEMISYMMETRIC SPACES AND CONCIRCULAR VECTOR FIELDS

JOSEF MIKEŠ, LUKÁŠ RACHŮNEK

ABSTRACT. In this paper we prove that concircular vector fields in proper T-semisymmetric (pseudo-) Riemannian spaces are convergent. Further, these results are generalized and applied to Kenmotsu manifolds.

1. INTRODUCTION

This paper is concerned about certain questions of concircular vector fields in T-semisymmetric Riemannian spaces. The analysis is carried out in tensor form, locally in a class of sufficiently smooth real functions.

One of the most studied classes of special (pseudo-) Riemannian spaces V_n are semisymmetric spaces, which were introduced by N. S. Sinyukov in 1954 (see [3], [11], [18]) and which generalize symmetric spaces. Semisymmetric spaces are investigated e.g. in [3], [21], [22].

Generalizations of semisymmetric spaces are Ricci semisymmetric spaces (see the review [16]), and these are further generalized by the spaces introduced by J. Mikeš as *T-semisymmetric* and studied in [10], [11], [12].

A Riemannian space V_n is called *T-semisymmetric* ([10], [11]), if for a tensor T the condition

$$R(X,Y) \circ T = 0 \tag{1}$$

holds for all tangent vectors X, Y in tangent space TM of V_n, where $R(X,Y)$ denotes the corresponding curvature transformation and the symbol $\circ$ indicates the corresponding derivation on the algebra of all tensor fields. We can write this condition in the local transcription as $T^{\cdots}_{\cdots,[lm]} = 0$, where "," denotes the covariant derivative with respect to a (possibly *indefinite*) metric tensor g_{ij} of a Riemannian space V_n and $[jk]$ denotes the alternation with respect to j and k.

2000 *Mathematics Subject Classification*: 53B20, 53B30.

Key words and phrases: torse-forming vector field, concircular vector field, convergent vector field, recurrent vector field, T-semisymmetric (pseudo-) Riemannian spaces, Kenmotsu manifolds.

Supported by grant No. 201/99/0265 of The Grant Agency of Czech Republic.

The paper is in final form and no version of it will be submitted elsewhere.

Evidently, a T-semisymmetric space is *semisymmetric*, or *Ricci semisymmetric* if T is the Riemannian tensor R, or the Ricci tensor Ric, respectively (see [1], [2], [3], [10], [11], [16], [18]).

The study of recurrent, convergent and torse-forming vector fields has a long history starting in 1925 by the works of H.W. Brinkmann, P.A. Shirokov and K. Yano (see [11], [18], [23], [24]. In Riemannian spaces V_n with the above vector fields there exists a metric of a special form; these spaces are now called (*almost*) *warped products* [4]. These vector fields have been used in many areas of differential geometry, for example in conformal, geodesic and holomorphically projective mappings and transformations (see [8] - [24]), in the theory of subprojective spaces by Kagan [18], Kenmotsu manifolds [1], [2], [6] and others.

In the papers [1], [2], [5], [7], [18] there were studied semisymmetric and Ricci semisymmetric spaces which contain concircular and torse-forming vector fields satisfying some other assumptions.

In our papers [13], [14] we have proved that all torse-forming vector fields in T_i- and T_{ij}-semisymmetric spaces (provided $T_i \neq 0$, $T_{ij} \neq \varrho g_{ij}$, where g_{ij} is a metric tensor in V_n and ϱ is a function) are convergent.

In this paper we present the following analogical assertion: Any concircular vector field in a T-semisymmetric space, where T is a tensor with an arbitrary valency, which cannot be decomposed in a tensor sum of products of functions, the Kronecker delta symbols and the metric components (with lower or upper indices) is convergent. This assertion follows from the more general theorem, which is proved here.

2. On the theory of concircular vector fields

Now we will recall results concerning concircular vector fields and their special case - convergent vector fields, which have been obtained in [4] - [7], [8] - [24].

A vector field ξ in a Riemannian space V_n is called *concircular* if it satisfies $\nabla_X \xi = \varrho X$ where $X \in TM$ and ϱ is a function. In the local transcription this reads

$$\xi^h_{,i} = \varrho\, \delta^h_i \tag{2}$$

where ξ^h are the components of ξ and δ^h_i is the Kronecker symbol. Throughout this paper we assume $\xi^h \neq 0$.

A concircular vector field ξ is called *convergent*, if $\varrho = \text{const}$. A vector field ξ is called *isotropic* if $g(\xi, \xi) = 0$, where g is a metric on V_n.

It is well known (see [18]) that, if a Riemannian space V_n admits a non-isotropic concircular vector field ξ, then in V_n there exists a coordinate system x, in which the metric takes the form

$$ds^2 = e\,(dx^1)^2 + f(x^1)\, d\tilde{s}^2$$

where $e = \pm 1$, $f(\neq 0)$ is a function, and $d\tilde{s}^2(x^2, \dots, x^n)$ is the metric form of the associated Riemannian space $\tilde{V}_{n-1}$.

We can write the equation (2) for a concircular vector field in the following form [18]:

$$\xi_{i,j} = \varrho\, g_{ij}\,, \tag{3}$$

where $\xi_i \equiv \xi^\alpha g_{\alpha i}$ is a locally gradient covector, i.e. $\xi_i = f_{,i}$ where f is a function, g_{ij} is the metric tensor in V_n. Evidently, a concircular vector field with $\varrho \neq 0$ is non-isotropic. This implies:

Lemma 1. *Any non-convergent concircular vector field is non-isotropic.*

In the following we shall study non-isotropic concircular vector fields. The integrability condition arising from (3) can be written in the form

$$\xi_\alpha R^\alpha_{ijk} = g_{ij}\varrho_{,k} - g_{ik}\varrho_{,j} \tag{4}$$

where R^h_{ijk} is the Riemannian tensor of V_n.

When contracting (4) with such a ξ^i, we arrive to the formula

$$\xi_j\varrho_{,k} - \xi_k\varrho_{,j} = 0\,.$$

In view of $\xi_i \neq 0$, we get

$$\varrho_{,i} = \tau\xi_i \tag{5}$$

where τ is a function. Then (4) can be written in the form

$$\xi_\alpha R^\alpha_{ijk} = \tau(g_{ij}\xi_k - g_{ik}\xi_j)\,. \tag{6}$$

According to (5) we can see that if ξ is not convergent, i.e. $\varrho \neq$ const, then $\varrho_{,i} \neq 0$ and therefore $\tau \neq 0$ is true.

3. On the operator $R(X,Y)$

In this section we shall be interested in T-semisymmetric Riemannian spaces where T is an arbitrary tensor field of type $(0,m)$, i.e. T will be an m-linear form

$$T(X_1, X_2, \ldots, X_m)\,,$$

where $X_1, X_2, \ldots, X_m \in TM$.

If V_n contains a convergent field ξ, then instead of the condition (1) one can consider a weaker condition

$$R(X,\xi)\circ T = 0 \quad \text{for each } X \in TM\,. \tag{7}$$

In the local transcription this condition can be written in the form $T_{\ldots,[l\alpha]}\xi^\alpha = 0$.

Because the operator $R(X,Y)$ is a derivation on the tensor algebra, we have for any two tensor fields U and V the equalities

$$\begin{aligned} R(X,Y)\circ(U \pm V) &= R(X,Y)\circ U \pm R(X,Y)\circ V \\ R(X,Y)\circ(U\otimes V) &= (R(X,Y)\circ U)\otimes V + U\otimes(R(X,Y)\circ V)\,. \end{aligned} \tag{8}$$

Under the assumption that the tensor T is the tensor composition of functions on V_n and the metric tensor g we can get by means of (8) that (7) is satisfied.

Let us denote by $\tilde{T}$ an arbitrary contraction of the tensor T with g^{ij} where $\|g^{ij}\| \equiv \|g_{ij}\|^{-1}$. Then the above properties of the operator $R(X,Y)$ imply the following lemma.

Lemma 2. *Let the tensor T satisfy $R(X,\xi)\circ T = 0$, $\forall X \in TM$. Then $R(X,\xi)\circ\tilde{T} = 0$, $\forall X \in TM$ holds.*

4. On concircular vector fields in T-semisymmetric spaces and their generalization

The classical result that all concircular vector fields in semisymmetric (with non constant curvature) and those in non-Einsteinian Ricci semisymmetric spaces are convergent was generalized by J. Mikeš for arbitrary T_i- and T_{ij}-semisymmetric spaces such that $T_i \neq 0$ and $T_{ij} \neq \varrho g_{ij}$, respectively, see [10]. This was proved also in the case of torse-forming vector fields by J. Mikeš and L. Rachůnek in [13] and [14].

Let us show further generalization of the above assertions.

Theorem 1. *Let V_n be a T-semisymmetric space, where the corresponding m-covariant tensor field T is of degree $m < n$. If there exists a concircular vector field ξ on V_n, then either* a) *ξ is convergent, or* b) *T can be expressed as the tensor sum of products of functions and metric components. Moreover, in the last case, $T = 0$ for m odd.*

This theorem follows from the more general theorem

Theorem 2. *Let T be a m-covariant tensor field in a space V_n $(m < n)$ and ξ is concircular vector field on V_n. If the condition* (7) *is satisfied, then either ξ is convergent or T can be expressed as the tensor sum of products of functions and metric components. In the last case, $T = 0$ for m odd.*

Proof. Let there exist a concircular vector field ξ on V_n which is not convergent. We will use the induction. Theorem 2 is valid for $m = 1, 2$, see [13] and [14]. Supposing the validity of Theorem 2 for $m - 1$ we prove it for $m > 2$.

Let (7) for m-covariant tensor T be satisfied. In the local transcription (7) can be written in the form

$$\sum_{\sigma=1}^{m} R^{\alpha}_{i_\sigma k\beta}\xi^{\beta} \cdot T_{i_1\ldots i_{\sigma-1}\alpha i_{\sigma+1}\ldots i_m} = 0 \tag{9}$$

where $T_{i_1 i_2 \ldots i_m}(x)$ are local components of tensor T and α, β are summation indices too.

Using the assumption that ξ is not convergent ($\tau \neq 0$) and the properties of the Riemann tensor we get by means of (6) and (9)

$$\sum_{\sigma=1}^{m}(g_{ki_\sigma}\xi^{\alpha}T_{i_1\ldots i_{\sigma-1}\alpha i_{\sigma+1}\ldots i_m} - \xi_{i_\sigma}T_{i_1\ldots i_{\sigma-1}k i_{\sigma+1}\ldots i_m}) = 0\,. \tag{10}$$

Contracting (10) with g^{ki_ε}, $\varepsilon = 1, 2, \ldots, m$, we obtain a system of equations whose left-hand sides contain the terms

$$\xi^{\alpha}T_{\alpha i_2\ldots i_m}\,,\quad \xi^{\alpha}T_{i_1\alpha i_3\ldots i_m}\,,\quad \ldots,\quad \xi^{\alpha}T_{i_1 i_2\ldots\alpha} \tag{11}$$

and right-hand sides contain the metric components, the covariant components ξ_i of ξ and the components of all contracted tensors $\tilde{T}$ of T. The last tensors $\tilde{T}$ according to the induction assumption (since they have their valency less than $m - 1$) and due to Lemma 2 can be expressed as the tensor sum of products of functions and the metric tensor.

We can show that under the assumption $m < n$ the system of equations with the unknown variables (11) has a nonzero determinant and thus it has a unique solution, which means that the unknown variables (11) can be determined as a linear combination of the right-hand sides. Therefore

$$\xi^\alpha T_{i_1\dots i_{s-1}\alpha i_{s+1}\dots i_m} = \sum_{\alpha=1}^{m} \xi_{i_\alpha} \overset{(\alpha)}{\tilde{g}}{}_{i_1\dots i_{\alpha-1} i_{\alpha+1}\dots i_m} \tag{12}$$

holds, where the tensors $\overset{(\alpha)}{\tilde{g}}$ have the form of the tensor sum of products of functions and the metric components.

If we substitute (12) to (10) we get the system of equations

$$\sum_{\sigma=1}^{m} \xi_{i_\sigma} \overset{\sigma}{T}_{i_1\dots i_{\sigma-1} k i_{\sigma+1}\dots i_m} = 0 \tag{13}$$

where $\overset{\sigma}{T}_{i_1 i_2\dots i_m} = T_{i_1 i_2\dots i_m} + \overset{\sigma}{g}_{i_1 i_2\dots i_m}$ and $\overset{\sigma}{g}_{i_1 i_2\dots i_m}$ are the tensor sums of products of functions with the metric components. Applying the differential operator $R(X,Y)$ on (13) and using its properties we obtain

$$\sum_{\sigma=1}^{m} (\xi_{i_\sigma}\xi_l - \xi_\alpha \xi^\alpha g_{i_\sigma l}) \cdot \overset{\sigma}{T}_{i_1\dots i_{\sigma-1} k i_{\sigma+1}\dots i_m} = 0\,. \tag{14}$$

Let us suppose that $\overset{1}{T} \neq 0$. Then there exist vectors $\underset{1}{a}^{i_1}, \underset{2}{a}^{i_2}, \dots, \underset{m}{a}^{i_m}$ such that

$$\overset{1}{T}_{i_1 i_2\dots i_m} \underset{1}{a}^{i_1} \underset{2}{a}^{i_2} \cdots \underset{m}{a}^{i_m} \neq 0\,.$$

According to Lemma 1 we have $\xi_\alpha \xi^\alpha \neq 0$. Therefore, contracting (14) with $\underset{1}{a}^{k} \underset{2}{a}^{i_2} \cdots \underset{m}{a}^{i_m}$, we deduce that $\operatorname{rank}\|g_{ij}\| \leq m$ which contradicts the assumption $m < n$. This means that $\overset{1}{T} = 0$. Therefore T is the tensor sum of products of functions on V_n and the metric components.

Remarks. It is clear that Theorems 1 and 2 are not valid in a general case for $m \leq n$.

5. On T-semisymmetric Kenmotsu manifolds

A $(2n+1)$-dimensional Riemannian space V_{2n+1} is said to be a Kenmotsu manifold if it admits a vector field ξ, a 1-form η and an endomorphism ϕ of its tangent bundle TM for which:

1) (ϕ, η, ξ, g) is metric almost contact structure, i.e.: $\phi^2 = \mathrm{id} + \eta \otimes \xi$, $\eta(\xi) = 1$, $\varphi\xi = 0$, $\eta \circ \varphi = 0$, $g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$, $\eta(X) = g(X,\xi)$, and

2) $(\nabla_X \varphi)Y = -g(X, \varphi Y)\xi - \eta(X)\varphi Y$, $\nabla_X \xi = X - \eta(X)\xi$ for any $X, Y \in TM$, where ∇ denotes the Riemannian connection of g.

Kenmotsu manifold is an example of almost contact manifold, which is not a K-contact (and hence not a Sasakian-manifold), as shown by Kenmotsu [6].

T.Q. Binh, U.C. De, L. Tamassy and M. Tarafdar [1], [2] studied Ricci-semisymmetric and semisymmetric Kenmotsu manifolds. In Kenmotsu manifolds there exists a unit vector field ξ satisfying the condition $\nabla_X \xi = X - \eta(X)\xi$, where $\eta(X) = g(X,\xi)$. By simple observation we convince ourselves that this vector field is concircular, but it is not convergent.

Therefore we can apply the results of Theorems 1 and 2 on Kenmotsu manifolds and in this way we can generalize the results of [1] and [2]. On the base of the above Theorems we get

Theorem 3. *Let T be a m-covariant tensor field in a Kenmotsu manifold V_n $(m < n)$ and let ξ be a concircular vector field on V_n which generates this manifold. For a Kenmotsu manifold the following conditions are equivalent:*

1) $R(X,Y) \circ T = 0$ *for any* $X, Y \in TM$,
2) $R(X,\xi) \circ T = 0$ *for any* $X \in TM$.
3) T *is the tensor sum of products of functions and the metric components.*

References

[1] Binh, T. Q., *Some remarks on Kenmotsu manifolds*, Abstract in Int. Congress on Diff. Geom. in memory of Alfred Gray, Sept. 18–23, 2000, Univ. del País Vasco, Bilbao, p. 12.

[2] Binh, T. Q., De, U. C., Tamassy, L., Tarafdar, M., *Some remarks on Kenmotsu manifolds*, In poster of Int. Congress on Diff. Geom. in memory of Alfred Gray, Sept. 18–23, 2000, Univ. del País Vasco, Bilbao, 2000.

[3] Boeckx, E., Kowalski, O., Vanhecke, L., *Riemannian manifolds of conullity two*, World Scientific, Singapore, xvii, 1996.

[4] Defever, F., Deszcz, R., *On warped product manifolds satisfying a certain curvature condition*, Atti Accad. Peloritana Pericolanti, Cl. Sci. Fis. Mat. Nat. **69** (1991), 213–236.

[5] Deszcz, R., *On some Riemannian manifolds admitting a concircular vector field*, Demonstratio Math. **9** No. 3, (1976), 487–495.

[6] Kenmotsu, K., *A class of almost contact Riemannian manifolds*, Tohoku Math. J. (2) (1972), 93–103.

[7] Kowolik, J., *On some Riemannian manifolds admitting torse-forming vector fields*, Demonstratio Math. **18** No. 3 (1985), 885–891.

[8] Mikeš, J., *Geodesic Ricci mappings of two-symmetric Riemann spaces*, Math. Notes **28** (1981), 622–624.

[9] Mikeš, J., *On Sasaki spaces and equidistant Kaehler spaces*, Sov. Math., Dokl., **34** (1987), 428–431; translation from Dokl. Akad. Nauk **291** (1986), 33–36.

[10] Mikeš, J., *Geodesic mappings of special Riemannian spaces*, Coll. Math. Soc. J. Bolyai. 46. Top. in diff. geom. Debrecen **2** (1984), Amsterdam etc. (1988), 793–813.

[11] Mikeš, J., *Geodesic mappings of affine-connected and Riemannian spaces*, J. Math. Sci. New York, **78** 3 (1996), 311–333.

[12] Mikeš, J., *Holomorphically projective mappings and their generalizations*, J. Math. Sci., New York **89** No.3 (1998), 1334–1353.

[13] Mikeš, J., Rachůnek, L., *On T-semisymmetric Riemannian spaces admitting torse-forming vector fields*, Abstr. of Contrib. Papers. Int. Summer School Seminar on Recent Problems in Theoret. and Math. Physics. The XI Petrov School - 1999. Kazan (1999), 64–65.

[14] Mikeš, J., Rachůnek, L., *Torse-forming vector fields in T-semisymmetric Riemannian spaces*, Proc. of Coloquium. on Diferential Geometry, Debrecen, Hungary, (2000) (to appear).

[15] Mikeš, J., Radulović, Ž., *On projective transformations of Riemannian spaces with harmonic curvature*, New developments in differential geometry, Budapest 1996. Proceedings of the conference, Budapest, Hungary, July 27-30, 1996. Kluwer Academic Publishers, Dordrecht, 1999, 279–283.

[16] Mirzojan, V. A., *Ric-semisymmetric submanifolds*, J. Math. Sci., New York **70** No.2 (1994), 1624–1646; translation from Itogi Nauki Tekh., Ser. Probl. Geom. **23** (1991), 29–66.

[17] Shirokov, P. A., *Collected works of geometry*, Kazan Univ. Press, 1966, 432 pp.

[18] Sinyukov, N. S., *Geodesic mappings of Riemannian spaces*, Nauka, Moscow, 1979, 256 pp.

[19] Sobchuk, V. S., *Ricci generalized symmetric Riemannian spaces admit nontrivial geodesic mappings*, Sov. Math. Dokl. **26** (1982), 699-701; translation from Dokl. Akad. Nauk **267** (1982), 793–795.

[20] Solodovnikov, A. S., *Spaces with common geodesics*, Tr. Semin. Vektor. tensor. Anal., **11** (1961), 43–102.

[21] Szabo, Z. I., *Structure theorems on Riemannian spaces satisfying $R(X,Y) \cdot R = 0$. I: The local version,* J. Differential Geom. **17** (1982), 531–582.

[22] Szabo, Z. I., *Structure theorems on Riemannian spaces satisfying $R(X,Y) \cdot R = 0$. II. Global versions,* Geom. Dedicata **19** (1985), 65–108.

[23] Yano, K., *Concircular geometry, I-IV*, Proc. Imp. Acad. Tokyo **16** (1940), 195–200, 354–360, 442–448, 505–511.

[24] Yano, K., *On torse-forming directions in Riemannian spaces*, Proc. Imp. Acad. Tokyo **20** (1944), 701–705.

J. Mikeš, Department of Algebra and Geometry
Faculty of Science, Palacky University
Tomkova 40, 779 00 Olomouc
Czech Republic
E-mail: Mikes@risc.upol.cz

L. Rachůnek, Department of Mathematics
Faculty of Technology, UTB
Masaryk Square 275, 762 72 Zlín
Czech Republic
E-mail: Rachunek@zlin.vutbr.cz