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GENERALIZED JACOBI MORPHISMS IN VARIATIONAL SEQUENCES

MAURO FRANCAVIGLIA* AND MARCELLA PALESE**

ABSTRACT. We provide a geometric interpretation of generalized Jacobi morphisms in the framework of finite order variational sequences.

Jacobi morphisms arise classically as an outcome of an invariant decomposition of the second variation of a Lagrangian. Here they are characterized in the context of *generalized Lagrangian symmetries* in terms of *variational Lie derivatives* of generalized Euler–Lagrange morphisms. We introduce the *variational vertical derivative* and stress its link with the classical concept of *variation*. The relation with generalized Helmholtz morphisms is also clarified.

1. INTRODUCTION

Our framework is the calculus of variations on finite order jets of a fibered manifold. More precisely, we consider the geometrical formulation of this framework in terms of *variational sequences* introduced by Krupka [17, 18]. As it is well known, in this formulation the variational sequence is defined as a quotient of the de Rham sequence on a finite order jet prolongation of a fibered manifold with respect to an intrinsically defined subsequence, the *contact subsequence*. Standard objects of the calculus of variations can be interpreted as sheaf sections and morphisms in the variational sequence, which turns out to be an exact resolution of the constant sheaf $\mathbb{R}$ over the relevant fibered manifold.

This work belongs to a series of papers concerned with the geometric and algebraic characterization of variational objects in the framework of variational sequences in field theories. Starting from a suitable representation of generalized Lagrangian, Euler–Lagrange morphisms and Helmholtz–Sonin morphisms as sections of sheaves of vector bundles (earlier provided in [14, 21, 24, 25, 26]) we introduced the *variational Lie derivative* as an operator acting on the sheaves of the variational sequence and we gave a suitable representation of symmetries and Noether Theorems together with

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their interpretation in terms of conserved currents [7] as well as a geometric proof of the existence and globality of superpotentials in gauge–natural theories [8, 5]. Furthermore, in a recent paper [6] we provided a suitable geometric description of the second variation of a generalized Lagrangian, by showing that the *variational Lie derivative* is a functor on the category of variational sequences. In this note the Jacobi morphisms, which arise as an outcome of an invariant decomposition of the second variation of a Lagrangian (see e.g. [1, 23] and [2, 3, 4, 9]) are characterized in the context of *generalized Lagrangian symmetries* in terms of *variational Lie derivatives* of generalized Euler–Lagrange morphisms.

As it is well known from a classical view point, the Jacobi equations of an action functional arise from the second variation , which governs the behaviour of the action itself in the neighbourood of critical sections, allowing to distinguish between minima, maxima and degenerate extremals. Historically they were related to the so–called *accessory problem* (see, *e.g.* [1, 23]), where they are directly obtained as the variation of the Euler–Lagrange equations of a given Lagrangian. In other words, the solutions of the Jacobi equations are vector fields which 'deform' (families of) solutions of the Euler–Lagrange equations into (families of) solutions. More recently in [2, 3, 4] it was shown how to recast (up to divergencies) the system formed by the Euler–Lagrange equations together with the Jacobi equations for a given Lagrangian as the Euler Lagrange equations for a *'deformed Lagrangian'.*

We shall here provide a geometrical characterization of the second variation (see *e.g.* [1, 2, 3, 4, 9, 23]) of a Lagrangian in the framework of finite order variational sequences. To this purpose, we introduce the notion of iterated variation of a section as an i–parameter 'deformation' of the section by means of vertical flows and thus define the i-th variation of a morphism which is very simply related to the iterated Lie derivative of the morphism itself. Relying on previous results of us [7] on the representation of the Lie derivative operator in the variational sequence we can then define an operator on the quotient sheaves of the sequence, the *variational vertical derivative.* We stress some linearity properties of this operator and show that it is a functor on the category of variational sequences. Making use of suitable representations of the variational vertical derivative we relate the second order variation of a generalized Lagrangian with the variational Lie derivative of generalized Euler–Lagrange operators associated with the Lagrangian itself. In [6] we stated the Theorem concerning the functoriality of the variational vertical derivative operator. Here we give a more explicit and detailed proof of this result and stress the aspects concerned with variational symmetries.

We show then that Euler–Lagrange equations as well as Jacobi equations, for a given Lagrangian λ, can be obtained from a unique variational problem for the Lagrangian $\hat{\delta}\lambda$ (the variational vertical derivative of λ), in terms of its generalized symmetries, which turn out to be solutions of the classical Jacobi equations along critical sections. In this way we recover the recent result formulated in the geometric framework of jet bundles [2, 3, 4] as well as a classical result of e.g. [9]. The Lagrangian characterization of the second variation of a Lagrangian in the framework of jet bundles has been in fact considered in [2, 3, 4, 9].

As an outcome of the intrinsic representation of the second order variation of a generalized Lagrangian, the afore mentioned invariant decomposition is geometrically interpreted here as a simple and direct application of the representation of the variational Lie derivative of variational morphisms provided in [7], together with a suitable version of a global decomposition formula of vertical morphisms due to Kolář [10, 11]. The generalized Jacobi morphism is then represented as a new geometric object in the variational sequence. It turns out that the generalized Jacobi morphism is closely related with the generalized Helmholtz morphism. The latter is the skew–symmetric part of a morphism uniquely defined in terms of the generalized Euler–Lagrange morphism, as introduced by Vitolo [25] and studied in [14]. The former arises from an invariant integration by parts of the morphism mentioned above.

2. Variational sequences on jets of fibered manifolds

Our framework is a fibered manifold $\pi : \boldsymbol{Y} \to \boldsymbol{X}$, with $\dim \boldsymbol{X} = n$ and $\dim \boldsymbol{Y} = n + m$ (see *e.g.* [22]).

For $r \geq 0$ we are concerned with the r–jet space $J_r\boldsymbol{Y}$; in particular, we set $J_0\boldsymbol{Y} \equiv \boldsymbol{Y}$. We recall the natural fiberings $\pi^r_s : J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$, $r \geq s$, $\pi^r : J_r\boldsymbol{Y} \to \boldsymbol{X}$, and, among these, the *affine* fiberings π^r_{r-1}. We denote by $V\boldsymbol{Y}$ the vector subbundle of the tangent bundle $T\boldsymbol{Y}$ of vectors on $\boldsymbol{Y}$ which are vertical with respect to the fibering π.

Charts on $\boldsymbol{Y}$ adapted to π are denoted by (x^λ, y^i). Greek indices $\lambda, \mu, \ldots$ run from 1 to n and they label base coordinates, while Latin indices $i, j, \ldots$ run from 1 to m and label fibre coordinates, unless otherwise specified. We denote by $(\partial_\lambda, \partial_i)$ and (d^λ, d^i) the local bases of vector fields and 1–forms on $\boldsymbol{Y}$ induced by an adapted chart, respectively.

We denote multi–indices of dimension n by boldface Greek letters such as $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, with $0 \leq \alpha_\mu$, $\mu = 1, \ldots, n$; by an abuse of notation, we denote with λ the multi–index such that $\alpha_\mu = 0$, if $\mu \neq \lambda$, $\alpha_\mu = 1$, if $\mu = \lambda$. We also set $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_n$ and $\boldsymbol{\alpha}! := \alpha_1! \ldots \alpha_n!$. The charts induced on $J_r\boldsymbol{Y}$ are denoted by $(x^\lambda, y^i_{\boldsymbol{\alpha}})$, with $0 \leq |\boldsymbol{\alpha}| \leq r$; in particular, we set $y^i_{\boldsymbol{0}} \equiv y^i$. The local vector fields and forms of $J_r\boldsymbol{Y}$ induced by the above coordinates are denoted by $(\partial^{\boldsymbol{\alpha}}_i)$ and $(d^i_{\boldsymbol{\alpha}})$, respectively.

For $r \geq 1$, we consider the natural complementary fibered morphisms over the affine fibering $J_r\boldsymbol{Y} \to J_{r-1}\boldsymbol{Y}$ induced by *contact maps* on jet spaces

$$\mathcal{D} : J_r\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X} \to TJ_{r-1}\boldsymbol{Y}, \qquad \vartheta : J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} TJ_{r-1}\boldsymbol{Y} \to VJ_{r-1}\boldsymbol{Y},$$

with coordinate expressions, for $0 \leq |\boldsymbol{\alpha}| \leq r-1$, given by

$$\mathcal{D} = d^\lambda \otimes \mathcal{D}_\lambda = d^\lambda \otimes (\partial_\lambda + y^j_{\boldsymbol{\alpha}+\lambda} \partial^{\boldsymbol{\alpha}}_j), \qquad \vartheta = \vartheta^j_{\boldsymbol{\alpha}} \otimes \partial^{\boldsymbol{\alpha}}_j = (d^j_{\boldsymbol{\alpha}} - y^j_{\boldsymbol{\alpha}+\lambda} d^\lambda) \otimes \partial^{\boldsymbol{\alpha}}_j .$$

We have the following natural fibered splitting

$$J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*J_{r-1}\boldsymbol{Y} = \left(J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*\boldsymbol{X} \right) \oplus \overset{*}{\mathcal{C}}_{r-1}[\boldsymbol{Y}], \tag{1}$$

where $\overset{*}{\mathcal{C}}_{r-1}[\boldsymbol{Y}] := \operatorname{im} \vartheta^*_r$ and the canonical isomorphism $\overset{*}{\mathcal{C}}_{r-1}[\boldsymbol{Y}] \simeq J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} V^*J_{r-1}\boldsymbol{Y}$ holds true (see [17, 18, 19, 22]).

The above splitting induces also a decomposition of the exterior differential on $\boldsymbol{Y}$, $(\pi^{r+1}_r)^* \circ d = d_H + d_V$, where d_H and d_V are called the *horizontal* and *vertical* *differential*, respectively. The action of d_H and d_V on functions and 1–forms on $J_r\boldsymbol{Y}$ uniquely characterizes d_H and d_V (see, *e.g.*, [22, 25] for more details).

If $f : J_r\boldsymbol{Y} \to \mathbb{R}$ is a function, then we set $D_\lambda f := \varPi_\lambda f$, $D_{\alpha+\lambda} f := D_\lambda D_\alpha f$, where the operator D_λ is the standard *formal derivative.*

A *projectable vector field* on $\boldsymbol{Y}$ is defined to be a pair (Ξ, ξ), where $\Xi : \boldsymbol{Y} \to T\boldsymbol{Y}$ and $\xi : \boldsymbol{X} \to T\boldsymbol{X}$ are vector fields and Ξ is a fibered morphism over ξ. A projectable vector field (Ξ, ξ) can be conveniently prolonged to a projectable vector field $(j_r\Xi, \xi)$, the coordinate expression of which can be found *e.g.* in [7] and [16, 19, 22]. A *vertical vector field* on $\boldsymbol{Y}$ is a projectable vector field on $\boldsymbol{Y}$ such that $\xi = 0$.

The following sheaves will be needed in the sequel.

i. For $r \geq 0$, the standard sheaves $\overset{p}{\Lambda}_r$ of p–forms on $J_r\boldsymbol{Y}$.

ii. For $0 \leq s \leq r$, the sheaves $\overset{p}{\mathcal{H}}_{(r,s)}$ and $\overset{p}{\mathcal{H}}_r$ of *horizontal forms*, i.e. of local fibered morphisms over π^r_s and π^r of the type $\alpha : J_r\boldsymbol{Y} \to \overset{p}{\wedge} T^*J_s\boldsymbol{Y}$ and $\beta : J_r\boldsymbol{Y} \to \overset{p}{\wedge} T^*\boldsymbol{X}$, respectively.

iii. For $0 \leq s < r$, the subsheaf $\overset{p}{\mathcal{C}}_{(r,s)} \subset \overset{p}{\mathcal{H}}_{(r,s)}$ of *contact forms*, i.e. of sections $\alpha \in \overset{p}{\mathcal{H}}_{(r,s)}$ with values into $\overset{p}{\wedge}(\overset{*}{\mathcal{C}}_s[\boldsymbol{Y}])$. There is a distinguished subsheaf $\overset{p}{\mathcal{C}}_r \subset \overset{p}{\mathcal{C}}_{(r+1,r)}$ of local fibered morphisms $\alpha \in \overset{p}{\mathcal{C}}_{(r+1,r)}$ such that $\alpha = \overset{p}{\wedge} \operatorname{im} \vartheta^*_{r+1}[\boldsymbol{Y}] \circ \tilde{\alpha}$, where $\tilde{\alpha}$ is a section of the fibration $J_{r+1}\boldsymbol{Y} \underset{J_r\boldsymbol{Y}}{\times} \overset{p}{\wedge} V^*J_r\boldsymbol{Y} \to J_{r+1}\boldsymbol{Y}$ which projects down onto $J_r\boldsymbol{Y}$.

According to [25], the fibered splitting (1) yields naturally the sheaf splitting

$$\overset{p}{\mathcal{H}}_{(r+1,r)} = \bigoplus_{t=0}^{p} \overset{p-t}{\mathcal{C}}_{(r+1,r)} \wedge \overset{t}{\mathcal{H}}_{r+1} \,,$$

which restricts to the inclusion $\overset{p}{\Lambda}_r \subset \bigoplus_{t=0}^{p} \overset{p-t}{\mathcal{C}}_r \wedge \overset{t}{\mathcal{H}}{}^h_{r+1}$, where $\overset{p}{\mathcal{H}}{}^h_{r+1} := h(\overset{p}{\Lambda}_r)$ for $0 < p \leq n$ and h is defined to be the restriction to $\overset{p}{\Lambda}_r$ of the projection of the above splitting onto the non–trivial summand with the highest value of t.

2.1. **Generalized Euler–Lagrange and Helmholtz–Sonin morphisms in variational sequences.** We recall now the theory of variational sequences on finite order jet spaces, as it was developed by Krupka in [17]. By an abuse of notation, denote by $d\ker h$ the sheaf generated by the presheaf $d\ker h$. Set $\overset{*}{\Theta}_r := \ker h + d\ker h$. The diagram following Definition 2.1 is commutative and its rows and columns are exact.

Definition 2.1. The bottom row of the below diagram is called the r-th order *variational sequence* associated with the fibered manifold $\boldsymbol{Y} \to \boldsymbol{X}$.

□

$$
\begin{array}{ccccccccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \overset{1}{\Theta}_r & \xrightarrow{d} & \overset{2}{\Theta}_r & \xrightarrow{d} & \dots & \xrightarrow{d} & \overset{I}{\Theta}_r & \xrightarrow{d} & 0 & \longrightarrow & \dots \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I\!R & \longrightarrow & \overset{0}{\Lambda}_r & \xrightarrow{d} & \overset{1}{\Lambda}_r & \xrightarrow{d} & \overset{2}{\Lambda}_r & \xrightarrow{d} & \dots & \xrightarrow{d} & \overset{I}{\Lambda}_r & \xrightarrow{d} & \overset{I+1}{\Lambda}_r & \xrightarrow{d} & \dots \xrightarrow{d} 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I\!R & \longrightarrow & \overset{0}{\Lambda}_r & \xrightarrow{\mathcal{E}_0} & \overset{1}{\Lambda}_r/\overset{1}{\Theta}_r & \xrightarrow{\mathcal{E}_1} & \overset{2}{\Lambda}_r/\overset{2}{\Theta}_r & \xrightarrow{\mathcal{E}_2} & \dots & \xrightarrow{\mathcal{E}_{I-1}} & \overset{I}{\Lambda}_r/\overset{I}{\Theta}_r & \xrightarrow{\mathcal{E}_I} & \overset{I+1}{\Lambda}_r & \xrightarrow{d} & \dots \xrightarrow{d} 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & & & & 0 & & 0 & &
\end{array}
$$

The quotient sheaves in the variational sequence can be conveniently represented as follows [25]:

i. If $k \leq n$, then the sheaf morphism h yields the natural isomorphism

$$I_k : \overset{k}{\Lambda}_r/\overset{k}{\Theta}_r \to \overset{k}{\mathcal{H}}{}^h_{r+1} := \overset{k}{\mathcal{V}}_r : [\alpha] \mapsto h(\alpha) .$$

ii. If $k > n$, then the projection h induces the natural sheaf isomorphism

$$I_k : \left(\overset{k}{\Lambda}_r/\overset{k}{\Theta}_r\right) \to \left(\overset{k-n}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}\right) /h(\overline{d\ker h}) := \overset{k}{\mathcal{V}}_r : [\alpha] \mapsto [h(\alpha)] .$$

We remark that a section $\lambda \in \overset{n}{\mathcal{V}}_r$ is just a *Lagrangian* of order $(r+1)$ according to the standard literature; furthermore, $\mathcal{E}_n(\lambda) \in \overset{n+1}{\mathcal{V}}_r$ coincides with the standard *higher order Euler–Lagrange morphism* $\mathcal{E}(\lambda)$ associated with λ.

Let $\alpha \in \overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}$. Then there is a unique pair of sheaf morphisms

$$E_\alpha \in \overset{1}{\mathcal{C}}_{(2r,0)} \wedge \overset{n}{\mathcal{H}}{}^h_{2r+1} , \qquad F_\alpha \in \overset{1}{\mathcal{C}}_{(2r,r)} \wedge \overset{n}{\mathcal{H}}{}^h_{2r+1} , \tag{2}$$

such that $(\pi^{2r+1}_{r+1})^*\alpha = E_\alpha - F_\alpha$, and F_α is locally of the form $F_\alpha = d_H p_\alpha$, with $p_\alpha \in \overset{1}{\mathcal{C}}_{(2r-1,r-1)} \wedge \overset{n-1}{\mathcal{H}}_{2r}$ (see *e.g.* [11, 25]).

Definition 2.2. Let $\gamma \in \overset{n+1}{\Lambda}_r$. The morphism $E_{h(\gamma)} \in \overset{n+1}{\mathcal{V}}_r$ is called the *generalised Euler–Lagrange morphism* associated with γ and the operator $\mathcal{E}_n$ is called the *generalised Euler–Lagrange operator.* Furthermore $p_{h(\gamma)}$ is a generalised momentum associated with $E_{h(\gamma)}$.

□

Recall (see [25]) that if $\beta \in \overset{1}{\mathcal{C}}_s \wedge \overset{1}{\mathcal{C}}_{(s,0)} \wedge \overset{n}{\mathcal{H}}_s$, then there is a unique morphism

$$\tilde{H}_\beta \in \overset{1}{\mathcal{C}}_{(2s,s)} \otimes \overset{1}{\mathcal{C}}_{(2s,0)} \wedge \overset{n}{\mathcal{H}}_{2s}$$

such that, for all $\Xi : \boldsymbol{Y} \to V\boldsymbol{Y}$, $E_{\hat{\beta}} = C^1_1\left(j_{2s}\Xi \otimes \tilde{H}_\beta\right)$, where $\hat{\beta} := j_s\Xi \rfloor \beta$, C^1_1 stands for tensor contraction and $\rfloor$ denotes inner product. Then there is a unique pair of sheaf morphisms

$$H_\beta \in \overset{1}{\mathcal{C}}_{(2s,s)} \wedge \overset{1}{\mathcal{C}}_{(2s,0)} \wedge \overset{n}{\mathcal{H}}_{2s}\,, \quad G_\beta \in \overset{2}{\mathcal{C}}_{(2s,s)} \wedge \overset{n}{\mathcal{H}}_{2s}\,, \tag{3}$$

such that $\pi^{2s*}_s \beta = H_\beta - G_\beta$ and $H_\beta = \frac{1}{2} A(\tilde{H}_\beta)$, where A stands for antisymmetrisation. Moreover, G_β is locally of the type $G_\beta = d_H q_\beta$, where $q_\beta \in \overset{2}{\mathcal{C}}_{2s-1} \wedge \overset{n-1}{\mathcal{H}}_{2s-1}$, hence $[\beta] = [H_\beta]$.

Definition 2.3. Let $\gamma \in \overset{n+1}{\Lambda}_r$. The morphism $H_{dE_{h(\gamma)}}$ is called the *generalised Helmholtz morphism* and the operator $\mathcal{E}_{n+1}$ is called the *generalised Helmholtz operator.* Furthermore $q_{dE_{h(\gamma)}}$ is a generalised momentum associated to the Helmholtz morphism. □

Coordinate expressions of the morphisms E_α and H_β can be found e.g. in [14, 25].

2.2. **Symmetries in variational sequences.** Making use of the sheaf isomorphisms aboveand of decomposition formulae (2) and (3), in [7] we proved that the Lie derivative operator with respect to the r-th order prolongation $j_r\Xi$ of a projectable vector field (Ξ, ξ) can be conveniently represented on the quotient sheaves of the variational sequence in terms of an operator, the *variational Lie derivative* $\mathcal{L}_{j_r\Xi}$. Here, we adapt the representation to the case when Ξ is a vertical vector field as follows:

i. If $p = n$ and $\lambda \in \overset{n}{\mathcal{V}}_r$, then

$$\mathcal{L}_{j_r\Xi}\lambda = \Xi \rfloor \mathcal{E}(\lambda) + d_H(j_r\Xi \rfloor p_{d_V\lambda})\,; \tag{4}$$

ii. If $p = n + 1$ and $\eta \in \overset{n+1}{\mathcal{V}}_r$, then

$$\mathcal{L}_{j_r\Xi}\eta = \mathcal{E}(\Xi \rfloor \eta) + \tilde{H}_{d\eta}(j_{2r+1}\Xi)\,. \tag{5}$$

Definition 2.4. Let Ξ be a variational vector field on $\boldsymbol{Y}$ and let $\lambda \in \overset{n}{\mathcal{V}}_r$ be a generalized Lagrangian. Then Ξ is called a variational *symmetry* of λ if $\mathcal{L}_{j_{r+1}\Xi}\,\lambda = 0$. □

Definition 2.5. Let Ξ be a variational vector field on $\boldsymbol{Y}$ and let $\eta \in \overset{n+1}{\mathcal{V}}_r$ be a generalized Euler–Lagrange morphism. Then Ξ is called a variational *generalized symmetry* of η if $\mathcal{L}_{j_{2r+1}\Xi}\,\eta = 0$. □

3. Variations

We shall here introduce *variations* of a morphism as *multiparameter deformations* showing that this is equivalent to take iterated variational Lie derivatives with respect to vertical vector fields.

We define the i-th variation of a section and introduce the i-th variation of a morphism of the kind $\alpha : J_r\boldsymbol{Y} \to \overset{k}{\wedge} T^*J_r\boldsymbol{Y}$ along the section σ in terms of the pull-back by means of the r-th prolongation of the i-th variation of a section $\sigma : \boldsymbol{X} \to \boldsymbol{Y}$.

Definition 3.1. Let $\sigma : X \to Y$ be a section and i any integer. An i-th variation of σ is a smooth section $\Gamma_i : I \times X \to Y$, $0 \in I \subset \mathbf{R}^i$, such that $\Gamma_i(0) = \sigma$. □

In other words, Γ_i is a i–parameter smooth deformation of σ.

Let $\Xi_1, \dots, \Xi_i$ be vertical vector fields on Y and let $\Gamma_i(t_1, \dots, t_i)$ be an i-th variation of the section σ such that

$$\begin{aligned}
&\frac{\partial \Gamma_i}{\partial t_1}(t_1, 0, \dots, 0)\big|_{t_1=0} = \Xi_1 \circ \sigma\,, \\
&\frac{\partial \Gamma_i}{\partial t_2}(t_1, t_2, 0, \dots, 0)\big|_{t_2=0} = \Xi_2 \circ \Gamma_i(t_1, 0, \dots, 0)\,, \\
&\dots, \\
&\frac{\partial \Gamma_i}{\partial t_i}(t_1, t_2, \dots, t_{i-1}, t_i)\big|_{t_i=0} = \Xi_i \circ \Gamma_i(t_1, t_2, \dots, t_{i-1}, 0)\,.
\end{aligned}$$

In this case we say that Γ_i is generated by the i–tuple $(\Xi_1, \dots, \Xi_i)$.

We have thus the following characterization of Γ_i as the variation of σ by means of vertical flows.

Proposition 3.2. *Let $\psi^k_{t_k}$, with $1 \leq k \leq i$, be the flows generated by the vertical vector fields Ξ_k. Then for the Γ_i generated by $(\Xi_1, \dots, \Xi_i)$ we have:*

$$\Gamma_i(t_1, \dots, t_i) = \psi^i_{t_i} \circ \dots \circ \psi^1_{t_1} \circ \sigma\,. \quad □ \tag{6}$$

Definition 3.3. Vertical vector fields on any fiber bundle which 'deform' sections as above are called *variation vector fields.* □

3.1. Variations of morphisms and Lie derivative.

We define the i-th variation of a fibered morphism $\alpha : J_r Y \to \overset{k}{\wedge} T^* J_r Y$ along a section σ. We prove a Lemma which relates the i-th variation with the iterated Lie derivative (see e.g. [13]) of the morphism itself. Furthermore we provide a proof of the relation between the variation of the morphism and the vertical exterior differential (see e.g. [11]) which can be used to show the relation between the second order variation of a Lagrangian morphism and Lie derivative of the associated first order variation.

Definition 3.4. Let $\alpha : J_r Y \to \overset{k}{\wedge} T^* J_r Y$ and let Γ_i be an i-th variation of the section σ generated by an i-tuple $(\Xi_1, \dots, \Xi_i)$. We define the i-th variation of the morphism α to be

$$\delta^i \alpha := \frac{\partial^i}{\partial t_1 \dots \partial t_i}\big|_{t_1, \dots, t_i = 0}(\alpha \circ j_r \Gamma_i(t_1, \dots, t_i))\,. \quad □ \tag{7}$$

The following Lemma states the relation between the i-th variation of a morphism and its iterated Lie derivative (see also [9]).

Lemma 3.5. *Let $\alpha : J_r Y \to \overset{k}{\wedge} T^* J_r Y$ and $L_{j_r \Xi_k}$ be the Lie derivative operator with respect to $j_r \Xi_k$.*

Let Γ_i be the i-th variation of the section σ by means of the variation vector fields $\Xi_1, \dots, \Xi_i$ on Y. Then we have

$$\delta^i \alpha = (j_r \sigma)^* L_{j_r \Xi_1} \dots L_{j_r \Xi_i} \alpha\,. \tag{8}$$

Proof. It follows from the application of Proposition 3.2 to Definition 3.4. QED

In the following Lemma we show that the first order variation $\delta\lambda := \delta^1\lambda$ of λ is simply related to the vertical differential of λ (see also [10, 16]).

Lemma 3.6. *Let σ be a section of $\boldsymbol{Y}$, Ξ a variation vector field on $\boldsymbol{Y}$ and $\lambda \in \overset{n}{\Lambda}_r$. Then we have*

$$\delta\lambda = j_r\sigma^*(j_r\Xi \rfloor d_V\lambda)\,. \tag{9}$$

Proof. In fact we have

$$\delta\lambda = j_r\sigma^*\mathcal{L}_{j_r\Xi}\lambda = j_r\sigma^*(\Xi\rfloor\mathcal{E}(\lambda) + d_H(j_r\Xi\rfloor p_{d_V\lambda})) =$$
$$j_r\sigma^*(\Xi\rfloor\mathcal{E}(\lambda) + (j_r\Xi\rfloor d_H p_{d_V\lambda})) = j_r\sigma^*(j_r\Xi\rfloor d_V\lambda)\,,$$

since $\Xi\rfloor d_H = d_H\Xi\rfloor$, for any vertical vector field Ξ on $\boldsymbol{Y}$. QED

For notational convenience, in the sequel we shall denote with a superimposed bar all objects defined on the vertical prolongation $V\boldsymbol{Y}$ and with two bars those defined on the iterated vertical prolongation $V(V\boldsymbol{Y})$.

We can then characterize the second order variation of λ as follows.

Proposition 3.7. *Let $\lambda \in (\overset{n}{\Lambda}_r)_{\boldsymbol{Y}}$ and σ be a section of $\boldsymbol{Y} \to \boldsymbol{X}$. Let Ξ_1, Ξ_2 be two variation vector fields on $\boldsymbol{Y}$ and let $\bar{\Xi}_2$ be a variation vector field on $V\boldsymbol{Y}$, which projects down onto Ξ_2. Moreover, let $\bar{d}$, $\bar{\mathcal{E}}$ and $\bar{p}$ be the exterior differential, the Euler–Lagrange morphism and the momentum morphism on $V\boldsymbol{Y}$, respectively. Then we have, with a slight abuse of notation:*

$$\delta^2\lambda = L_{j_r\bar{\Xi}_2}\delta\lambda = \bar{\delta}L_{j_r\Xi_2}\lambda\,. \tag{10}$$

Proof. We apply the above Lemma, the decomposition formula (2) and the fact that $d_H\delta = \delta d_H$, which follows directly from the analogous naturality property of the Lie derivative operator. QED

3.2. **The vertical derivative.** Let $\boldsymbol{Y} \to \boldsymbol{X}$ be a fibered manifold. For reasons which shall be clear later, in the following we shall denote by $V\boldsymbol{Y}$ and $\delta\boldsymbol{Y}$ the vertical subbundles of $T\boldsymbol{Y}$, with respect to the projections over $\boldsymbol{Y}$ and $\boldsymbol{X}$, respectively.

Definition 3.8. Let $\alpha : J_r\boldsymbol{Y} \to \overset{p}{\wedge}T^*J_r\boldsymbol{Y}$ be a p-form on $J_r\boldsymbol{Y}$. The *vertical derivative* of α is the morphism

$$\delta\alpha : \delta J_r\boldsymbol{Y} \to \delta(\overset{p}{\wedge}T^*J_r\boldsymbol{Y})\,, \tag{11}$$

defined as the restriction of the tangent map $T\alpha : TJ_r\boldsymbol{Y} \to T(\overset{p}{\wedge}T^*J_r\boldsymbol{Y})$ to $\delta J_r\boldsymbol{Y}$, where $\delta J_r\boldsymbol{Y}$ is the vertical subbundle of $TJ_r\boldsymbol{Y}$ with respect to the projection over $\boldsymbol{X}$. □

Proposition 3.7 admits the following equivalent formulation, which will be used in the sequel.

Proposition 3.9. *Let $\lambda \in (\overset{n}{\Lambda}_r)_Y$ and σ be a section of $Y \to X$. Let Ξ_1, Ξ_2, $\bar{\Xi}_2$, $\bar{d}$, $\bar{\mathcal{E}}$ and $\bar{p}$ be as defined in Proposition 3.7. Then we have*

(12) $\delta^2\lambda = j_r\sigma^*(\bar{\Xi}_2 \rfloor \mathcal{E}(\Xi_1 \rfloor \mathcal{E}(\lambda)) + d_H(j_{r+1}\bar{\Xi}_2 \rfloor \bar{p}_{d_V\Xi_2} \rfloor_{\mathcal{E}(\lambda)} + \delta(j_r\Xi_2 \rfloor p_{d_V\lambda}))\,.$

Remark 3.10. Let VJ_rY stand, as usual, for the vertical subbundle of the bundle $TJ_rY \to Y$. We have the following splitting.

$$\delta(V^*J_rY) \simeq V^*J_rY \oplus (V^*J_rY \underset{X}{\times} V^*J_rY)\,. \tag{13}$$

In fact, $\delta(V^*J_rY)$ is a vector bundle over V^*J_rY. The zero section of $\delta(V^*J_rY) \to V^*J_rY$ induces a splitting $V^*J_rY \to \delta(V^*J_rY)$ of the following exact sequence of vector bundles

$$0 \to V(V^*J_rY) \to \delta(V^*J_rY) \to V^*J_rY \to 0\,.$$

Furthermore, the isomorphism $V(V^*J_rY) \simeq (V^*J_rY \underset{X}{\times} V^*J_rY)$ holds true (see [9].) □

In the following $(x^\lambda, y^j_{\boldsymbol\lambda}, \dot{x}^\lambda, \dot{y}^j_{\boldsymbol\lambda}, \dot{\bar{y}}^j_{\boldsymbol\lambda}, \ddot{y}^j_{\boldsymbol\lambda})$ and $(\partial^\lambda, \partial^{\boldsymbol\lambda}_j, \dot{\partial}^{\boldsymbol\lambda}_j)$, with $0 \le |\boldsymbol\lambda| \le r$, are local coordinates on $T(VJ_rY)$ and a local basis of tangent vector fields on VJ_rY, respectively.

Example 3.11. If the coordinate expression of α is given by

$$\alpha = \alpha^{\beta_1 \ldots \beta_h}_{i_1 \ldots i_h \mu_{h+1} \ldots \mu_p} d^{i_1}_{\beta_1} \wedge \ldots \wedge d^{i_h}_{\beta_h} \wedge d^{\mu_{h+1}} \wedge \ldots \wedge d^{\mu_p}\,,$$

then we have

$$\delta\alpha = [\partial^{\lambda}_j(\alpha^{\beta_1 \ldots \beta_h}_{i_1 \ldots i_h \mu_{h+1} \ldots \mu_p})\dot{y}^j_{\lambda} + \dot{\partial}^{\lambda}_j(\alpha^{\beta_1 \ldots \beta_h}_{i_1 \ldots i_h \mu_{h+1} \ldots \mu_p})\ddot{y}^j_{\lambda}] d^{i_1}_{\beta_1} \wedge \ldots \wedge d^{i_h}_{\beta_h} \wedge d^{\mu_{h+1}} \wedge \ldots \wedge d^{\mu_p}\,.$$

□

Remark 3.12. The sequence $VJ_rY \to J_rY \to X$ induces a sequence

$$T^*X \to T^*(J_rY) \to T^*(VJ_rY)\,,$$

and then we have the inclusions $\overset{p}{\mathcal{H}}_r \hookrightarrow \overset{p}{\Lambda}(T^*J_rY) \hookrightarrow \overset{p}{\Lambda}(T^*VJ_rY)$. □

We can then state the following Lemma.

Lemma 3.13. *The vertical derivative δ preserves the contact structure, i.e. if $\beta \in (\overset{p}{\mathcal{C}}_r)_Y$ then $\delta\beta \in (\overset{p}{\mathcal{C}}_r)_{VY}$.*

By a straightforward calculation it is easy now to prove the following Theorem.

Theorem 3.14. *Let $\alpha : J_rY \to \overset{p}{\Lambda}T^*J_rY$. Let d, $\bar{d}$ be the exterior differentials and δ, $\bar{\delta}$ the vertical derivatives on J_rY and δJ_rY, respectively. We have:*

$$\bar{d}\delta\alpha = \bar{\delta}d\alpha\,. \tag{14}$$

Proof. We have

$$\bar{\delta}d\alpha = \partial^{\beta}_k(\partial^{\lambda}_i(\alpha))\dot{y}^k_{\beta}\dot{y}^i_{\lambda} + \partial^{\lambda}_i(\alpha)\ddot{y}^i_{\lambda} + \partial_\mu(\partial^{\lambda}_i(\alpha))\dot{y}^i_{\lambda}\dot{x}^\mu = \bar{d}\delta\alpha\,.$$ QED

Remark 3.15. Owing to the linearity properties of $d_V\lambda$, we can think of the operator δ as a *linear* morphism with respect to the vector bundle structure $J_rVY \to Y$ of the kind

$$\delta : J_rY \underset{Y}{\times} J_rVY \to \overset{n}{\wedge}T^*X\,. \tag{15}$$

This property can be obviously iterated for each integer i, so that one can analogously define an i–linear morphism δ^i. □

4. Variational vertical derivatives

In this Section we restrict our attention to morphisms which are sections of sheaves in the variational sequence. We shall recall some results of ours [6] by defining the i-th order *variational vertical derivative* of morphisms, since the i-th variation operator passes to the quotient in the variational sequence, due to the fact that δ^i, as well as L_{Ξ_i} (see [7]), preserve the contact structure.

Proposition 4.1. *Let* $\alpha \in (\overset{n}{\mathcal{V}}_r)_Y$. *We have*

$$[\delta^i\alpha] = \hat{\delta}^i[\alpha]\,,$$

where we set $\hat{\delta}^i := \mathcal{L}_{\Xi_i} \ldots \mathcal{L}_{\Xi_1}$.

Proof. In fact, from the above Lemma, we have

$$[\delta^i\alpha] = j_r\sigma^*[L_{\Xi_i} \ldots L_{\Xi_1}\alpha] = j_r\sigma^*\mathcal{L}_{\Xi_i} \ldots \mathcal{L}_{\Xi_1}[\alpha] := \hat{\delta}^i[\alpha]\,,$$

QED

Definition 4.2. We call the operator $\hat{\delta}^i$ the *i-th variational vertical derivative operator.*

□

This enables us to represent variations of morphisms in the variational sequence.

Proposition 4.3. *Let* $\alpha \in (\overset{n}{\mathcal{V}}_r)_Y$. *Let* d, $\bar{d}$ *be the exterior differentials and* $\hat{\delta}$, $\bar{\hat{\delta}}$ *the variational vertical derivative on* J_rY *and* $\hat{\delta}J_rY$, *respectively. We have:*

$$\bar{d}\hat{\delta}\alpha = \bar{\hat{\delta}}d\alpha\,. \tag{16}$$

We can thus state the following important result (see also [6]).

Theorem 4.4. *The operator* $\hat{\delta}$ *is a functor defined on the category of variational sequences.*

Proof. It follows from Proposition 3.7, Equation (10), Remark 3.15 and Lemma 4.1.

QED

Then the functor $\hat{\delta}$ associates the sequence associated with the fibration $Y \to X$ to the sequence associated with the fibration $VY \to X$ and this can be summarized in the following diagram.

$$
\begin{array}{ccccccccc}
0 \longrightarrow \dots\dots & \xrightarrow{d_H} & (\overset{n}{\mathcal{V}}_r)_Y & \xrightarrow{\mathcal{E}_n} & (\overset{n+1}{\mathcal{V}}_r)_Y & \xrightarrow{\mathcal{E}_{n+1}} & \mathcal{E}_{n+1}(\overset{n+1}{\mathcal{V}}_r)_Y & \xrightarrow{\mathcal{E}_{n+2}} & 0 \\
 & & \Big\downarrow{\scriptstyle \hat{\delta}} & & \Big\downarrow{\scriptstyle \hat{\delta}} & & \Big\downarrow{\scriptstyle \hat{\delta}} & & \\
0 \longrightarrow \dots\dots & \xrightarrow{\bar{d}_H} & (\overset{n}{\mathcal{V}}_r)_{VY} & \xrightarrow{\bar{\mathcal{E}}_n} & (\overset{n+1}{\mathcal{V}}_r)_{VY} & \xrightarrow{\bar{\mathcal{E}}_{n+1}} & \bar{\mathcal{E}}_{n+1}(\overset{n+1}{\mathcal{V}}_r)_{VY} & \xrightarrow{\bar{\mathcal{E}}_{n+2}} & 0 \\
 & & \Big\downarrow{\scriptstyle \hat{\bar{\delta}}} & & \Big\downarrow{\scriptstyle \hat{\bar{\delta}}} & & \Big\downarrow{\scriptstyle \hat{\bar{\delta}}} & & \\
0 \longrightarrow \dots\dots & \xrightarrow{\bar{\bar{d}}_H} & (\overset{n}{\mathcal{V}}_r)_{V(VY)} & \xrightarrow{\bar{\bar{\mathcal{E}}}_n} & (\overset{n+1}{\mathcal{V}}_r)_{V(VY)} & \xrightarrow{\bar{\bar{\mathcal{E}}}_{n+1}} & \bar{\bar{\mathcal{E}}}_{n+1}(\overset{n+1}{\mathcal{V}}_r)_{V(VY)} & \xrightarrow{\bar{\bar{\mathcal{E}}}_{n+2}} & 0
\end{array}
$$

Proposition 3.7, Remark 3.15 and Equation (10) give us the following characterization of the second order variation of a generalized Lagrangian in the variational sequence in terms of generalized symmetries of the Euler–Lagrange operator associated with the first order variation of the Lagrangian itself.

Proposition 4.5. *Let* $\lambda \in (\overset{n}{\mathcal{V}}_r)_Y$, $\hat{\delta}\lambda \in (\overset{n}{\mathcal{V}}_r)_{VY}$. *We have*

$$
\hat{\delta}^2\lambda = \bar{\mathcal{E}}(\bar{\Xi}_2 \rfloor \hat{\delta}\lambda) + \tilde{H}_{d\hat{\delta}\lambda}(\bar{\Xi}_2)\,, \tag{17}
$$

where $\tilde{H}_{d\hat{\delta}\lambda}$ *is the unique morphism belonging to* $\overset{1}{\mathcal{C}}_{(2r,r)} \otimes \overset{1}{\mathcal{C}}_{(2r,0)} \wedge \overset{n}{\mathcal{H}}_{2r}$ *such that, for all* $\Xi_1 : Y \to VY$, $E_{j_r\Xi \rfloor d\hat{\delta}\lambda} = C^1_1(j_{2r}\Xi_1 \otimes \tilde{H}_{d\hat{\delta}\lambda})$, *and* C^1_1 *stands for tensor contraction.*

Proof. In fact we have

$$
\hat{\delta}^2\lambda = \hat{\bar{\delta}}\mathcal{L}_{j_r\Xi_2}\lambda = \mathcal{L}_{j_r\Xi_2}\hat{\delta}\lambda\,, \tag{18}
$$

so that the assertion follows by a straightforward application of the representation provided by Equation (5) and by linearity properties stressed in Remark 3.15. QED

Remark 4.6. Notice that the same invariant decomposition can be obtained by applying Remark 3.10 to Proposition 3.9.

□

4.1. **Generalized Jacobi morphisms.** The following is an application of an abstract result due to Kolář [12], concerning a global decomposition formula for vertical morphisms. Hereafter the canonical isomorphism $J_rVY \simeq VJ_rY$ is obviously understood.

Lemma 4.7. *Let* $\hat{\mu} : J_sY \to \overset{*}{\mathcal{C}}_k[VY] \wedge \overset{p}{\wedge} T^*X$, *with* $0 \leq p \leq n$ *and let* $\bar{d}_H\hat{\mu} = 0$. *Then we have* $\hat{\mu} = E_{\hat{\mu}} + F_{\hat{\mu}}$, *where*

$$
E_{\hat{\mu}} : J_{2s+k}VY \to \overset{*}{\mathcal{C}}_0[VY] \wedge \overset{p}{\wedge} T^*X\,, \tag{19}
$$

and locally, $F_{\hat{\mu}} = \bar{d}_H M_{\hat{\mu}}$, *with* $M_{\hat{\mu}} : J_{2s+k-1}VY \to \overset{*}{\mathcal{C}}_{k-1}[VY] \wedge \overset{p-1}{\wedge} T^*X$.

Proof. Following *e.g.* [11, 12, 25], the global morphisms $E_{\hat{\mu}}$ and $\bar{d}_H M_{\hat{\mu}}$ can be evaluated by means of a backwards procedure.

QED

Now, it is very easy to see from Remark 3.15 and by means of a simple calculation that the following holds true.

Lemma 4.8. *Let* $\chi(\lambda) := \tilde{H}_{d\hat{\delta}\lambda}$. *We have* $\chi(\lambda) : J_{2r}\boldsymbol{Y} \to \overset{*}{\mathcal{C}}_r[V\boldsymbol{Y}] \wedge (\overset{n}{\wedge} T^*\boldsymbol{X})$ *and* $\bar{d}_H\chi(\lambda) = 0$.

Thus, as a straightforward application of Lemma 4.7, we obtain our main result which consists in a suitable geometrical interpretation of the second variation of a generalized Lagrangian and provides a new characterization of the Jacobi morphism in the framework of variational sequences.

Theorem 4.9. *Let* $\chi(\lambda)$ *be as in the above Lemma. Then we have*

$$\chi(\lambda) = E_{\chi(\lambda)} + F_{\chi(\lambda)},$$

where

$$E_{\chi(\lambda)} : J_{4r}\boldsymbol{Y} \to \overset{*}{\mathcal{C}}_0[V\boldsymbol{Y}] \wedge (\overset{n}{\wedge} T^*\boldsymbol{X}), \tag{20}$$

and locally, $F_{\chi(\lambda)} = \bar{d}_H M_{\chi(\lambda)}$, *with* $M_{\chi(\lambda)} : J_{4r-1}\boldsymbol{Y} \to \overset{*}{\mathcal{C}}_{r-1}[V\boldsymbol{Y}] \wedge \overset{n-1}{\wedge} T^*\boldsymbol{X}$.

Remark 4.10. If the coordinate expression of $\chi(\lambda)$ is given by

$$\chi(\lambda) = \chi_i^\alpha \vartheta_\alpha^i \wedge \omega,$$

where ϑ_α^i are contact forms on $J_k V\boldsymbol{Y}$, the corresponding coordinate expressions of $E_{\chi(\lambda)}$ and $M_{\chi(\lambda)}$ are respectively given by

$$E_{\chi(\lambda)} = E_i \vartheta^i \wedge \omega,$$
$$M_{\chi(\lambda)} = M_i^{\alpha+\lambda} \vartheta_\alpha^i \wedge \omega_\lambda .$$

We have, in particular

$$E_{\chi(\lambda)} = (-1)^{|\beta|} D_\beta \chi_i^\beta \vartheta^i \wedge \omega,$$

with $0 \leq |\beta| \leq k$. □

Definition 4.11. We call the morphism $\mathcal{J}(\lambda) := E_{\chi(\lambda)}$ the *generalized Jacobi morphism* associated with the Lagrangian λ.

□

Remark 4.12. The Euler–Lagrange equations together with the Jacobi equations for a given Lagrangian λ can be then obtained out of a unique variational problem for the Lagrangian $\hat{\delta}\lambda$, in terms of its generalized symmetries, which are solutions of the classical Jacobi equations along critical sections. In this way a classical result [2, 3, 4, 9] is recovered in a very general setting and represented in terms of sections of sheaves in the variational sequence. □

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