Petr Somberg
Crystallizing patterns of BGG sequences

In: Jan Slovák and Martin Čadek (eds.): Proceedings of the 21st Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 2002. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 69. pp. [233]–243.

Persistent URL: http://dml.cz/dmlcz/701700

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 69 (2002), pp. 233-243

CRYSTALLIZING PATTERNS OF BGG SEQUENCES

PETR SOMBERG

ABSTRACT. The aim of this article is to show the, as we shall argue not accidental, coincidence of two structures. The first one is the underlying combinatorial structure of Bernstein-Gelfand-Gelfand resolution carried by Hasse graph of weight graph, which is given by purely representational-theoretical data. The second structure is the crystal graph associated to the couple - quantum universal enveloping algebra of (classical series) Lie algebra and its integrable highest weight (finite dimensional) module. This approach should in turn pinch out the natural candidates on quantum BGG sequences, or more precisely on the category of geometrical objects whose underlying representational patterns we are going to unfold. The hope is that this category is equivalent to the category of (graded) quantum groups.

1. THE STRUCTURE OF "CLASSICAL" BERNSTEIN-GELFAND-GELFAND RESOLUTIONS

In this section we recall the structure of Bernstein-Gelfand-Gelfand resolution (*BGG* for short) associated to parabolic invariant theory. Then we shall be interested in $|1|$-graded Lie algebras of special type (all weight spaces of $\mathfrak{g}_0$-module $\mathfrak{g}_1$ are extremal), for which we recall the bijection between BGG resolutions for ($|1|-graded$) $\mathfrak{g}$ and Hasse graphs of $\mathfrak{g}_0$-module $\mathfrak{g}_1$. This section is only slight modification of [1],[6], but it is summarized in compact form useful for further generalizations.

Definition 1.1. Let $\mathcal{G}$ be a principal P-bundle on a manifold M, and let for $X \in \mathfrak{g}$ be $X^\sharp$ the corresponding fundamental vector field. The left (right) action of $p \in P$ on $\mathcal{G}$ will be denoted $L_p : P \times \mathcal{G} \to \mathcal{G}$ ($R_p : \mathcal{G} \times P \to \mathcal{G}$).

A Cartan connection on $\mathcal{G}$ is a 1-form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ satisfying:

- $\omega(X^\sharp) = X$ for all $X \in \mathfrak{g}$,
- $(R_p)^\star\omega = Ad_p^{-1}\omega$ for all $p \in P$,
- $\omega|_{T_u\mathcal{G}} : T_u\mathcal{G} \xrightarrow{\sim} \mathfrak{g}$ is isomorphism for all $u \in \mathcal{G}$.

A parabolic structure on M is given by data $(\mathcal{G}, P, M, \omega)$ (ω is assumed to be normal connection).

2000 *Mathematics Subject Classification*: 17B37, 17B10, 17B05.

Key words and phrases: deformed universal enveloping algebra, crystal graph, Hasse graph, Bernstein-Gelfand-Gelfand sequence.

This work was supported by grant GAČR No. 201/00/P070.

The paper is in final form and no version of it will be submitted elsewhere.

We shall be interested in the case of $|1|$-graded simple Lie algebras $\mathfrak{g}$ (an Almost Hermitean Symmetric structure on M), i.e. $\mathfrak{g} \simeq \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$$

with parabolic subalgebra $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_1 \subset \mathfrak{g}, \mathfrak{p}_+ := \mathfrak{g}_1$. The Killing-Cartan form on $\mathfrak{g}$ induces isomorphism $\mathfrak{g}_i^\star \xrightarrow{\sim} \mathfrak{g}_{-i}$ for all i.

The subclass of invariant differential operators, so called standard invariant operators, can be described explicitly as follows. The maximal ideal $\mathfrak{p}_+$ of $\mathfrak{p}$, generated by Lie subalgebra with elements of (strictly) positive grading, acts on any irreducible P-module $\mathbb{V}$ by nilpotent endomorphisms. As a (finite dimensional) module of reductive Lie algebra $\mathfrak{g}_0$, $\mathbb{V}$ is determined by dominant weight λ of semi-simple subalgebra $\mathfrak{g}_0^s = [\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0$, and as $Z(\mathfrak{g}_0) = Ker\{\mathfrak{g}_0 \to [\mathfrak{g}_0, \mathfrak{g}_0]\} \subset \mathfrak{g}_0$-module by complex number w (called generalized conformal weight). Any such P-module $\mathbb{V}$ will be shortly written as $\mathbb{V}(\lambda, w)$. Let β be a weight of $\mathfrak{g}_0^s$-module $\mathfrak{g}_1$, for which λ and $\lambda + k\beta$ are dominant weights of $\mathfrak{g}_0^s$ ($k \in \mathbb{N}$). Then there exists a unique value of generalized conformal weight w, such that there exists a (unique) standard invariant differential operator D,

$$D : C^\infty(M, \mathbb{V}(\lambda, w))^P \longrightarrow C^\infty(M, \mathbb{V}(\lambda + k\beta, w + k))^P ,$$

for which k is the order of D. The operator $D := \pi_k \circ (\nabla^\omega)^{\otimes k}$ acts between associated bundles induced from homomorphisms of P-modules

$$\mathbb{V}(\lambda) \longrightarrow \otimes^k \mathfrak{g}_1 \otimes \mathbb{V}(\lambda) \xrightarrow{\pi_k} \mathbb{V}(\lambda + k\beta) .$$

Because the construction of standard operators is independent of the manifold M (e.g. curvature) and what only matters are the source and target representation spaces, we write simply $D : \mathbb{V}(\lambda, w) \longrightarrow \mathbb{V}(\lambda + k\beta, w + k)$.

Let us consider, for particular ($|1|$-graded) parabolic invariant theory $(\mathfrak{g}, P)$, the set of all standard invariant differential operators:

$$\left\{ \begin{array}{c} D(\lambda, \beta, k) : \mathbb{V}(\lambda, w) \longrightarrow \mathbb{V}(\lambda + k\beta, w + k) , \\ \beta \text{ is a weight of } \mathfrak{g}_1;\ k \in \mathbb{N};\ \lambda, \lambda + k\beta \text{ are dominant weights for } \mathfrak{g}_0^s . \end{array} \right\}$$

Let us consider a graph with the set of vertices given by representation spaces $\mathbb{V}(\lambda, w)$ and arrows given by $D(\lambda, \beta, k)$. Note, that we explicitly suppressed Cartan connection ω (and corresponding covariant derivative ∇^ω).

Definition 1.2. Let $(\mathfrak{g}, P)$ be parabolic invariant theory. The BGG resolution is defined to be connected component of the graph of all standard invariant differential operators. The underlying graph of BGG sequence, given by forgetting orders of operators and representation content of all places, i.e. keeping just the W-labeled graph structure of 1-dimensional CW complex, will be called B.

The definition of W-labeled graphs is given in the following paragraph.

Let S be a set. Then the graph with S-labeled arrows (S-graph) is a finite (as CW-complex) oriented graph $G \equiv (V, E)$; V is 0-dimensional subcomplex of G, and $E \simeq (G \setminus V)$ is 1-dimensional subcomplex of G (the set of oriented arrows). The set of arrows $E \subset V \times V$ defines the structure of partially ordered set by

$$u \geq v \Longleftrightarrow u \longrightarrow v , \qquad (1)$$

and there is canonical map $\varphi : E \subset V \times V \longrightarrow S$, such that if $u, v \in V, e \equiv (u, v) \in E$ and $\varphi(e) = s \in S$, we write $u \xrightarrow{s} v$.

Definition 1.3. Let R be an irreducible representation of simple Lie algebra $\mathfrak{a}$ with highest weight $\beta_{\max}$. The weight graph associated with couple $(\mathfrak{a}, R)$ is the graph labeled by the (subset of the) set of simple roots of $\mathfrak{a}$ (i.e. S is the subset of the set of simple roots of $\mathfrak{a}$):

1. the set of vertices is the set of all weights of R,
2. there is an arrow $\beta_1 \xrightarrow{\alpha} \beta_2$ labeled by α iff there exists a simple root α of $\mathfrak{a}$ such that $\beta_2 = \beta_1 - \alpha$.

The poset structure on weight graph is induced from the standard one of weight lattice (e.g. the highest weight $\beta_{\max}$ is the greatest element).

The subgraph W_{sub} of the weight graph W of representation R is called acceptable iff W_{sub} contains with every vertex $v \in W$ (corresponding to a weight space of R) all vertices u with the property that there exists arrow $e \in W$, $u \xrightarrow{e} v$, contained in W_{sub}. This means, that acceptable subgraph of W contains, as a partially ordered set, with every element $v \in W_{sub}$ all elements $u \in W$ such that $u > v$. The set of all acceptable subgraphs of weight graph forms another graph.

Definition 1.4. Let W be a weight graph of a representation R of a Lie algebra $\mathfrak{a}$. Then the Hasse graph (diagram) for R is graph labeled by vertices of weight diagram (e.g. S is now the set of weights of R) such that:

1. vertices are acceptable subgraphs of W,
2. if $V_1 \neq V_2$ are acceptable subgraphs of W and $\beta \in V_2$ such $V_1 \cup \beta = V_2$, then there is an arrow $V_1 \xrightarrow{\beta} V_2$

Also Hasse diagram has the structure of poset (of acceptable subsets of weight graph W).

The relation of equivalence on S-labeled graphs is the standard one.

Definition 1.5. Two S-labeled graphs W_1, W_2 are **isomorphic** iff there is bijection $\psi : W_1 \equiv (V_1, E_1) \longrightarrow W_2 \equiv (V_2, E_2)$, such that for all arrows ($s \in S$) holds

$$u \xrightarrow{s} v \implies \psi(u) \xrightarrow{s} \psi(v) .$$

The previous two structures - Hasse graphs of $\mathfrak{a} = \mathfrak{g}_0^s$-module $R = \mathfrak{g}_1$ and BGG sequence of invariant parabolic theory on $(\mathfrak{g}, P)$ - are isomorphic.

Theorem 1.6. ([6],Theorem 4.12, p.25) *Let $\mathfrak{g}$ be a $|1|$-graded Lie algebra, such that all weights of $\mathfrak{g}_0$-module $\mathfrak{g}_1$ are extremal (=are of the same length); the highest weight will be denoted $\beta_{\max}$. Introduce the number $r = \frac{1+|\beta|^2}{2}$, independent on weight β. Let λ^0 be a dominant weight for $\mathfrak{g}_0^s$, $k \in \mathbb{N}$, and let $w^0 = -(\lambda^0, \beta_{\max}) + (1 - k^0)r$. Denote by B the BGG sequence containing the subgraph*

$$D(\beta_{\max}, k^0) : (\lambda^0, w^0) \longrightarrow (\lambda^0 + k^0\beta_{\max}, w^0 + k^0) .$$

Denote by W the weight graph of $\mathfrak{g}_1$, and by H the Hasse diagram of W. Then there is isomorphism φ of W-labeled graphs:

$$\varphi : H \xrightarrow{\sim} B \ , \ \varphi(0) = (\lambda^0, w^0) .$$

The moral is, that on the conditions given by assumptions of this Theorem, one can construct BGG sequences using weight graphs of $\mathfrak{g}_0$-module $\mathfrak{g}_1$. For example, this bijection doesn't hold true in the |1|-graded case of B_n series of Lie algebra with first crossed node.

Having this equivalence, we abstract from geometry. By this we mean the search for the structure, which is uniform in q and which corresponds, for $q \to 1$, to the couple given by simple Lie algebra and the weight graph of its finite dimensional irreducible module.

2. Crystal bases and crystal graphs of quantized universal enveloping algebras of simple Lie algebras

We have left the last section with the description of BGG sequence encoded in purely representational-theoretical data carried by (all acceptable subgraphs of) weight graph of (semi)simple Lie algebra $\mathfrak{g}_0^s$. Thus the starting point will be the series of (finite dimensional) integrable highest weight representations of $\mathcal{U}_q(\mathfrak{g})$; let us describe it briefly.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra with Cartan subalgebra $\mathfrak{t}$. For finite set I, let us denote the set of simple roots by $\{\alpha_i \in \mathfrak{t}^\star\}_{i\in I}$ and the set of simple coroots by $\{\mathfrak{h}_i \in \mathfrak{t}\}_{i\in I}$. The standard inner product $(-,-)$ on $\mathfrak{t}^\star$ has properties

$$(\alpha_i, \alpha_j) \in \mathbb{Z}_> \,,$$
$$< h_i, \lambda >= \tfrac{2}{(\alpha_i,\alpha_i)}(\alpha_i, \lambda)\,,\ \lambda \in \mathfrak{t}^\star\,. \tag{2}$$

Let $\{\lambda_i\}_{i\in I}$ be the dual base of $\{h_i\}_{i\in I}$, and define two $\mathbb{Z}$-modules $P = \sum_i \mathbb{Z}\lambda_i$ resp. $P^\star = \sum_i \mathbb{Z}h_i$. Then the q-analogue $\mathcal{U}_q(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ is the algebra over $\mathbb{Q}(q)$ generated by $\{e_i, f_i, q^h\}$ $(h \in P^\star)$ satisfying

$$q^{h_1}q^{h_2} = q^{h_1+h_2}\,,\ q^h = 1 \text{ if } h = 0\,,$$
$$q^h e_j q^{-h} = q^{<h,\alpha_j>} e_j\,,$$
$$q^h f_j q^{-h} = q^{-<h,\alpha_j>} f_j\,,$$
$$[e_i, f_j] = \delta_{i,j}\tfrac{q^{(\alpha_i,\alpha_j)h_i} - q^{-(\alpha_i,\alpha_i)h_i}}{q^{(\alpha_i,\alpha_i)} - q^{-(\alpha_i,\alpha_i)}}\,. \tag{3}$$

We also have for $i \neq j$

$$\sum_{j=0}^{1-<h_i,\alpha_j>} \frac{1}{[j]_i![j-1+<h_i,\alpha_j>]_i!} e_i^j e_j e_i^{j-1+<h_i,\alpha_j>}$$
$$= \sum_{j=0}^{1-<h_i,\alpha_j>} \frac{1}{[j]_i![j-1+<h_i,\alpha_j>]_i!} e_i^j e_j e_i^{j-1+<h_i,\alpha_j>} = 0\,, \tag{4}$$

where $[j]_i! := \Pi_{m=1}^{j} \frac{q^{m(\alpha_i,\alpha_i)} - q^{-m(\alpha_i,\alpha_i)}}{q^{(\alpha_i,\alpha_i)} - q^{-(\alpha_i,\alpha_i)}}$.

Let M be a finite dimensional integrable $\mathcal{U}_q(\mathfrak{g})$-module, $\lambda \in P$, and set $M_\lambda := \{v \in M\,,\ q^h v = q^{<h,\lambda>}v\}$ $(\forall h \in \mathfrak{t})$, e.g. $M = \oplus_\lambda M_\lambda$. Then

$$M_\lambda = \oplus_{k \geq max(k,-<h_i,\lambda>)} \frac{1}{[k]_i!} f_i^k (M_{\lambda+k\alpha_i} \cap Ker\, e_i)\,. \tag{5}$$

We introduce operators $\widetilde{e}_i, \widetilde{f}_i$, acting on M by

$$\widetilde{e}_i \frac{1}{[k]_i!} f_i^k v = \frac{1}{[k-1]_i!} f_i^{k-1} v ,$$
$$\widetilde{f}_i \frac{1}{[k]_i!} f_i^k v = \frac{1}{[k+1]_i!} f_i^{k+1} v ,$$

where $v \in (M_\lambda \cap Ker\, e_i)$, and the couple (λ, k) is as in the previous summation.

Let q be transcendental over $\mathbb{Q}$, and let $R = \mathbb{Q}[q]$ be a ring with quotient field $k = \mathbb{Q}(q)$. Then $A \equiv \{\frac{f}{g}, f,g \in \mathbb{Q}[q], , g(0) \neq 0\}$ is a discrete valuation ring (i.e. local ring, which is principal ideal domain). The (unique) maximal ideal $I \subset A$ is $I = qA$, and $A/I = A/qA \simeq \mathbb{Q}$. For a simple Lie algebra $\mathfrak{g}$, let M be a finite dimensional $\mathcal{U}_q(\mathfrak{g})$-module.

We have two important definitions. The first one introduces the notion of admissible lattice, the second one the notion of crystal base.

Definition 2.1. An *admissible lattice* in M is an A-submodule $\mathcal{M} \subset M$ with properties:

- $\mathcal{M}$ is finitely generated free A-module, such that $\mathcal{M} \otimes_A k \simeq M$ (as k-modules);
- $\mathcal{M} = \oplus_{\lambda \in \Lambda} \mathcal{M}_\lambda$, where $\mathcal{M}_\lambda = \mathcal{M} \cap M_\lambda$;
- $\widetilde{f}_\alpha \mathcal{M} \subset \mathcal{M}\, , \widetilde{e}_\alpha \mathcal{M} \subset \mathcal{M}\, , \forall\, \alpha \in \Delta^+$.

Definition 2.2. A crystal base of M is a pair $(\mathcal{M}, \mathcal{B})$ where $\mathcal{M}$ is an admissible lattice in M and $\mathcal{B}$ is a basis of the vector space $\mathcal{M}/q\mathcal{M}$ over $\mathbb{Q}$ such that

- $\mathcal{B} = \coprod_\lambda \mathcal{B}_\lambda$, where $\mathcal{B}_\lambda = \mathcal{B} \cap (\mathcal{M}_\lambda / q\mathcal{M}_\lambda)$;
- $\widetilde{f}_\alpha \mathcal{B} \subset \mathcal{B} \cup \{0\}\, , \widetilde{e}_\alpha \mathcal{B} \subset \mathcal{B} \cup \{0\}\, , \forall\, \alpha \in \Delta^+$;
- for any $u_1, u_2 \in \mathcal{B}$ and $\alpha \in \Delta^+$,

$$u_1 = \widetilde{e}_\alpha u_2 \Longleftrightarrow u_2 = \widetilde{f}_\alpha u_1 \, . \tag{6}$$

Let us consider classical series of Lie algebras, i.e. $\mathfrak{g} = A_n, B_n, C_n, D_n$. The following results are proved in [4]. Let $\lambda \in P_+$ and $M(\lambda)$ be an integrable highest weight $\mathcal{U}_q(\mathfrak{g})$-module generated by cyclic vector u_λ. Let $\mathcal{M}(\lambda)$ be A-submodule of $M(\lambda)$ generated by vectors of the form $\widetilde{f}_{\alpha_{i_1}} \dots \widetilde{f}_{\alpha_{i_k}} u_\lambda$, and let $\mathcal{B}(\lambda)$ be the base of $\widetilde{f}_{\alpha_{i_1}} \dots \widetilde{f}_{\alpha_{i_k}} u_\lambda \; mod \; q\mathcal{M}(\lambda)$.

Theorem 2.3. ([4]) *The couple* $(\mathcal{M}(\lambda), \mathcal{B}(\lambda))$ *is crystal base of* $\mathcal{M}(\lambda)$.

The last definition introduces the notion of crystal graph.

Definition 2.4. The crystal graph of crystal base $(\mathcal{M}(\lambda), \mathcal{B}(\lambda))$ of integrable irreducible (finite dim.) $\mathcal{U}_q(\mathfrak{g})$-module $M(\lambda)$ is the graph labeled by simple roots, with the arrows

$$u_1 \xrightarrow{\alpha} u_2 \text{ iff } u_2 = \widetilde{f}_\alpha u_1 \, . \tag{7}$$

The aim of following subsections is to analyze the q-analog of the correspondence presented in the first section and give a basic evidence for it. In our vocabulary, the associated crystal graph (i.e. the structure uniform in q) of lattice $\mathcal{M}$ in M plays the rôle of quantized analog of the weight graph in the classical case ($q \to 1$, see the first section). In other words, we shall analyse the quantum analog of maximal parabolic subalgebras of Lie algebras, i.e. consider q-analogs of weight graphs of quantized universal envelopping algebras $\mathcal{U}_q(\mathfrak{g}_0^s)$. The natural candidates on these q-analogs are crystal graphs of suitable $\mathcal{U}_q(\mathfrak{g}_0^s)$-modules.

2.1. **Crystal graph of vector representation of $U_q(A_n)$.** We start by definition of $\mathcal{U}_q(A_n)$. Let $\mathfrak{t} \equiv \oplus_{i=1}^n \mathbb{Q}h_i$, $\forall i = 1, ..., n$, $h_i \in \mathfrak{h}$, and let $\{\lambda_i \in \mathfrak{h}^\star, i = 1, ..., n\}$ be the dual base of fundamental weights. We set $e_i = \lambda_i - \lambda_{i-1}$, $\forall i = 1, ..., n$, $e_1 = \lambda_1$, such that $e_{i+1} = -\sum_{i=1}^n e_i$. The simple roots are $\alpha_i = e_i - e_{i+1}$, $i = 1, ..., n$, and the inner product $(-, -)$ on $\mathfrak{h}^\star$ is such that

$$\begin{aligned}
&(\alpha_i, \alpha_i) = 1, \forall i = 1, ..., n \\
&(\alpha_i, \alpha_j) = -\frac{1}{2}, \forall i, j = 1, ..., n, |i - j| = 1 \\
&(\alpha_i, \alpha_j) = 0, \forall i, j = 1, ..., n, |i - j| > 1 \qquad (8)
\end{aligned}$$

Let us construct crystal graph of the vector representation $\mathbb{V}_{\mathcal{U}_q(A_n)}$. We consider $\mathbb{V}_{\mathcal{U}_q(A_n)}$ to be $(n + 1)$-dimensional $\mathbb{Q}(q)$-vector space with basis $\{\boxed{\text{i}}, i = 1, ..., n + 1\}$. The structure of $\mathcal{U}_q(A_n)$-module is given by

$$\begin{aligned}
&q^h\boxed{\text{i}} = q^{e_i(h)}\boxed{\text{i}}, i = 1, ..., n + 1 \\
&e_j\boxed{\text{i}} = \delta_{j,i-1}\boxed{\text{i-1}}, i = 1, ..., n + 1, j = 1, ..., n \\
&f_j\boxed{\text{i}} = \delta_{j,i}\boxed{\text{i+1}}, i = 1, ..., n + 1, j = 1, ..., n, \qquad (9)
\end{aligned}$$

and $\boxed{\text{i}} = 0$ unless $1 \leq i \leq n + 1$. $\boxed{1}$ is the highest weight vector with highest weight $\lambda_1 = e_1$. The crystal base $(L(\mathbb{V}_{\mathcal{U}_q(A_n)}), B(\mathbb{V}_{\mathcal{U}_q(A_n)}))$ of $\mathbb{V}_{\mathcal{U}_q(A_n)}$ is generated

$$\begin{aligned}
&L(\mathbb{V}_{\mathcal{U}_q(A_n)}) = \oplus_{i=1}^{n+1} A\boxed{\text{i}}, \\
&B(\mathbb{V}_{\mathcal{U}_q(A_n)}) = \{\boxed{\text{i}} \; mod \; qL(\mathbb{V}_{\mathcal{U}_q(A_n)}), i = 1, ..., n + 1\}, \qquad (10)
\end{aligned}$$

such that

$$\begin{aligned}
&\widetilde{e}_j\boxed{\text{i}} = \delta_{j,i-1}\boxed{\text{i-1}}, i = 1, ..., n + 1, j = 1, ..., n \\
&\widetilde{f}_j\boxed{\text{i}} = \delta_{j,i}\boxed{\text{i+1}}, i = 1, ..., n + 1, j = 1, ..., n. \qquad (11)
\end{aligned}$$

The crystal graph $B(\mathbb{V}_{\mathcal{U}_q(A_n)})$ of $\mathbb{V}_{\mathcal{U}_q(A_n)}$ is given by

$$\bullet\boxed{1} \xrightarrow{1} \bullet\boxed{2} \xrightarrow{2} \bullet\boxed{3} \cdots \xrightarrow{n-1} \bullet\boxed{\text{n}} \xrightarrow{n} \bullet\boxed{\text{n+1}}$$

Note, that this case corresponds to the underlying pattern for A_{n+1}-series $|1|$-graded parabolic geometry with crossed Dynkin diagram $\overset{1}{\times}\!\!-\!\!\overset{2}{\bullet}\!\!- \cdots -\!\!\overset{n+1}{\bullet}$.

2.2. **Crystal graph of vector representation of $U_q(B_n)$.** Let $\{e_1, ..., e_n\}$ be the ON-base of the dual of Cartan subalgebra of B_n, with simple roots $\alpha_i = e_i - e_{i+1}$, $i = 1, ..., n - 1$ and $\alpha_n = e_n$ (α_n is the shorter simple root). The set of fundamental weights (as a basis of dual of $\mathfrak{h}$) is

$$\begin{aligned} \lambda_i &= e_1 + \cdots + e_i\,, \qquad i = 1, ..., n-1\,, \\ \lambda_n &= \frac{1}{2}(e_1 + \cdots + e_n)\,. \end{aligned} \tag{12}$$

We shall construct crystal graph of vector representation $\mathbb{V}_{\mathcal{U}_q(B_n)}$. We consider $\mathbb{V}_{\mathcal{U}_q(B_n)}$ to be $(2n+1)$-dimensional $\mathbb{Q}(q)$-vector space with basis $\{\boxed{i}, i = 0, ..., n\}\{\overline{\boxed{i}}, i = 1, ..., n\}$. The structure of $\mathcal{U}_q(B_n)$-module $\mathbb{V}_{\mathcal{U}_q(B_n)}$ is given by

$$\begin{aligned} & q^h\boxed{i} = q^{e_i(h)}\boxed{i}\,,\ q^h\overline{\boxed{i}} = q^{e_i(h)}\overline{\boxed{i}}\,,\ q^h\boxed{0} = \boxed{0}\,,\ i = 1, ..., n\,, \\ & e_j\boxed{i} = \delta_{j,i-1}\boxed{i\text{-}1}\,,\ e_j\overline{\boxed{i}} = \delta_{j,i-1}\overline{\boxed{i\text{-}1}}\,,\ e_j\boxed{0} = 0\,,\ i = 1, ..., n\,,\ j = 1, ..., n-1\,, \\ & f_j\boxed{i} = \delta_{j,i}\boxed{i+1}\,,\ f_j\overline{\boxed{i}} = \delta_{j,i}\overline{\boxed{i+1}}\,,\ f_j\boxed{0} = 0\,,\ i = 1, ..., n\,,\ j = 1, ..., n-1\,, \\ & e_n\boxed{i} = 0\,,\ e_n\overline{\boxed{i}} = \delta_{n,i}\boxed{0}\,,\ e_n\boxed{0} = [2]_n\boxed{n}\,,\ i = 1, ..., n \\ & f_n\overline{\boxed{i}} = \delta_{n,i}\boxed{0}\,,\ f_n\overline{\boxed{i}} = 0\,,\ f_n\boxed{0} = [2]_n\overline{\boxed{n}}\,,\ i = 1, ..., n\,. \end{aligned} \tag{13}$$

In the case i does not belong to $\{0, ..., n\}$, $\boxed{i} = 0$ and $\overline{\boxed{i}} = 0$. The crystal base $(L(\mathbb{V}_{\mathcal{U}_q(B_n)}), B(\mathbb{V}_{\mathcal{U}_q(B_n)}))$ of $\mathbb{V}_{\mathcal{U}_q(B_n)}$ is

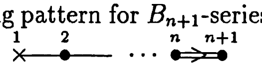
$$\begin{aligned} L(\mathbb{V}_{\mathcal{U}_q(B_n)}) &= \oplus_{i=1}^{n}(A\boxed{i} + A\overline{\boxed{i}}) \oplus A\boxed{0}\,, \\ B(\mathbb{V}_{\mathcal{U}_q(B_n)}) &= \{\boxed{i}, \overline{\boxed{i}};\ i = 1, ..., n\} \cup (\boxed{0})\,. \end{aligned} \tag{14}$$

Replacing the couple (e_j, f_j) by $(\widetilde{e}_j, \widetilde{f}_j)$ $(j = 1, ..., n)$, the only changes are

$$\begin{aligned} \widetilde{e}_n\boxed{0} &= \boxed{n}\,, \\ \widetilde{f}_n\boxed{0} &= \overline{\boxed{n}}\,. \end{aligned} \tag{15}$$

Summarizing all above, the crystal graph is given by

$$\bullet\boxed{1} \xrightarrow{1} \bullet\boxed{2} \xrightarrow{2} \bullet\boxed{3} \cdots \bullet\boxed{n} \xrightarrow{n} \bullet\boxed{0} \xrightarrow{n} \bullet\overline{\boxed{n}} \cdots \bullet\overline{\boxed{3}} \xrightarrow{2} \bullet\overline{\boxed{2}} \xrightarrow{1} \bullet\overline{\boxed{1}}$$

Note, that this case corresponds to the underlying pattern for B_{n+1}-series $|1|$-graded parabolic geometry with crossed Dynkin diagram  . However, the main Theorem1.6 does not hold true for this $|1|$-graded (odd conformal) case, because the weights of $\mathfrak{g}$ are not of the same length. Now this correspondence suggests to overcome all technical obstructions and prove the Theorem in full generality.

2.3. **Crystal graph of vector representation of $U_q(C_n)$.** Let $\{e_1, ..., e_n\}$ be the ON-base of the dual of Cartan subalgebra of C_n, such that the simple roots are $\alpha_i = e_i - e_{i+1}\,, i = 1, ..., n$ and $\alpha_n = 2e_n$ (α_n is the longer simple root). The set of fundamental weights (as a basis of dual of $\mathfrak{h}$) is

$$\lambda_i = e_1 + \cdots + e_i\,, i = 1, ..., n\,. \tag{16}$$

We shall construct crystal graph of vector representation $\mathbb{V}_{\mathcal{U}_q(C_n)}$. We consider $\mathbb{V}_{\mathcal{U}_q(C_n)}$ to be $(2n)$-dimensional $\mathbb{Q}(q)$-vector space with basis $\{\boxed{i}, \overline{\boxed{i}}, i = 1, ..., n\}$. The structure of $\mathcal{U}_q(C_n)$-module $\mathbb{V}_{\mathcal{U}_q(C_n)}$ is given by

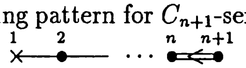
$$\begin{aligned}
& q^h\boxed{i} = q^{e_i(h)}\boxed{i}, \; q^h\overline{\boxed{i}} = q^{-e_i(h)}\overline{\boxed{i}}, \; q^h\boxed{0} = \boxed{0}, \; i = 1, ..., n, \\
& e_j\boxed{i} = \delta_{j,i}\boxed{i-1}, \; e_j\overline{\boxed{i}} = \delta_{j,i}\overline{\boxed{i-1}}, \; i = 1, ..., n, \; j = 1, ..., n, \\
& f_j\boxed{i} = \delta_{j,i}\boxed{i+1}, \; f_j\overline{\boxed{i}} = \delta_{j+1,i}\overline{\boxed{i+1}}, \; i = 1, ..., n, \; j = 1, ..., n, \\
& e_n\boxed{i} = 0, \; e_n\overline{\boxed{i}} = \delta_{n,i}\boxed{n}, \; i = 1, ..., n \\
& f_n\boxed{i} = \delta_{n,i}\overline{\boxed{n}}, \; f_n\overline{\boxed{i}} = 0, \; i = 1, ..., n. \qquad (17)
\end{aligned}$$

In the case i does not belong to $\{1, ..., n\}$, $\boxed{i} = 0$ and $\overline{\boxed{i}} = 0$. The crystal base $(L(\mathbb{V}_{\mathcal{U}_q(C_n)}), B(\mathbb{V}_{\mathcal{U}_q(C_n)}))$ of $\mathbb{V}_{\mathcal{U}_q(C_n)}$ is

$$\begin{aligned}
& L(\mathbb{V}_{\mathcal{U}_q(C_n)}) = \oplus_{i=1}^{n}(A\boxed{i} + A\overline{\boxed{i}}), \\
& B(\mathbb{V}_{\mathcal{U}_q(C_n)}) = \{\boxed{i}, \overline{\boxed{i}}; \; i = 1, ..., n\}. \qquad (18)
\end{aligned}$$

The crystal graph is

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \ldots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \overline{\boxed{n}} \xrightarrow{n-1} \ldots \overline{\boxed{3}} \xrightarrow{2} \overline{\boxed{2}} \xrightarrow{1} \overline{\boxed{1}}$$

This case corresponds to the underlying pattern for C_{n+1}-series $|2|$-graded parabolic geometry with crossed Dynkin diagram  . The main Theorem1.6 does not hold true for this $|2|$-graded (contact) case. Now similar remark applies as to the previous case.

2.4. **Crystal graph of vector representation of $U_q(D_n)$.** Let $\{e_1, ..., e_n\}$ be the ON-base of the dual of Cartan subalgebra of B_n, such that the simple roots are $\alpha_i = e_i - e_{i+1}$, $i = 1, ..., n-1$ and $\alpha_n = e_{n-1} + e_n$ (α_n is the longer simple root). The set of fundamental weights (as a basis of dual of $\mathfrak{h}$) is

$$\begin{aligned}
& \lambda_i = e_1 + \cdots + e_i, \; i = 1, ..., n-2, \\
& \lambda_{n-1} = \frac{1}{2}(e_1 + \cdots + e_{n-1} - e_n), \\
& \lambda_n = \frac{1}{2}(e_1 + \cdots + e_{n-1} + e_n). \qquad (19)
\end{aligned}$$

We shall construct crystal graph of vector representation $\mathbb{V}_{\mathcal{U}_q(D_n)}$. Let $\boxed{i}, \overline{\boxed{i}}$, $i = 1, ..., n$ be the base of $\mathbb{Q}(q)^{\oplus 2n}$. The vector representation $\mathbb{V}_{\mathcal{U}_q(D_n)}$ is given by

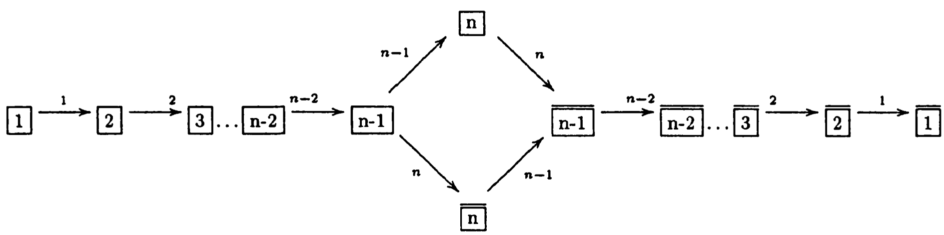
$$\begin{aligned}
& q^h\boxed{i} = q^{e_i(h)}\boxed{i}, \; q^h\overline{\boxed{i}} = q^{-e_i(h)}\overline{\boxed{i}}, \; i = 1, ..., n, \\
& e_j\boxed{i} = \delta_{j+1,i}\boxed{i-1}, \; e_j\overline{\boxed{i}} = \delta_{j,i}\overline{\boxed{i+1}}, \; e_j\boxed{0} = 0, \; i = 1, ..., n, \; j = 1, ..., n-1, \\
& f_j\boxed{i} = \delta_{j,i}\boxed{i+1}, \; f_j\overline{\boxed{i}} = \delta_{j+,i}\overline{\boxed{i-1}}, \quad i = 1, ..., n, \; j = 1, ..., n-1, \\
& e_n\boxed{i} = 0, \; e_n\overline{\boxed{n}} = \boxed{n-1}, \; e_n\overline{\boxed{n-1}} = \boxed{n}, \; e_n\overline{\boxed{i}} = 0, \; i = 1, ..., n-2, \\
& f_n\overline{\boxed{i}} = 0, \; f_n\boxed{n-1} = \overline{\boxed{n}}, \; f_n\boxed{n} = \overline{\boxed{n-1}}, \; f_n\boxed{i} = 0, \; i = 1, ..., n-2. \qquad (20)
\end{aligned}$$

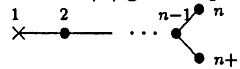
If $i \neq 1, \ldots, n$, we understand $\boxed{i} = 0, \overline{\boxed{i}} = 0$. The crystal base $(L(\mathbb{V}_{\mathcal{U}_q(D_n)}), B(\mathbb{V}_{\mathcal{U}_q(C_n)}))$ of $\mathbb{V}_{\mathcal{U}_q(D_n)}$ is

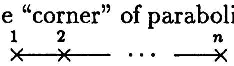
$$L(\mathbb{V}_{\mathcal{U}_q(D_n)}) = \oplus_{i=1}^{n}(A\boxed{i} + A\overline{\boxed{i}}),$$

$$B(\mathbb{V}_{\mathcal{U}_q(D_n)}) = \{\boxed{i}, \overline{\boxed{i}};\ i = 1, \ldots, n\}. \tag{21}$$

In this case, $\widetilde{e}_i = e_i$ and $\widetilde{f}_i = f_i$ for all i. The crystal graph is described by



In this case the underlying pattern corresponds to D_{n+1}-series $|1|$-graded parabolic (even conformal) geometry with crossed Dynkin diagram  .

2.5. **Quantum Bernstein-Gelfand-Gelfand resolutions for $\mathcal{U}_q(A_n)$.** The last piece of evidence comes from the article [7]. Opposite to the previous $|1|$ or $|2|$-graded cases, it corresponds to opposite "corner" of parabolic invariant theory - the Borel case with crossed Dynkin diagram . Using Drinfeld basis instead of the previously mentioned Jimbo one, $(\{h_i, e_i, f_i\} \to \{h_i, h_i e_i, h_i f_i\})$, the construction relies on the triangular decomposition of $\mathcal{U}_q(\mathfrak{g} = A_n)$ (in Drinfeld basis!), i.e.

$$\mathcal{U}_q(\mathfrak{g}) \simeq \mathcal{U}_q(\mathfrak{n}^-) \otimes \mathbb{C}[h_i] \otimes \mathcal{U}_q(\mathfrak{n}^+),$$

using the notation for triangular decomposition $\mathfrak{g} \simeq \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. A representation of commutative algebra $\mathbb{C}[h_i]$ is given by character χ, i.e. the one dimensional representation $\mathbb{C}_\lambda$ is determined by $\chi(\mathfrak{h}_i) = \lambda_i \in \mathbb{C}$, $i = 1, \ldots, n$.

The quantum Verma module is defined by

$$M(\lambda) = \mathcal{U}_q(\mathfrak{g}) \otimes_{\mathcal{U}_q(\mathfrak{h} \oplus \mathfrak{n}^+)} \mathbb{C}_\lambda\ , \tag{22}$$

where the action of $\mathcal{U}_q(\mathfrak{h} \oplus \mathfrak{n}^+)$ is given by $(S \otimes 1) \otimes \Delta$, such that the action on $\mathbb{C}_\lambda$ is trivial for $\mathfrak{n}^+$ and given by character χ for $\mathfrak{h}$. Δ is standard comultiplication (homomorphically extended on universal enveloping algebra), and S is the antipode (acting only on the left part of the tensor product).

The standard homomorphisms of quantum Verma modules enter the definition of quantum BGG sequence:

$$0 \leftarrow \mathbb{V}(\lambda) \leftarrow C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow 0\ , \tag{23}$$

where $C_i = \oplus_{l(w)=i} M(w(\lambda + \rho) - \rho)$ are free $\mathcal{U}_q(\mathfrak{n}^-)$-modules, and $\mathbb{V}(\lambda)$ is the unique irreducible finite dimensional quotient of $M(\lambda)$.

The main result of [7] is, that the sequence (23) is the resolution (denote it Res_q) of $\mathbb{V}(\lambda)$; in particular it is exact.

In the sense of structural morphism $Res_q \longrightarrow q$, we have uniform family of patterns called Hasse graphs. Even in the classical case $q \to 1$ it is not explicitly clear what structure, in the sense of $\mathfrak{g}_0^s$-module $\oplus_{i>0}\mathfrak{g}_i$, classifies Hasse graphs of parabolic geometries (it must be an object more complicated then the weight graph in $|1|$-graded cases). However for our purposes it is sufficient the result of uniform structure $Res_q \longrightarrow q$, in particular the coincidence of "classical"pattern $(q \to 1)$ with the ones for $q \neq 1$. The main problem of this construction - recovering of full category of quantum groups - is accomplished.

3. Final remarks

There are many premature questions, closely related to many natural branches of investigations. The most interesting related to this (conjectural) extension of the construction of quantum analogs of BGG sequences (Section 1.) via crystal graphs of quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g}_0^s)$ are of pure algebro-geometrical nature.

Let us consider the (structural) morphism $\mathcal{U}_q(\mathfrak{g}_0^s) \to q$, and associated morphism $M_q \to q$, where M_q is finite dimensional $\mathcal{U}_q(\mathfrak{g}_0^s)$-module.

Definition 3.1. Let A be an associative (non-commutative, non-cocommutative) algebra over the field k of characteristic zero, e.g. $A = \mathcal{U}_q(\mathfrak{g}_0^s)$ and $k = \mathbb{Q}(q)$ (q is transcendental (indeterminate) over $\mathbb{Q}$). Then the A-module M is called **flat** if for every couple of A-modules K, L holds

$$K \hookrightarrow L \implies K \otimes_A M \hookrightarrow L \otimes_A M\ . \tag{24}$$

Applying the procedure as in the (proved) case $q = 1$ (Sectionl), we arrive at the question not (only) of the flatness of ..., $M_{i-1}, M_i, M_{i+1}, \ldots$ (sitting in the sequence $\cdots \to M_{i-1} \to M_i \to M_{i+1} \ldots$), but (also) of $Im\{M_{i-1} \to M_i\}$ and $Ker\{M_i \to M_{i+1}\}$, for all i.

Question: Let us propose, that we have (for $q = 1$ on i-th place) the property of complex, i.e. $Im\{M_{i-1} \to M_i\}|_{q=1} \hookrightarrow Ker\{M_i \to M_{i+1}\}|_{q=1}$; let $M_i, Ker\{M_i \to M_{i+1}\}$ and $Im\{M_{i-1} \to M_i\}$ be flat A-modules, then

$$K \otimes_A M_i \hookrightarrow L \otimes_A M_i\ ,$$
$$K \otimes_A Ker\{M_i \to M_{i+1}\} \hookrightarrow L \otimes_A Ker\{M_i \to M_{i+1}\}\ ,$$
$$K \otimes_A Im\{M_{i-1} \to M_i\} \hookrightarrow L \otimes_A Im\{M_{i-1} \to M_i\}$$

$$\overset{?}{\Longrightarrow}$$

$$K \otimes_A Ker\{M_i \to M_{i+1}\}/Im\{M_{i-1} \to M_i\} \hookrightarrow L \otimes_A Ker\{M_i \to M_{i+1}\}/Im\{M_{i-1} \to M_i\}$$
$$\wedge$$
$$(Im\{M_{i-1} \to M_i\} \hookrightarrow Ker\{M_i \to M_{i+1}\})|_{q=1} \Longrightarrow (Im\{M_{i-1} \to M_i\} \hookrightarrow Ker\{M_i \to M_{i+1}\})|_q \tag{25}$$

for all A-modules K, L.

The author of this article is for example not aware of the constructions similar to [7] for other classical series of Lie algebras; he does not know, whether this is only coincident with similar restricting (technical) problems in the classical realm.

[1] Cap, A., Slovák, J., Souček, V., *Bernstein-Bernstein-Bernstein Sequences*, ESI Preprint 722, www-page "http://www.esi.ac.at/ESI-Preprints.html".
[2] Dixmier, J., *Universal enveloping algebras*, North-Holland Publishing Company, Series 0 7204 2450 X, 1977.
[3] Jantzen, J. C., *Lectures on Quantum Groups*, Graduate Texts in Mathematics, Volume 6, AMS.
[4] Kashiwara, M., *Crystalizing the q-Analogue of Universal Enveloping Algebras*, Comm. Math. Physics **133** (1990), 249–260.
[5] Kashiwara, M. and Nakashima, T., *Crystal Graphs for Representations of the q-Analogues of Classical Lie algebras* Journal of Algebra **165** (1994), 295–345.
[6] Krump, L., *Construction of Bernstein-Gelfand-Gelfand Resolutions for Almost Hermitean Symmetric Structures*, Dissertation, Prague 2000.
[7] Rosso, M., *Koszul Resolutions and Quantum Groups*, Nuclear Physics B (Proc. Suppl.) **18B** (1990), 269–276.

MATHEMATICAL INSTITUTE OF CHARLES UNIVERSITY
SOKOLOVSKÁ 83
PRAHA 8 - KARLÍN
CZECH REPUBLIC
E-mail: somberg@karlin.mff.cuni.cz