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SOME SPECIAL GEOMETRY IN DIMENSION SIX

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ABSTRACT. We generalize the notion of contact manifold by allowing the contact distribution to have codimension two. There are special features in dimension six. In particular, we show that the complex structure on a three-dimensional complex contact manifold is determined solely by the underlying contact distribution.

DEFINITIONS

Suppose M is a 6-dimensional connected oriented smooth manifold and H is a rank 4 smooth subbundle of its tangent bundle TM. Let Q denote the quotient bundle TM/H. There is a homomorphism of vector bundles $\mathcal{L} : H \wedge H \to Q$ induced by Lie bracket:

$$\mathcal{L}(\xi, \eta) = [\xi, \eta] \bmod H \quad \text{for } \xi, \eta \in \Gamma(H).$$

Regard $\mathcal{L}$ as a tensor $\mathcal{L} \in \Gamma(\Lambda^2 H^* \otimes Q)$. Then $\mathcal{L} \wedge \mathcal{L} \in \Gamma(\Lambda^4 H^* \otimes \bigodot^2 Q)$ may be regarded as a quadratic form on Q^* defined up to scale. We shall say that (M, H) is *non-degenerate* if and only if $\mathcal{L} \wedge \mathcal{L}$ is non-degenerate as such a quadratic form. Since Q has rank two, there are only two cases:

- (M, H) is *elliptic* $\iff$ $\mathcal{L} \wedge \mathcal{L}$ is definite;
- (M, H) is *hyperbolic* $\iff$ $\mathcal{L} \wedge \mathcal{L}$ is indefinite.

An elliptic example may be obtained by taking a 3-dimensional complex contact manifold and forgetting its complex structure. A hyperbolic example may be obtained by taking the product of two 3-dimensional real contact manifolds. These two examples will be referred to as the 'flat' models. The motivations for our investigation are discussed in the end of this article.

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THE ELLIPTIC CASE

Theorem. *Suppose (M,H) is elliptic. Then M admits a unique almost complex structure $J : TM \to TM$ characterized by the following properties:*

- *J preserves H;*
- *the orientation on M induced by J is the given one;*
- *$\mathcal{L} : H \times H \to Q$ is complex bilinear for the induced structures, or equivalently $[\xi,\eta] + J[J\xi,\eta] \in \Gamma(H)$ for $\xi,\eta \in \Gamma(H)$;*
- *$[\xi,\eta] + J[J\xi,\eta] - J[\xi,J\eta] + [J\xi,J\eta] \in \Gamma(H)$ for $\xi \in \Gamma(TM)$, $\eta \in \Gamma(H)$.*

Furthermore, the tensor $S : Q \otimes H \to Q$ induced by

$$S(\xi,\eta) = [\xi,\eta] + J[J\xi,\eta] \bmod H \quad \textit{for } \xi \in \Gamma(TM),\ \eta \in \Gamma(H)$$

is the obstruction to J being integrable.

Proof. Fix $x \in M$. Since $\mathcal{L}_x \wedge \mathcal{L}_x$ is definite, there is no $\psi \in Q^*_x$ for which $(\psi \circ \mathcal{L}_x) \wedge (\psi \circ \mathcal{L}_x)$ vanishes—as a quadratic polynomial $\mathcal{L}_x \wedge \mathcal{L}_x$ has no real roots. Instead it has two complex roots, related by complex conjugation. Each of these roots gives $\psi \in Q^*_x \otimes \mathbb{C}$ defined up to complex scale, so that $(\psi \circ \mathcal{L}_x) \wedge (\psi \circ \mathcal{L}_x)$ vanishes as an element of $\Lambda^4 H^* \otimes \mathbb{C}$. In this case, according to the Plücker criterion, $\psi \circ \mathcal{L}_x$ is simple as an element of $\Lambda^2 H^* \otimes \mathbb{C}$. The corresponding complex 2-plane in $H^* \otimes \mathbb{C}$ defines a complex structure $J : H_x \to H_x$. At the same time $\psi \in Q^*_x \otimes \mathbb{C}$ identifies Q_x with $\mathbb{C}$ and, in particular, defines a complex structure $J : Q_x \to Q_x$. These complex structures on H_x and Q_x are unchanged if ψ is multiplied by any complex number. In other words, they are determined by choosing one of the two roots of $\mathcal{L}_x \wedge \mathcal{L}_x$ as a quadratic polynomial. The other root replaces J by $-J$ but only one of these choices induces the given orientation on M. To summarize, we now have uniquely determined almost complex structures on H and Q so that

$$\mathcal{L}(\xi,\eta) + J\mathcal{L}(J\xi,\eta) = 0 \quad \text{for } \xi,\eta \in \Gamma(H) \tag{1}$$

and inducing the given orientation on M. Choose any extension of these almost complex structures to an almost complex structure $\tilde{J} : TM \to TM$. This $\tilde{J}$ satisfies the first three properties claimed in the statement of the theorem. Define $\tilde{S} : TM \otimes H \to Q$ by

$$\tilde{S}(\xi,\eta) = [\xi,\eta] + J[\tilde{J}\xi,\eta] \bmod H \quad \text{for } \xi \in \Gamma(TM),\ \eta \in \Gamma(H). \tag{2}$$

This homomorphism depends on the choice of the extension $\tilde{J}$. For fixed $\xi \in TM$ consider the map $H \to Q$ defined by $\eta \mapsto \frac{1}{2}(-\tilde{S}(\xi,\eta) + J\tilde{S}(\xi,J\eta))$. By construction this map is complex linear, so non-degeneracy of $\mathcal{L}$ implies that there is a unique element $K\xi \in H$ such that

$$\mathcal{L}(K\xi,\eta) = \frac{-\tilde{S}(\xi,\eta) + J\tilde{S}(\xi,J\eta)}{2} \quad \text{for } \xi \in \Gamma(TM),\ \eta \in \Gamma(H) \tag{3}$$

and this defines a homomorphism $K : TM \to H$.

We claim that $J = \tilde{J} + K$ is the almost complex structure whose existence is asserted in the statement of the theorem. If $\xi \in \Gamma(H)$, then (1) implies that $\tilde{S}(\xi,\eta) = 0$ so $K\xi = 0$, and in particular $K^2 = 0$. Therefore, J preserves H. Also

$$(\tilde{J} + K)^2 = \tilde{J}^2 + \tilde{J}K + K\tilde{J} + K^2 = -\mathrm{Id} + \tilde{J}K + K\tilde{J}$$

so we must check that $\tilde{J}K + K\tilde{J} = 0$. By the non-degeneracy of $\mathcal{L}$ it suffices to check that

$$\mathcal{L}(\tilde{J}K\xi, \eta) + \mathcal{L}(K\tilde{J}\xi, \eta) = 0 \quad \text{for } \xi \in \Gamma(TM),\ \eta \in \Gamma(H).$$

This is easily verified using (1), (2), and (3). Thus, J is an almost complex structure. Moreover, it satisfies the first three requirements listed in the theorem as a consequence of $\tilde{J}$ doing so. Moreover, the tensor S corresponding to $J = \tilde{J} + K$ is visibly given by $S(\xi, \eta) = \tilde{S}(\xi, \eta) + \mathcal{L}(K\xi, \eta)$. By construction, this is just the component of $\tilde{S}$ which is conjugate linear in the second variable. But the final requirement is immediately seen to be equivalent to the fact that the corresponding tensor S (which is conjugate linear in the first variable by construction), is conjugate linear in the second variable, too. In fact, this forces (3) as the correct modification so J is uniquely characterized by having all four properties.

It remains to show that the tensor S is the obstruction to integrability of J. The Nijenhuis tensor of J is

$$N(\xi, \eta) = [\xi, \eta] + J[J\xi, \eta] + J[\xi, J\eta] - [J\xi, J\eta] \quad \text{for } \xi, \eta \in \Gamma(TM).$$

Notice that N is skew and $N(\xi, J\eta) = -JN(\xi, \eta)$. In particular,

$$N(\xi, J\xi) = -JN(\xi, \xi) = 0 \quad \text{for } \xi \in \Gamma(TM). \tag{4}$$

Firstly, consider the case when $\xi \in \Gamma(TM),\ \eta \in \Gamma(H)$. The vanishing of S means that

$$[\xi, \eta] + J[J\xi, \eta] \in \Gamma(H) \quad \text{for } \xi \in \Gamma(TM),\ \eta \in \Gamma(H). \tag{5}$$

In particular, this implies $N(\xi, \eta) \in \Gamma(H)$, so we may consider the tensor $R : TM \otimes H \otimes H \to Q$ defined by

$$R(\xi, \eta, \mu) = \mathcal{L}(N(\xi, \eta), \mu) \quad \text{for } \xi \in \Gamma(TM),\ \eta, \mu \in \Gamma(H).$$

We claim that R vanishes. Once this is proved, non-degeneracy of $\mathcal{L}$ implies that $N(\xi, \eta) = 0$ for $\xi \in \Gamma(TM),\ \eta \in \Gamma(H)$ and so N descends to $N : \Lambda^2 Q \to TM$. Then, as Q has complex rank one, (4) forces N to vanish.

To complete the proof, therefore, it suffices to show that R vanishes. In the following calculation $\equiv$ denotes equality modulo H and in passing from one line to the next we are using either the Jacobi identity, or (5), or the fact that S is conjugate linear in both variables.

$$\begin{aligned}
R(\xi, \eta, \mu) &\equiv [[\xi, \eta], \mu] + [J[J\xi, \eta], \mu] + [J[\xi, J\eta], \mu] - [[J\xi, J\eta], \mu] \\
&\equiv [[\xi, \eta], \mu] + J[[J\xi, \eta], \mu] + J[[\xi, J\eta], \mu] - [[J\xi, J\eta], \mu] \\
&= [[\xi, \mu], \eta] + J[[J\xi, \mu], \eta] + J[[\xi, \mu], J\eta] - [[J\xi, \mu], J\eta] \\
&\quad + [[\mu, \eta], \xi] + J[[\mu, \eta], J\xi] + J[[\mu, J\eta], \xi] - [[\mu, J\eta], J\xi] \\
&\equiv [[\xi, \mu], \eta] + [J[J\xi, \mu], \eta] + [J[\xi, \mu], J\eta] - [[J\xi, \mu], J\eta] \\
&\quad + [[\mu, \eta], \xi] + J[[\mu, \eta], J\xi] + J[[\mu, J\eta], \xi] - [[\mu, J\eta], J\xi] \\
&= [[\xi, \mu] + J[J\xi, \mu], \eta] + [J[\xi, \mu] - [J\xi, \mu], J\eta] \\
&\quad + [[\mu, \eta], \xi] + J[[\mu, \eta], J\xi] + J[[\mu, J\eta], \xi] - [[\mu, J\eta], J\xi] \\
&\equiv [[\mu, \eta], \xi] + J[[\mu, \eta], J\xi] + J[[\mu, J\eta], \xi] - [[\mu, J\eta], J\xi].
\end{aligned}$$

Therefore,

$$R(\xi, \eta, \mu) + R(\xi, \mu, \eta) \equiv J[[\mu, J\eta] + [\eta, J\mu], \xi] - [[\mu, J\eta] + [\eta, J\mu], J\xi]$$

and since $[\mu, J\eta] + [\eta, J\mu] \in \Gamma(H)$, this expression vanishes by (5). We conclude that $R : TM \otimes H \otimes H \to Q$ is skew in its last two entries. But by definition R is conjugate linear in the middle variable and complex linear in the last variable, which together with skew symmetry in these two variables forces R to vanish as required. □

Corollary. *The only local invariant of an elliptic (M, H) is the tensor S.*

Proof. If S vanishes, then (M, H) is a complex contact manifold. The Darboux theorem in the holomorphic setting says that all 3-dimensional complex contact manifolds are locally isomorphic. □

The Hyperbolic Case

There is an entirely parallel story for the hyperbolic case with almost complex structure replaced by almost product structure. The corresponding theorem may be stated as follows.

Theorem. *Suppose (M, H) is hyperbolic. Then H admits a canonical splitting $H = H_+ \oplus H_-$ characterized by the following properties:*

- $[\xi, \eta] \in \Gamma(H)$ *for* $\xi \in \Gamma(H_+)$, $\eta \in \Gamma(H_-)$;
- *the orientation on M induced by $\xi_1 \wedge \xi_2 \wedge [\xi_1, \xi_2] \wedge \eta_1 \wedge \eta_2 \wedge [\eta_1, \eta_2]$ for $\xi_1, \xi_2 \in \Gamma(H_+)$, $\eta_1, \eta_2 \in \Gamma(H_-)$ is the given one.*

Let $Q_\pm$ be the range of $\mathcal{L}|_{\Lambda^2 H_\pm}$. Non-degeneracy of $\mathcal{L}$ implies that $Q = Q_+ \oplus Q_-$. By setting $T_\pm M = [H_\pm, H_\pm]$, we obtain a canonical splitting $TM = T_+M \oplus T_-M$ such that $Q_\pm = T_\pm M / H_\pm$. Furthermore, the tensors $S_+ : Q_+ \otimes H_+ \to Q_-$ and $S_- : Q_- \otimes H_- \to Q_+$ induced by

$$S_\pm(\xi, \eta) \equiv [\xi, \eta] \operatorname{mod}(T_\pm \oplus H_\mp) \quad \text{for } \xi \in \Gamma(T_\pm M)\ , \eta \in \Gamma(H_\pm)$$

are the respective obstructions to T_+M and T_-M being Frobenius integrable.

If $S_\pm$ both vanish, then locally we obtain the flat local model, namely a product of two 3-dimensional real contact manifolds. The Darboux theorem, applied to each such contact manifold separately, implies that the flat model is locally unique. Again, the tensors $S_\pm$ provide the only local structure.

Motivations

Our motivation for this article comes from the theory of CR submanifolds of codimension 2 in $\mathbb{C}^4$. This theory was pioneered by Loboda [5] and Ezhov-Schmalz [4] who found normal forms for such submanifolds paralleling the Moser normal form for CR hypersurfaces. In this context, the distribution H is formed by the maximal complex subspaces of the tangent spaces. More generally, to make an elliptic or hyperbolic (M, H) into a partially integrable almost CR manifold, one has to specify an almost complex structure $\tilde{J}$ on H such that $\mathcal{L}(\tilde{J}\xi, \tilde{J}\eta) = \mathcal{L}(\xi, \eta)$ for all $\xi, \eta \in H$. In the hyperbolic case, this implies in particular that $H = H_+ \oplus H_-$ is a decomposition of H as a sum of two complex line bundles. On the other hand, in the elliptic case the second almost complex structure $\tilde{J}$ can also be rephrased as a decomposition $H = H_+ \oplus H_-$ as a sum of complex line bundles characterized by $\tilde{J} = \pm J$ on $H_\pm$.

Clearly, these additional structures lead to additional obstructions against being CR-isomorphic to the flat models (which are just appropriate quadrics). For example, one has the Nijenhuis tensor corresponding to $\tilde{J}$, or the obvious obstructions against integrability of the subbundles $H_\pm$ in the elliptic case. But in fact, in the CR setting, one gets much more structure: In [7] and [3] it is shown that one gets a parabolic geometry parallel to the Chern-Moser-Tanaka theory for CR hypersurfaces and thus in particular canonical Cartan connections. This article may be viewed as some remnant of the parabolic theory.

As pointed out to us by Peter Vassiliou, there is another context in which (M, H) with these special dimensions arise. The general pair of smooth first order partial differential equations in two independent variables (x, y) and two dependent variables (u, v) may be regarded as a codimension 2 submanifold M in the 8-dimensional jet space with coördinates $(x, y, u, v, u_x, u_y, v_x, v_y)$. This jet space has a natural distribution of rank 6 defined as common kernel of the two 1-forms

$$du - u_x\,dx - u_y\,dy \quad \text{and} \quad dv - v_x\,dx - v_y\,dy.$$

Generically, M will meet this distribution transversally and so will itself inherit a rank 4 distribution H. The elliptic flat model is obtained from the Cauchy-Riemann equations

$$u_x = v_y \quad \text{and} \quad u_y = -v_x. \tag{6}$$

The hyperbolic flat model arises from the equations

$$u_y = 0 \quad \text{and} \quad v_x = 0. \tag{7}$$

Further discussion may be found in [2, Chapter VII, §1], [8], and [9]. Indeed, the almost complex structure of our elliptic theorem and the almost product structure of our hyperbolic theorem are constructed in this context in Subcase 2.5 and Subcase 2.4 of [2, Chapter VII, §1]. The constructions there in terms of Pfaffian systems are equivalent to ours and it is also pointed out that integrability of these Pfaffian systems gives (6) and (7), respectively. It is also mentioned in [2, Chapter VII, §1] that a further classification of these Pfaffian systems could be obtained by applying Élie Cartan's method of equivalence. This would result in the tensor S in the elliptic case (or $S_\pm$ in the hyperbolic case: the two cases coincide when complexified). This was carried out by Robert Bryant [1] who also classified the homogeneous examples.

Higher Dimensions

If we start with a $(2n+1)$-dimensional complex contact manifold M with contact distribution H, then $\mathcal{L} \in \Gamma(\Lambda^2 H^* \otimes Q)$ may be defined as before but now we should consider $\mathcal{L}^{\wedge 2n} \in \Gamma(\Lambda^{4n} H^* \otimes \bigodot^{2n} Q)$ as a polynomial of degree $2n$ defined up to scale. Only when $n = 1$ is this polynomial generic In general it has only two roots, each complex and of multiplicity n. As regards the generalized Cauchy Riemann equation in higher dimensions, Benjamin McKay [6] has solved the equivalence problem for these equations in four dimensions, finding all the homogeneous examples and integrating the Darboux integrable ones.

References

[1] Bryant, R.L., *Aisenstadt lectures*, Centre de Recherche Mathématiques, Université de Montréal, 1994, unpublished.

[2] Bryant, R.L., Chern, S.S., Gardner, R., Goldschmidt, H. and Griffiths, P.A., *Exterior Differential Systems*, Springer, 1991.

[3] Čap, A. and Schmalz, G., *On elliptic partially integrable almost CR manifolds of CR dimension and codimension two*, to appear in Lie Groups, Geometric Structures, and Differential Equations — One Hundred Years after Sophus Lie, Adv. Stud. Pure Math., electronically available as Preprint ESI 937 at `http://www.esi.ac.at`.

[4] Ezhov, V.V. and Schmalz, G., *Normal form and 2-dimensional chains of an elliptic CR surface in* $\mathbb{C}^4$, J. Geom. Anal. **6** (1996), 495–529.

[5] Loboda, A.V., *Generic real analytic manifolds of codimension* 2 *in* $\mathbb{C}^4$ *and their biholomorphic mappings*, Math. USSR Izv. **33** (1989), 295–315.

[6] McKay, B., *Duality and Integrable Systems of Pseudoholomorphic Curves*, PhD Thesis, Duke University, 1999.

[7] Schmalz, G. and Slovák, J., *The geometry of hyperbolic and elliptic CR-manifolds of codimension two*, Asian. J. Math. **4** (2000), 565–597.

[8] Vassiliou, P.J., *Tangential characteristic symmetries and first order hyperbolic systems*, Appl. Algebra Engrg. Comm. Comput. **11** (2001), 377–395.

[9] Vassiliou, P.J., *Intrinsic geometry of first order partial differential equations in the plane*, in preparation.

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