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# NOETHER'S VARIATIONAL THEOREM II AND THE BV FORMALISM

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## 1. Introduction

Lagrangian physics derives 'equations of motion' from a variational principle of least action. Here an action refers to an integral

$$S(\phi) = \int_M L((j^n\phi)(x))vol_M$$

over some manifold $M$ where $\phi$ is a (possibly vector valued) function on $M$ or section of a bundle $E$ over $M$ and $L$ is a 'local function' on $E$, meaning a function on some finite jet space $J^nE$.

The Euler-Lagrange equations describe the critical points of $S$ with respect to variations in $\phi$. The action may have symmetries, i.e. variations in $\phi$ which do not change the value of $S$ and hence are physically irrelevant in the sense that $\phi$ and its transformed value encode the same physical information. Noether's second variational theorem establishes a correspondence between symmetries and differential algebraic relations among the Euler-Lagrange equations.

These symmetries create difficulties for quantization of such physical theories. The method of Batalin and Vilkovisky [2], [1] was invented to handle these difficulties, but turns out to also be of interest in a classical context. Their method extends the BRST cohomological approach by introducing anti-fields (independently and previously due to Zinn-Justin [7], [8]) dual to the original fields and anti-ghosts which correspond to the Noether relations and are dual to the ghosts which generate the BRST complex. A key ingredient in their approach is to use the duality to give an anti-bracket (independently due to Zinn-Justin [7], [8] and also known as an odd Poisson or Gerstenhaber bracket) in their construction.

The relevance of Noether's theorem is not emphasized in most of the literature using the BV approach. One aim of the present paper is to restore such an emphasis: Part of the BV complex is the Koszul-Tate resolution of the differential ideal generated by Euler-Lagrange equations. The anti-fields generate the Koszul complex, which is not a resolution; the anti-ghosts provide the next level of generators as described

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by Tate corresponding to the relations among the Euler-Lagrange equations. Rather than carrying out this analysis in the abstract, we illustrate it explicitly in terms of the Poisson sigma models of Cattaneo and Felder.

The higher order terms in the BV differential in these examples can be related to parts of an $L_\infty$-algebra structure, as we will explain elsewhere.

In Section 2, we review the basics of the Lagrangian approach and establish the notation we will use. Section 3 is devoted to Noether's Second Theorem with a slight modernization of language and notation. In Section 4, we present the Cattaneo-Felder sigma model and work out the Noether identities. In Section 5, we begin the description of the Batalin-Vilkovisky formalism, pointing out the initial Chevalley-Eilenberg (or BRST) part of the differential and especially the Koszul-Tate part. The latter shows explicitly how the anti-ghosts encode the Noether identities. We also recall how to extend the gauge symmetries to act on the anti-fields and anti-ghosts. To combine the Koszul-Tate and Chevalley-Eilenberg differentials into a total differential of square zero requires 'terms of higher order', which are created via the Batalin-Vilkovisky anti-bracket as worked out in Section 6.

## 2. Preliminaries

Let $\Sigma$ be an $s$-dimensional manifold and $\pi : E \to \Sigma$ a vector bundle of fiber dimension $k$ over $\Sigma$. Let $J^\infty E$ denote the infinite jet bundle of $E$ over $\Sigma$ with $\pi_E^\infty : J^\infty E \to E$ and $\pi_\Sigma^\infty : J^\infty E \to \Sigma$ the canonical projections. The vector space of smooth sections of $E$ with compact support will be denoted $\Gamma E$. For each section $\phi$ of $E$, let $j^\infty \phi$ denote the induced section of the infinite jet bundle $J^\infty E$. We will consider 'local' functions defined on a finite jet space (see below), but refer to $J^\infty E$ to avoid specifying some finite jet.

The restriction of the infinite jet bundle over an appropriate open $U \subset \Sigma$ is trivial with fibre an infinite dimensional vector space $V^\infty$. The bundle

$$\pi^\infty : J^\infty E_U = U \times V^\infty \to U \tag{1}$$

then has induced coordinates given by

$$(x^i, u^a, u_i^a, u_{i_1 i_2}^a, \ldots,). \tag{2}$$

We use multi-index notation and the summation convention throughout the paper. If $j^\infty\phi$ is the section of $J^\infty E$ induced by a section $\phi$ of the bundle $E$, then $u^a \circ j^\infty \phi = u^a \circ \phi$ and

$$u_I^a \circ j^\infty \phi = (\partial_{i_1} \partial_{i_2} \ldots \partial_{i_r})(u^a \circ j^\infty \phi)$$

where $r$ is the order of the symmetric multi-index $I = \{i_1, i_2, \ldots, i_r\}$, with the convention that, for $r = 0$, there are no derivatives.

**Definition 1.** *We say that a real-valued function on the jet space $J^\infty E$ is a* ***local function*** *if it is the composite of the projection from $J^\infty E$ onto $J^k E$ and a smooth real-valued function on $J^k E$ for some $k$. Thus such functions are pull-backs of functions in $C^\infty(J^k E)$ under the projection $\pi_k^\infty : J^\infty E \longrightarrow J^k E$.*

Let

$$D_i = \frac{\partial}{\partial x^i} + u_{iJ}^a \frac{\partial}{\partial u_J^a} \tag{3}$$

be the total differential operator acting on the space $Loc_E$ of local functions defined on the jet space $J^\infty E$.

More generally, **total differential operators** are mappings from $Loc_E$ into $Loc_E$ defined in local coordinates by $Z = Z^I D_I$ where $Z^I \in Loc_E$ and $D_I = D_{i_1} \circ \cdots \circ D_{i_r}$ for each symmetric multi-index $I$.

It can be shown that the complex $\Omega^*(J^\infty E, d)$ of differential forms splits as a bicomplex (though the finite level complexes $\Omega^*(J^p E)$ do not). The bigrading is described by writing a differential $p$-form $\alpha = \alpha^{\mathbf{J}}_{IA}(\theta^A_{\mathbf{J}} \wedge dx^I)$ as an element of $\Omega^{r,t}(J^\infty E)$, with $p = r + t$, where

$$(4) \qquad dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_r}, \qquad \theta^A_{\mathbf{J}} = \theta^{a_1}_{J_1} \wedge \cdots \wedge \theta^{a_t}_{J_t}$$

and

$$\theta^a_J = du^a_J - u^a_{J\mu} dx^\mu.$$

We restrict the complex $\Omega^*$ by requiring that the functions $\alpha^{\mathbf{J}}_{IA}$ be local functions. In this context, the horizontal differential is obtained by noting that $d\alpha$ is in $\Omega^{r+1,t} \oplus \Omega^{r,t+1}$ and then denoting the two pieces by, respectively, $d_H\alpha$ and $d_V\alpha$.

We will work exclusively with the $d_H$ subcomplex, the algebra of horizontal forms $\Omega^{*,0}$, which is the exterior algebra in the $dx^i$ with coefficients that are local functions. In this case we often use Olver's notation $D$ for the horizontal differential $d_H = dx^i D_i$ where $D_i$ is the total derivative defined above. It is well-known that in this language, the Poincaré lemma asserts that on an appropriate open subset of $J^\infty E$, $d_H\alpha = 0$ for $\alpha \in \Omega^{s,0}$ iff

$$\alpha = \partial_\mu j^\mu (dx^1 \wedge \cdots \wedge dx^s)$$

for some choice of local functions $\{j^\mu\}$.

**Definition 2.** *A **local functional** is a function $S$ from $\Gamma E$ into the reals such that, for each section $\phi \in \Gamma E$, we have*

$$(5) \qquad S(\phi) = \int_\Sigma L(x, \phi^{(p)}(x)) dvol_\Sigma = \int_\Sigma (j^\infty \phi)^* L(x, u^{(p)}) dvol_\Sigma$$

*is the integral over $\Sigma$ of the pull-back $(j^\infty \phi^*)L$ of some local function $L$ on $J^\infty E$. Recall that the elements of $\Gamma E$ have compact support so that the integral is well-defined.*

These definitions reflect the fact that we identify the fields $\phi$ of a physical theory with sections of an appropriate vector bundle $E \longrightarrow \Sigma$. With this identification, the *Lagrangian* $L$ of the theory is a local function on $J^\infty E$. We work on $J^\infty E$ for convenience but the Lagrangian, being local, only depends on finitely many derivatives of the fields. Finally, the *action* $S$ corresponding to the Lagrangian is simply the local functional defined by $L$ as in the definition above.

**Definition 3.** The Euler-Lagrange operator*: For $1 \le a \le k$, let $E_a$ denote the a-th component of the* Euler-Lagrange operator *defined for $F \in Loc_E$ by*

$$(6) \qquad E_a(F) = \frac{\partial F}{\partial u^a} - \partial_i \frac{\partial F}{\partial u^a_i} + \partial_i \partial_j \frac{\partial F}{\partial u^a_{ij}} - \cdots = (-D)_I (\frac{\partial F}{\partial u^a_I}).$$

We say that $Q$ is an *evolutionary vector field on $E$* if it is a mapping from $J^\infty E$ into the vertical vector fields on $E$. In local coordinates $Q = Q^a \frac{\partial}{\partial u^a}$ where the functions $Q^a$ are local functions on $J^\infty E$. For every evolutionary vector field $Q$ on $E$, there exists its *prolongation*, denoted $pr(Q)$, the unique vector field on $J^\infty E$ such that $(d\pi_E^\infty)(pr(Q)) = Q$ and $\mathcal{L}_{pr(Q)}(C) \subseteq C$. Here $\mathcal{L}_{pr(Q)}$ denotes the Lie derivative operator with respect to the vector field $pr(Q)$. The ideal $C$ is the ideal of forms on $J^\infty E$ generated by the contact forms $\{\theta^a_J\}$ used above in the definition of the bicomplex.

In local adapted coordinates, the prolongation of an evolutionary vector field $Q = Q^a\partial/\partial u^a$ assumes the form $pr(Q) = (D_J Q^a)\partial/\partial u^a_J$.

Given a total differential operator $Z$, define a new total differential operator $Z^+$ called the (formal) *adjoint* of $Z$ by

$$\int_M (j^\infty\phi)^*(FZ(G))dvol_\Sigma = \int_\Sigma (j^\infty\phi)^*(Z^+(F)G)dvol_\Sigma \tag{7}$$

for all sections $\phi \in \Gamma E$ and all $F, G \in Loc_E$. It follows that

$$FZ(G)dvol_\Sigma = Z^+(F)Gdvol_\Sigma + d_H\zeta \tag{8}$$

for some $\zeta \in \Omega^{n-1,0}(E)$. If $Z{=}Z^J D_J$ in local coordinates, then $Z^+(F){=}(-D)_J(Z^J F)$. This follows from an integration by parts in (8) and the fact that (8) must hold for all $G$.

## 3. Gauge symmetries and Noether identities

Recall that if a Lie group $G$ acts as automorphisms of a vector bundle $E \longrightarrow \Sigma$ (over the identity of $\Sigma$) in such a way that it leaves the action $S$ of a Lagrangian $L : J^\infty E \longrightarrow \mathbf{R}$ invariant, then the group action induces a vertical vector field $\tilde\eta$ on $E$, for each element $\eta$ of the Lie algebra of $G$, such that the prolongation $pr(\tilde\eta)$ of $\tilde\eta$ to $J^\infty E$ has the property that $dL(pr(\tilde\eta))vol_M$ is $d_H$ exact. Here $vol_\Sigma$ denotes both a volume on $\Sigma$ and its pullback to $J^\infty E$ via the projection $J^\infty E \longrightarrow \Sigma$. An evolutionary vector field $Q_E$ on $E$ is called a *variational symmetry* of a Lagrangian $L$ iff it has the property that $dL(pr(Q_E))vol_\Sigma = pr(Q_E)(L)vol_\Sigma$ is $d_H$ exact. In local coordinates, $Q_E$ is a variational symmetry iff

$$pr(Q_E)(L) := D_K(Q^a_E)\frac{\partial L}{\partial u^a_K} \tag{9}$$

is a divergence, i.e., iff it is equal to $D_\mu j^\mu$ for some set $\{j^\mu\}$ of local functions defined on $J^\infty E$. "Integrating by parts" shows that this condition is equivalent to requiring that $Q^a_E(-D)_K(\frac{\partial L}{\partial u^a_K})$ be a divergence. But the Euler Lagrange operator $E_a$ acting on the Lagrangian $L$ is defined by the equation $E_a(L) = (-D)_K(\frac{\partial L}{\partial u^a_K})$. Thus an evolutionary vector field $Q_E$ is a variational symmetry of a Lagrangian $L$ iff $Q^a_E E_a(L)$ is a divergence.

Finally, a gauge symmetry of a Lagrangian $L$ is defined when there is a linear mapping from $Loc_E$ into the variational symmetries. To be more precise, there must exist local functions $R^{aI} : J^\infty E \longrightarrow \mathbf{R}$ such that $R^{aI}(D_I\epsilon)\frac{\partial}{\partial u^a}$ is a variational symmetry of $L$ for each local function $\epsilon : J^\infty E \longrightarrow \mathbf{R}$. Notice that the coefficients of the vector field depend linearly on both $\epsilon$ and its derivatives. It follows that being a gauge symmetry is equivalent to requiring that $R^{aI}(D_I\epsilon)E_a(L)$ be a divergence for each $\epsilon$. This in turn is equivalent to saying that $\epsilon(R^{aI}D_I)^+(E_a(L))$ is a divergence for each $\epsilon$. Here $(R^{aI}D_I)^+$ is the formal adjoint of the differential operator $R^{aI}D_I$ which was defined in Section 2. The adjoint of a differential operator is also a differential operator and consequently there exist local functions $R^{+aI} : J^\infty E \longrightarrow \mathbf{R}$ such that $(R^{aI}D_I)^+ = R^{+aI}D_I$. These functions are found by working out the iterated total derivatives $(-D)_I(R^{aI}F)$.

In many cases it is easier to use an "integration by parts" procedure to obtain the coefficients $\{R^{+aI}\}$. This is what we do for the Poisson $\sigma$-model below.

It follows easily that $\epsilon \mapsto R^{aI}(D_I\epsilon)\frac{\partial}{\partial u^a}$ defines a gauge symmetry iff $\epsilon R^{+aI}D_I(E_a(L))$ is a divergence for each $\epsilon$. Finally, this condition is equivalent to saying that $R^{+aI}D_I(E_a(L))$ is identically zero on the jet bundle. Such identities are called *Noether identities* or *dependencies* in the translation of Noether's original term. One thus has a one-one correspondence between gauge symmetries of a Lagrangian and Noether identities.

The original version of Noether in 'Invariant variation problems' [6], was written in terms of an infinite continuous group, $G_{\infty\rho}$, 'understood to be a group whose most general transformations depend on $\rho$ essential arbitrary functions and their derivatives'. Noether's Theorem II refers to an integral $I$ $(= S$ in our notation) and reads:

> If the integral I is invariant with respect to a $G_{\infty\rho}$ in which the arbitrary functions occur up to the $\sigma$-th derivative, there subsist $\rho$ identity relationships between the Lagrange expressions and their derivatives up to the $\sigma$-th order. ... the converse holds.

Later in that paper these relations are called *dependencies.*

To recast and summarize in our notation and terminology, we have:

**Theorem 1.** (Noether) *For a given Lagrangian $L$ defined on the jet bundle $J^\infty E$ and for local real-valued functions $\{R^{aI}\}$ defined on $J^\infty E$, the following statements are equivalent:*

1. *the functions $\{R^{aI}\}$ define a gauge symmetry of $L$, i.e., $R^{aI}(D_I\epsilon)\frac{\partial}{\partial u^a}$ is a variational symmetry of $L$ for each local function $\epsilon : J^\infty E \longrightarrow \mathbf{R}$;*
2. *$R^{aI}(D_I\epsilon)E_a(L)$ is a divergence for each $\epsilon$.*
3. *the functions $\{R^{aI}\}$ define Noether identities of $L$, i.e., $R^{+aI}D_I(E_a(L))$ is identically zero on the jet bundle.*

## 4. The Poisson sigma model

To provide a specific example of this correspondence and how it relates to the Batalin-Vilkovisky machinery, we turn to a Poisson sigma model of Cattaneo and Felder [3].

The fields of this Poisson $\sigma$-model are ordered pairs $(X, \eta)$ such that $X$ is a mapping from a 2-dimensional manifold $\Sigma$ into a Poisson manifold $M$ and $\eta$ is a section of the bundle $Hom(T\Sigma, X^*T^*M) \longrightarrow \Sigma$. These fields are subject to boundary conditions, namely they should satisfy the conditions: $X(u) = 0$ and $\eta(u)(v) = 0$ for arbitrary $u$ in the boundary of $\Sigma$ and for $v$ tangent to the boundary of $\Sigma$ at $u$. Observe that for each $u \in \Sigma$, we can regard $\eta(u)$ as a linear mapping from $T_u\Sigma$ into $T^*_{X(u)}M$. In local coordinates $\{u^\mu\}$ on $\Sigma$ and $\{x^i\}$ on $M$, we write $dX = (dX^j)\frac{\partial}{\partial x^j}$ and $\eta(\frac{\partial}{\partial u^\mu}) = \eta_{i,\mu}dx^i$. The Poisson structure is given by a *Poisson tensor* which is a skew-symmetric tensor on $M$

$$\alpha = \alpha^{ij}(\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}) \tag{10}$$

which satisfies a Jacobi condition:

$$\alpha^{il}\partial_l\alpha^{jk} + \alpha^{jl}\partial_l\alpha^{ki} + \alpha^{kl}\partial_l\alpha^{ij} = 0. \tag{11}$$

The action $S$ of the model is defined in such local coordinates by

$$S(X,\eta) = \int_\Sigma (\eta_i \wedge dX^i) + \frac{1}{2}(\alpha^{ij} \circ X)(\eta_i \wedge \eta_j). \tag{12}$$

To understand this action in a more invariant notation, recall that for each $u \in \Sigma$, $dX$ is a linear mapping from $T_u\Sigma$ into $T_{X(u)}M$ and so one may define a two-form $\eta \wedge dX$ on $\Sigma$ by

$$(\eta \wedge dX)(v_1, v_2) = \eta(v_1)(dX(v_2)) - \eta(v_2)(dX(v_1)) \tag{13}$$

for $v_1, v_2 \in T\Sigma$. We may also define a two-form $\alpha_X(\eta \wedge \eta)$ on $\Sigma$ by

$$\alpha_X(\eta \wedge \eta)(v_1, v_2) = (\alpha \circ X)(\eta(v_1), \eta(v_2)). \tag{14}$$

Using the coordinates defined above, we see that:

$$\eta \wedge dX = \eta_i \wedge dX^i \tag{15}$$

$$\alpha_X(\eta \wedge \eta) = (\alpha \circ X)(\eta \wedge \eta) = \frac{1}{2}\alpha_X^{ij}(\eta_i \wedge \eta_j). \tag{16}$$

*For the remainder of the paper, we will restrict to $M = \mathbf{R}^k$ to avoid inserting 'in local coordinates' repeatedly.*

According to the variational principle, we obtain extrema of $S$ as those fields $(X, \eta)$ which satisfy the Euler-Lagrange equations:

$$E_{X^i} := d\eta_i + \frac{1}{2}\partial_i\alpha^{jk}(\eta_j \wedge \eta_k) = 0 \tag{17}$$

and

$$E_{\eta_i} := -dX^i - \alpha^{ij}\eta_j = 0. \tag{18}$$

In terms of the components of the fields, we write

$$E_{X^i} = (\partial_\mu \eta_{i,\nu} + \frac{1}{2}\partial_i\alpha^{jk}\eta_{j,\mu}\eta_{k,\nu})\epsilon^{\mu\nu} \tag{19}$$

and

$$E_{\eta_{i,\nu}} = -(\partial_\mu X^i + \alpha^{ij}\eta_{j,\mu})\epsilon^{\mu\nu}. \tag{20}$$

The gauge symmetries of the action are parameterized by all sections $\beta$ of the bundle $X^*T^*M \longrightarrow \Sigma$ which vanish on the boundary of $\Sigma$. For each such $\beta$, define $\delta_\beta$ acting on the fields by

$$(\delta_\beta X)^i = (\alpha \circ X)(dx^i, \beta) \tag{21}$$

$$(\delta_\beta \eta)(W \circ X) = -(d\beta)(W \circ X) - ((\mathcal{L}_W\alpha) \circ X)(\eta, \beta) \tag{22}$$

where $W$ is a vector field on $M$, and $\mathcal{L}_W\alpha$ is the Lie derivative of $\alpha$ with respect to $W$. Observe that $\delta_\beta X$ and $\delta_\beta \eta$ are indeed again fields since $\delta_\beta X$ is a mapping from $\Sigma$ to $M$ and $\delta_\beta \eta$ is a section of the bundle $Hom(T\Sigma, X^*T^*M) \longrightarrow \Sigma$.

If we regard $X^i$ and $\eta_{i,\nu}$ as jet coordinates on an appropriate jet bundle, we may write $\delta_\beta$ as a variational symmetry

$$\delta_\beta = (\alpha_X^{ij}\beta_j)\frac{\partial}{\partial X^i} - (\partial_\mu\beta_i + ((\partial_i\alpha^{jk})\circ X)\eta_{j,\mu}\beta_k)\frac{\partial}{\partial\eta_{i,\mu}}. \tag{23}$$

For notational convenience, we will not show the explicit $X$ dependence throughout the remainder of this section except when it is misleading to fail to do so.

It follows from Noether's theorem that

$$(\alpha^{ij}\beta_j)E_{X^i} - (\partial_\mu\beta_i + \partial_i\alpha^{jk}\eta_{j,\mu}\beta_k)E_{\eta_{i,\mu}} \tag{24}$$

is a divergence. To find the corresponding Noether identity, we must be able to factor out the gauge parameters $\beta_k$, so we transform the term $(\partial_\mu\beta_i)E_{\eta_{i,\mu}}$ via the identity

$$(\partial_\mu\beta_i)E_{\eta_{i,\mu}} = \partial_\mu(\beta_i E_{\eta_{i,\mu}}) - \beta_i\partial_\mu E_{\eta_{i,\mu}} = div - \beta_i\partial_\mu E_{\eta_{i,\mu}}. \tag{25}$$

If $N^k := \alpha^{ik}E_{X^i} + \partial_\mu E_{\eta_{k,\mu}} - \partial_i\alpha^{jk}\eta_{j,\mu}E_{\eta_{i,\mu}}$, then

$$N^k\beta_k = (\alpha^{ik}E_{X^i} + \partial_\mu E_{\eta_{k,\mu}} - \partial_i\alpha^{jk}\eta_{j,\mu}E_{\eta_{i,\mu}})\beta_k \tag{26}$$

is a divergence for every $\beta_k$. It follows that the integral of $N^k\beta_k$ vanishes for all $\beta_k$ and consequently $N^k = 0$ for each $k$. From this, we see that the equations

$$\alpha^{ik}E_{X^i} + \partial_\mu E_{\eta_{k,\mu}} - \partial_i\alpha^{jk}\eta_{j,\mu}E_{\eta_{i,\mu}} = 0 \tag{27}$$

are the Noether identities corresponding to the gauge symmetry $\delta_\beta$ defined above.

To write this identity in differential form notation, multiply the last equation by $du^1 \wedge du^2$ and use the identity $\epsilon^{\mu\nu}(du^1 \wedge du^2) = du^\mu \wedge du^\nu$ to get

$$\alpha^{ij}(\partial_\mu\eta_{i,\nu} + \frac{1}{2}\partial_i\alpha^{rs}\eta_{r,\mu}\eta_{s,\nu})(du^\mu \wedge du^\nu) - \partial_i\alpha^{kj}\eta_{k,\mu}(\partial_\nu X^i + \alpha^{ir}\eta_{r,\nu})(du^\mu \wedge du^\nu)$$
$$+\partial_\mu(\partial_\nu X^j + \alpha^{jr}\eta_{r,\nu})(du^\mu \wedge du^\nu) = 0,$$

which in turn implies that

$$\alpha^{ij}[d\eta_i + \frac{1}{2}\partial_i\alpha^{rs}(\eta_r \wedge \eta_s)] - \partial_i\alpha^{kj}[\eta_k \wedge (dX^i + \alpha^{ir}\eta_r)] + d[dX^j + \alpha^{ji}\eta_i] = 0. \tag{28}$$

Now utilize the formulas (17) and (18) for $E_{X^i}$ and $E_{\eta_{i,\mu}}$ above to obtain the Noether identities in the form

$$\alpha^{ij}E_{X^i} + \partial_i\alpha^{kj}(\eta_k \wedge E_{\eta_i}) - dE_{\eta_j} = 0. \tag{29}$$

## 5. First steps of the Batalin-Vilkovisky formalism

Rather than review the Batalin-Vilkovisky formalism in general as in [5], [2], [1], we illustrate it by example: the Poisson sigma model we have been considering. Batalin and Vilkovisky first construct a graded commutative algebra over $Loc_E$ with generators $X_i^+$ and $\eta^{+i}$, called 'anti-fields', $\gamma_i$ called 'ghosts' and $\gamma^{+i}$, called 'anti-ghosts. (If only the ghosts were used as generators, this would be a BRST algebra.)

These generators are bigraded, as indicated in the following table where the form degree is displayed as the top row and the ghost degree as the first column. *The graded commutativity is with respect to the sum of the ghost degree and the form degree (which we call the total degree).*

The assignments of degree (from left to right) and ghost number (from top to bottom) are given by

$$\begin{array}{cccc} & 0 & 1 & 2 \\ -2 & & & \gamma^{+i} \\ -1 & & \eta^{+i} & X_i^+ \\ 0 & X^i & \eta_i & \\ 1 & \gamma_i & & \end{array}$$

Ultimately, this algebra is given a differential $D$ which is a derivation with respect to the ghost degree, but initially has just two such derivations which need not square to zero.

One of the derivations $\delta$ looks like the Chevalley-Eilenberg differential for Lie algebra cohomology, even though Batalin and Vilkovisky need not have a Lie algebra, and is often called a BRST operator. It is defined initially by

$$\begin{aligned} \delta X^i &= \alpha^{ij}(X)\gamma_j, \\ \delta\eta_i &= -d\gamma_i - \partial_i\alpha^{jk}(X)\eta_j\gamma_k, \\ \delta\gamma_i &= \frac{1}{2}\partial_i\alpha^{jk}(X)\gamma_j\gamma_k. \end{aligned} \tag{30}$$

The other derivation, $d_{KT}$, does square to zero. It is the Koszul-Tate differential for the differential ideal generated by the Euler-Lagrange equations. The Koszul complex is graded by the ghost number. This means the anti-fields generate the Koszul complex with

$$\begin{aligned} d_{KT}\, X_i^+ &= d\eta_i + \frac{1}{2}\partial_i\alpha^{kl}(X)\eta_k \wedge \eta_l = E_{X^i} \\ d_{KT}\, \eta^{+i} &= -dX^i - \alpha^{ij}(X)\eta_j = E_{\eta_i}. \end{aligned} \tag{31}$$

Because of the Noether identities, the Koszul complex has non-trivial cohomology in ghost degree $-1$, namely the classes given by the formulas for the identities with $E_{X^i}$ and $E_{\eta_i}$ replaced by $X_i^+$ and $\eta^{+i}$:

$$-\alpha^{ij}(X)X_j^+ - \partial_k\alpha^{ij}(X)\eta_j \wedge \eta^{+k} - d\eta^{+i}. \tag{32}$$

These classes can be killed by adjoining the anti-ghosts $\gamma^{+i}$ and defining

$$d_{KT}\gamma^{+i} = -\alpha^{ij}(X)X_j^+ - \partial_k\alpha^{ij}(X)\eta_j \wedge \eta^{+k} - d\eta^{+i}. \tag{33}$$

Thus the anti-ghosts occur precisely because of the identities identified by Noether.

The pairing between symmetries and identities is now expressed as the pairing between ghosts and anti-ghosts, which plays a crucial role in the Batalin-Vilkovisky anti-bracket, but first the anti-fields and anti-ghosts are themselves subject to symmetries corresponding to $\delta_\beta$ as follows:

$$\begin{aligned} \delta X_i^+ &= \partial_i\alpha^{kj}(X)X_k^+\gamma_j \\ \delta\eta^{+i} &= \partial_k\alpha^{ij}(X)\eta^{+k}\gamma_j \\ \delta\gamma^{+i} &= \partial_k\alpha^{ij}(X)\gamma^{+k}\gamma_j. \end{aligned} \tag{34}$$

## 6. The Batalin-Vilkovisky anti-bracket and total differential

The hoped for total differential $D$ will be obtained by adding 'terms of higher order' to $d_{KT}+\delta$, which does not square to zero. To do this in general, Batalin and Vilkovisky introduce an 'anti-bracket' ( , ) which is defined in terms of distributional derivatives of functionals of the fields and anti-fields.

Before we define the anti-bracket, it is convenient to first consider the definition of the derivative of a functional $A$ of fields and antifields which are denoted collectively as $(\psi_\alpha)$. The derivative $\frac{\partial A}{\partial \psi^\alpha}$ is the distribution whose value at test forms $(\rho^\alpha)$ (of the same degree and ghost number as $(\psi^\alpha)$) is given by

$$\frac{d}{dt}A(\psi+t\rho)\bigg|_{t=0} = \int_\Sigma \rho^\alpha \wedge \frac{\partial A}{\partial \psi^\alpha}.$$

Consider the functional $A$ defined by

$$A(\phi,\phi^+) = \int_\Sigma (\phi \wedge \phi^+)$$

then we see that up to signs $\frac{\partial A}{\partial \phi}$ is in some sense identified with $\phi^+$ while $\frac{\partial A}{\partial \phi^+}$ is identified with $\phi$. In this way we see that $\phi$ and $\phi^+$ are "canonically conjugate".

Thus we have a canonical distributional pairing of each field or ghost with its 'anti':

$$(X^i, X_j^+) = \delta_j^i \qquad (35)$$
$$(\eta_j, \eta_+^i) = \delta_j^i$$
$$(\gamma_j, \gamma_+^i) = \delta_j^i.$$

The BV anti-bracket extends this as a graded biderivation with respect to ghost degree and in this example can be written as

$$(A,B) = \sum_\alpha \int_\Sigma (-1)^{|\phi_\alpha|(|\phi_\alpha|+|A|)} \left( \frac{\partial A}{\partial \phi^\alpha} \wedge \frac{\partial B}{\partial \phi_\alpha^+} - (-1)^{(deg(\phi_\alpha)+|A|+1)} \frac{\partial A}{\partial \phi_\alpha^+} \wedge \frac{\partial B}{\partial \phi^\alpha} \right) \qquad (36)$$

where $|C| = gh(C) + deg(C)$ denotes the Grassman parity of $C$ ($C$ is either a field or a function of fields). Note that physicists prefer to use both left and right derivatives and hence exhibit a different set of signs.

The antibracket obeys the graded commutativity relation

$$(A,B) = -(-1)^{(\mathrm{gh}(A)-1)(\mathrm{gh}(B)-1)}(B,A)$$

and the Leibnitz rule

$$(A,BC) = (A,B)C + (-1)^{(\mathrm{gh}(A)-1)\mathrm{gh}(B)}B(A,C), \qquad (37)$$

which emphasizes the resemblance to a Poisson bracket. The only difference from a graded Poisson bracket is that the bracket shifts the degree by 1 and the several identities (skew-commutativity, Jacobi and Leibniz) inherit certain signs. Such an 'odd' Poisson bracket is also known as a **Gerstenhaber bracket** [4].

Now it is possible to express $d_{KT}+\delta$ in the form $(S^0+S^1,\ \ )$ where

$$S^0 = (X,\eta) = \int_\Sigma (\eta_i \wedge dX^i) + \frac{1}{2}(\alpha^{ij}\circ X)(\eta_i \wedge \eta_j),$$

our original action, and $S^1$ is

$$\int_\Sigma X_i^+\alpha^{ij}(X)\gamma_j - \eta^{+i}\wedge(d\gamma_i + \partial_i\alpha^{kj}(X)\eta_k\gamma_j) - \frac{1}{2}\gamma^{+i}\partial_i\alpha^{jk}(X)\gamma_j\gamma_k. \tag{38}$$

Corresponding to the fact that $(d_{KT}+\delta)^2 \neq 0$, we have

$$(S^0+S^1, S^0+S^1)\neq 0.$$

The additional terms in the differential $D$ we seek will be found by extending $S^0+S^1$ by terms of higher order to achieve the full BV action $S_{BV}$. First, let us analyze the derivation $(S^0,\ )$.

Notice that $(\ ,X_i^+)$ is effectively (up to sign) $\partial_{X^i}$ and similarly for the other anti's, while $(\ ,X^i)$ is effectively $\partial_{X_i^+}$, etc. More precisely, for any of our basic variables, denoted collectively as $\phi^a$ and their anti's denoted $\phi_a^+$, we have

$$(S,\phi^a) = (-1)^{gh(\phi^a)}\frac{\partial S}{\partial\phi_a^+} \tag{39}$$

$$(S,\phi_a^+) = (-1)^{gh(\phi^a)+deg(\phi^a)}\frac{\partial S}{\partial\phi^a} \tag{40}$$

whenever the parity of $S(\phi^\alpha,\phi_\alpha^+)$ is even. The parities of $S^0, S^1, S^2$ are all 2. Recall that the parity was defined above to be the total degree.

Since $S^0$ has no anti's, $(S^0,S^0)=0$, in fact,

$$(S^0,X^i)=0,\quad (S^0,\eta_i)=0\ \text{ and }\ (S^0,\gamma_i)=0.$$

However, $(S^0,\ )$ does act non-trivially on some of the anti's:

$$\begin{aligned}(S^0,X_i^+) &= d\eta_i + 1/2\partial_i\alpha^{kl}(X)\eta_k\wedge\eta_j\\ (S^0,\eta^{+i}) &= -dX^i - \alpha^{ij}\eta_j,\end{aligned} \tag{41}$$

which reproduces part of $d_{KT}$, cf. (31), while $(S^0,\gamma^{+i})=0$.

Now consider $(S^1,\ )$:

$$\begin{aligned}(S^1,X^i) &= \alpha^{ij}\gamma_j\\ (S^1,\eta_i) &= -(d\gamma_i+\partial_i\alpha^{jk}(X))\wedge\eta_j\gamma_k\\ (S^1,\gamma_i) &= 1/2\partial_i\alpha^{jk}(X)\gamma_j\gamma_k\\ (S^1,X_i^+) &= \partial_i\alpha^{kl}(X)X_k^+\gamma_l - \partial_i\partial_j\alpha^{kl}(X)\eta^{+j}\wedge\eta_k\gamma_l - \frac{1}{2}\partial_i\partial_j\alpha^{kl}(X)\gamma^{+j}\gamma_k\gamma_l\\ (S^1,\eta^{+i}) &= \eta^{+k}\partial_k\alpha^{ij}(X)\gamma_j\\ (S^1,\gamma^{+i}) &= -\alpha^{ij}X_i^+ - d\eta^{+i} + \partial_k\alpha^{ij}(X)\eta^{+k}\wedge\eta_j + \partial_k\alpha^{ij}(X)\gamma^{+k}\gamma_j,\end{aligned} \tag{42}$$

reproducing (30) and (34).

Batalin and Vilkovisky show that, in much more general situations, one can add terms $S^i$ of ghost degree $i>1$ to achieve a total $S_{BV}$ such that

$$(S_{BV},S_{BV})=0.$$

The reason for this is that the $d_{KT}$ homology vanishes in appropriate degrees.

In the Cattaneo-Felder model, only one more term is needed:

$$S^2 = \int_\Sigma -\frac{1}{4}\eta^{+i} \wedge \eta^{+j}\partial_i\partial_j\alpha^{kl}(X)\gamma_k\gamma_l. \tag{43}$$

Thus the total Batalin-Vilkovisky generator is

$$\begin{aligned} S_{BV} \;=\; & \int_\Sigma \eta_i \wedge dX^i + \frac{1}{2}\alpha^{ij}(X)\eta_i \wedge \eta_j \\ & + X_i^+\alpha^{ij}(X)\gamma_j - \eta^{+i} \wedge (d\gamma_i + \partial_i\alpha^{kl}(X)\eta_k\gamma_l) - \frac{1}{2}\gamma_+^i\partial_i\alpha^{jk}(X)\gamma_j\gamma_k \\ & - \frac{1}{4}\eta^{+i} \wedge \eta^{+j}\partial_i\partial_j\alpha^{kl}(X)\gamma_k\gamma_l. \end{aligned} \tag{44}$$

## 7. Summary

We hope to have called deserving attention to Noether's second variational theorem and how it accounts for the anti-ghosts which are an essential part of the Batalin-Vilkovisky method. Beyond that, we are now able to show how the terms $S^i$ in the total $S_{BV}$ of the Catanneo-Felder sigma model correspond to the Koszul-Tate, Chevalley-Eilenberg and other parts of the total differential in the BV differential graded algebra. Consider the total differential as found in Cattaneo and Felder:

$$\delta X^i = \alpha^{ij}(X)\gamma_j, \tag{45}$$

$$\delta\eta^{+i} = -dX^i - \alpha^{ij}(X)\eta_j + \partial_k\alpha^{ij}(X)\eta^{+k}\gamma_j, \tag{46}$$

$$\begin{aligned} \delta\gamma^{+i} = & -d\eta^{+i} - \alpha^{ij}(X)X_j^+ + \frac{1}{2}\partial_k\partial_l\alpha^{ij}(X)\eta^{+k} \wedge \eta^{+l}\gamma_j \\ & + \partial_k\alpha^{ij}(X)\eta^{+k} \wedge \eta_j + \partial_k\alpha^{ij}(X)\gamma^{+k}\gamma_j \end{aligned} \tag{47}$$

and

$$\delta\gamma_i = \frac{1}{2}\partial_i\alpha^{kl}(X)\gamma_k\gamma_l, \tag{48}$$

$$\delta\eta_i = -d\gamma_i - \partial_i\alpha^{kl}(X)\eta_k\gamma_l - \frac{1}{2}\partial_i\partial_j\alpha^{kl}(X)\eta^{+j}\gamma_k\gamma_l, \tag{49}$$

$$\begin{aligned} \delta X_i^+ = & \; d\eta_i + \partial_i\alpha^{kl}(X)X_k^+\gamma_l - \partial_i\partial_j\alpha^{kl}(X)\eta^{+j} \wedge \eta_k\gamma_l + \frac{1}{2}\partial_i\alpha^{kl}(X)\eta_k \wedge \eta_l \\ & - \frac{1}{4}\partial_i\partial_j\partial_p\alpha^{kl}(X)\eta^{+j} \wedge \eta^{+p}\gamma_k\gamma_l - \frac{1}{2}\partial_i\partial_j\alpha^{kl}(X)\gamma^{+j}\gamma_k\gamma_l. \end{aligned} \tag{50}$$

These individual terms can be identified as coming from a particular $S^i$. For example, $\delta X^i$ comes from $S^1$, the first two terms of $\delta\eta^{+i}$ come from $S^0$ and the third from $S^1$, as do all the terms of $\delta\gamma^{+i}$ except for the middle term which comes from $S^2$. Similarly, $\delta\gamma_i$ comes from $S^1$, the first two terms of $\delta\eta_i$ come from $S^1$ and the third from $S^2$, while the five terms of $\delta X_i^+$ come from $S^i$ with $i$ respectively $0, 1, 1, 0, 2, 1$.

In contrast, if we identify terms as coming from $d_{KT}$ or $d_{CE}$ we find $\delta X^i$ comes from $d_{CE}$, the first two terms of $\delta\eta^{+i}$ come from $d_{KT}$ and the third from $d_{CE}$, while the first, second and fourth terms of $\delta\gamma^{+i}$ come from $d_{KT}$, the fifth from $d_{CE}$ and the third term is of neither origin. Similarly, $\delta\gamma_i$ comes from $d_{CE}$, as do the first two terms of $\delta\eta_i$ and the third is of neither origin. The first and fourth terms of $\delta X_i^+$ come from $d_{KT}$, the second term comes $d_{CE}$, and the remaining terms come from neither $d_{CE}$ nor $d_{KT}$.

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