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A CONSTRUCTION OF FINITE-DIMENSIONAL FAITHFUL REPRESENTATION OF LIE ALGEBRA

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ABSTRACT. The Ado theorem is a fundamental fact, which has a reputation to be a 'strange theorem'. We give its natural proof.

1. CONSTRUCTION OF FAITHFUL REPRESENTATION

Consider a finite-dimensional Lie algebra $\mathfrak{g}$. Assume that $\mathfrak{g}$ is a semidirect product $\mathfrak{p} \ltimes \mathfrak{n}$ of a subalgebra $\mathfrak{p}$ and a nilpotent ideal $\mathfrak{n}$. Assume that the adjoint action of $\mathfrak{p}$ on $\mathfrak{n}$ is faithful, i.e., for any $z \in \mathfrak{p}$, there exists $x \in \mathfrak{n}$ such that $[z, x] \neq 0$.

Consider the minimal k such that all the commutators

$$[\dots[[x_1, x_2], x_3], \dots, x_k], \qquad x_j \in \mathfrak{n}$$

are 0.

Denote by $\mathcal{U}(\mathfrak{n})$ the enveloping algebra of $\mathfrak{n}$. The algebra $\mathfrak{n}$ acts on $\mathcal{U}(\mathfrak{n})$ by the left multiplications. The algebra $\mathfrak{p}$ acts on $\mathcal{U}(\mathfrak{n})$ by the derivations

$$d_z x_1 x_2 x_3 \dots x_l = [z, x_1] x_2 x_3 \dots x_l + x_1 [z, x_2] x_3 \dots x_l + \dots, \qquad \text{where } z \in \mathfrak{p}.$$

This defines the action of the semidirect product $\mathfrak{p} \ltimes \mathfrak{n} = \mathfrak{g}$ on $\mathcal{U}(\mathfrak{n})$.

Denote by I the subspace in $\mathcal{U}(\mathfrak{n})$ spanned by all the products $x_1 x_2 \dots x_N$, where $N > k + 2$. Obviously,

1. I is the two-side ideal in $\mathcal{U}(\mathfrak{n})$.
2. Consider the linear span $\mathcal{A} \subset \mathcal{U}(\mathfrak{n})$ of 1 and all the $x \in \mathfrak{g}$. Obviously, $I \cap \mathcal{A} = 0$.
3. I is invariant with respect to the derivations d_z.

Obviously, the module $\mathcal{U}(\mathfrak{n})/I$ is a finite-dimensional faithful module over $\mathfrak{g}$.

2. THE ADO THEOREM

Lemma 1. *Any finite-dimensional Lie algebra $\mathfrak{q}$ admits an embedding to an algebra $\mathfrak{g}$ such that*

(a) *$\mathfrak{g}$ is a semidirect product of a reductive subalgebra $\mathfrak{p}$ and a nilpotent ideal $\mathfrak{n}$;*
(b) *the action of $\mathfrak{p}$ on $\mathfrak{n}$ is completely reducible.*

The paper is in final form and no version of it will be submitted elsewhere.

Obviously, Lemma 1 implies the Ado theorem. Indeed, $\mathfrak{g}$ admits a decomposition

$$\mathfrak{g} = \mathfrak{p}' \oplus (\mathfrak{p}'' \ltimes \mathfrak{n})$$

where $\mathfrak{p}'$, $\mathfrak{p}''$ are reductive subalgebras and the action of $\mathfrak{p}''$ on $\mathfrak{n}$ is faithful. After this, it is sufficient to apply the construction of p.1.

REMARK. The Ado theorem implies Lemma 1 modulo the Chevalley construction of algebraic envelope of a Lie algebra. But Lemma 1 itself can be easily proved directly.

3. Killing lemma

Let $\mathfrak{g}$ be a Lie algebra, let d be its derivation. For an eigenvalue λ, denote by $\mathfrak{g}_\lambda$ its root subspace $\mathfrak{g}_\lambda = \cup_k \ker(d-\lambda)^k$; we have $\mathfrak{g} = \oplus \mathfrak{g}_\lambda$. As it was observed by Killing, $x \in \mathfrak{g}_\lambda$, $y \in \mathfrak{g}_\mu$ implies $[x, y] \in \mathfrak{g}_{\lambda+\mu}$.

Thus the Lie algebra $\mathfrak{g}$ admits the gradation by the eigenvalues of d. Consider the gradation operator $d_s : \mathfrak{g} \to \mathfrak{g}$ defined by $d_s v = \lambda v$ if $v \in \mathfrak{g}_\lambda$. Obviously, d_s is a derivation, and $dd_s = d_s d$. We also consider the derivation $d_n := d - d_s$, this operator is nilpotent (the equality $d = d_n + d_s$ is called the Jordan-Chevalley decomposition). Clearly,

$$\ker d_s \supset \ker d; \qquad \ker d_n \supset \ker d; \tag{1}$$

$$\operatorname{im} d_s \subset \operatorname{im} d_s; \qquad \operatorname{im} d_n \subset \operatorname{im} d_s. \tag{2}$$

4. Elementary expansions

Let $\mathfrak{q}$ be a Lie algebra, let I be an ideal of codimension 1. Let $x \notin I$. Denote by d the operator $\mathrm{Ad}_x : I \to I$. Consider the corresponding pair of derivations d_s, d_n. Consider the space

$$\mathfrak{q}' = \mathbb{C}y + \mathbb{C}z + I$$

where y, z are formal vectors. We equip this space with a structure of a Lie algebra by the rule

$$[y, z] = 0, \qquad [y, u] = d_s u, \qquad [z, u] = d_n u, \qquad \text{for all } \; u \in I$$

and the commutator of $u, v \in I$ is the same as it was in I.

The subalgebra $\mathbb{C}(y+z) \oplus I \subset \mathfrak{q}'$ is isomorphic $\mathfrak{q}$. We say that $\mathfrak{q}'$ is an *elementary expansion* of $\mathfrak{q}$.

Obviously, $[\mathfrak{q}', \mathfrak{q}'] = [\mathfrak{q}, \mathfrak{q}]$.

For a general Lie algebra, the required embedding to a semidirect product can be obtained by a sequence of elementary expansions.

5. Proof of Lemma 1

Let $\mathfrak{q}$ be a Lie algebra. Let $\mathfrak{h}$ be its Levi part, and $\mathfrak{r}$ be the radical. Denote by $\mathfrak{m}$ the nilradical of $\mathfrak{q}$, i.e., $\mathfrak{m} = [\mathfrak{q}, \mathfrak{r}]$; recall that $\mathfrak{m}$ is a nilpotent ideal, and $[\mathfrak{q}, \mathfrak{q}] = \mathfrak{h} \ltimes \mathfrak{m}$ (see [1], 1.4.9).

Consider a nilpotent ideal $\mathfrak{n}$ of $\mathfrak{q}$ containing the nilradical $\mathfrak{m}$. Consider a subalgebra $\mathfrak{p} \supset \mathfrak{h}$ such that the adjoint action of $\mathfrak{p}$ on $\mathfrak{q}$ is completely reducible and $\mathfrak{p} \cap \mathfrak{n} = 0$; for instance, the can choice $\mathfrak{n} = \mathfrak{m}$, $\mathfrak{p} = \mathfrak{h}$.

Obviously, the $\mathfrak{q}$-module $\mathfrak{q}/(\mathfrak{p} \ltimes \mathfrak{n})$ is trivial. Consider any subspace I of codimension 1 containing $\mathfrak{p} \ltimes \mathfrak{n}$, obviously I is an ideal in $\mathfrak{q}$. Since the action of $\mathfrak{p}$ on $\mathfrak{q}$ is completely reducible, there exists a $\mathfrak{p}$-invariant complementary subspace for I. Let x be an element of this subspace. Since the $\mathfrak{p}$-module $\mathfrak{q}/I$ is trivial, x commutes with $\mathfrak{p}$. We apply the elementary expansion to these data.

We obtain the new algebra $\mathfrak{q}' = \mathbb{C}y + \mathbb{C}z + I$ with the nilpotent ideal $\mathfrak{n}' = \mathbb{C}z + \mathfrak{n}$ and with the reductive subalgebra $\mathfrak{p}' = \mathbb{C}y \oplus \mathfrak{p}$ (by (1), y commutes with $\mathfrak{p}$).

It remains to notice that

$$\dim \mathfrak{q}' - \dim \mathfrak{p}' - \dim \mathfrak{n}' = \dim \mathfrak{q} - \dim \mathfrak{p} - \dim \mathfrak{n} - 1$$

and we can repeat the same construction.

References

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