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Sigma models in geometry

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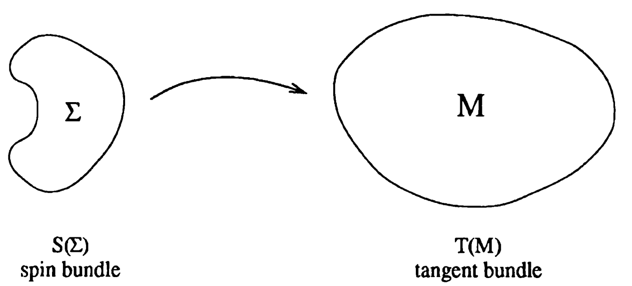
SIGMA MODELS IN GEOMETRY

MARTIN ROCEK

1. Lecture 1

These lectures are lightly edited versions of the actual lectures that I gave at the Srni Winter School in January 2003. I have tried to make them comprehensible to mathematicians, but being a physicist, I cannot say to what extent I have succeeded. There are plenty of good references for people who want to study more about supersymmetry – several books, recent lectures by Ulf Lindström at a previous Srni Winter School, etc.

The basic question that we will study in these lectures is: "What can we learn about the geometry of a manifold by studying maps into the manifold and/or into bundles over the manifold?" For example, studying maps from S^n into a manifold M tells us about the homotopy of the manifold M. The problem of maps of surfaces into a general manifold is known as the harmonic map problem. As shown in the figure, we consider a manifold Σ with possible additional structure such as the spin bundle $S(\Sigma)$; these are mapped into the manifold M, and into possible additional structure on M such as the tangent bundle $T(M)$:



By studying the properties of Σ and the maps of Σ into M we want to learn about the geometry of M.

An important example that we'll explore in detail is the case when Σ is a supermanifold. Then the maps of Σ into M tell us about the holonomy of M.

The paper is in final form and no version of it will be submitted elsewhere.

1.1. **Supersymmetry.** Supersymmetry can be viewed as a relation between the spectrum of first and second order operators, for example, between the Dirac operator $\not{D}$ which acts on $\Gamma(S)$ (that is on sections of the spin bundle) and the d'Alembertian $\Box$ which act on zero forms ω_0. This relation can be seen by taking a solution of the Dirac equation

$$(\not{D} - m)\,\psi = 0\,. \tag{1}$$

The square of the Dirac operator (in flat space) is the d'Alembertian. Using a parallel spinor ϵ, that is, $\epsilon \in \Gamma(S)$ such that $D\epsilon = 0$, we can construct a solution to the Klein-Gordon equation by picking out the zero form piece in the tensor product of ϵ and ψ. We can do this as sections of the spin bundle transform under the spin $\frac{1}{2}$ representation of the spin group; we are simply taking the spin 0 piece of the tensor product of spin $\frac{1}{2}$ with itself. In terms of gamma matrices, we write this as

$$\omega_0 = \bar{\epsilon}\psi\,. \tag{2}$$

Thus, for each parallel spinor, we get a solution to the Klein-Gordon equation

$$\left(\Box - m^2\right)\omega_0 = 0\,, \tag{3}$$

from a solution of the Dirac equation.

Why is this supersymmetry according to "standard" point of view? This is a slightly unusual way of introducing supersymmetry, which is usually described as an invariance of an action functional under certain transformations. Such an action leads to a variational problem that is solved by the solutions to certain differential equations; supersymmetry as introduced above is then a relation among such solutions. In the particular example that we just considered, the action functional is the sum of the Dirac and Klein-Gordon actions. The advantage of the description we used is that it requires less background machinery, but we shall need to discuss actions later on.

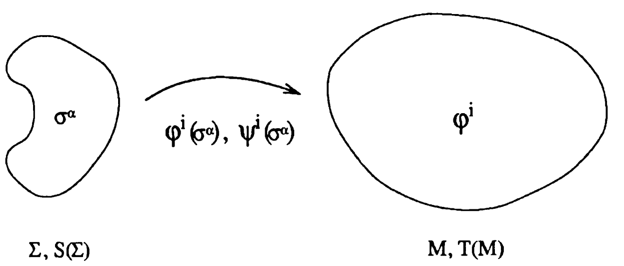
Another example of supersymmetry arises when we construct a one-form ω_1, satisfying $d \star d\omega_1 = 0$, from solutions of the Dirac equation ψ and a parallel spinor ϵ. This we get by choosing the one-form piece in the tensor product of ψ and ϵ. Using gamma matrices we can do this by taking the product

$$\bar{\epsilon}\gamma^{\mu}\psi\,. \tag{4}$$

This corresponds to a "vector" super-multiplet while the previous construction corresponds to a "scalar" super-multiplet.

What we have discussed so far is called rigid supersymmetry. Some of these notions can be generalized to the case where the spinor ϵ is not parallel, and can be an arbitrary section of the spin bundle; this requires more structure (i.e., supergravity) and is called local supersymmetry.

We now turn to supersymmetric sigma models. This gives us a nonlinear generalization of the scalar multiplet that we have just discussed. As shown in the figure below – which is a particular example of the general structure that we saw above – we define, in addition to the maps $\varphi^i(\sigma)$ from Σ into M, Grassmann odd (anticommuting) maps $\psi^i(\sigma)$ from $S(\Sigma)$ to $T(M)$ at the point $\varphi(\sigma)$:



For concreteness, we focus on the case when Σ is 3−dimensional. The supersymmetric multiplet involved is once more a scalar multiplet. The action is given by the pullback by $\varphi(\sigma)$ of the length of $d\varphi$, a Dirac-term involving the pullback of the connection on M, and a term constructed from the Riemann tensor:

$$\begin{aligned} S &= \frac{1}{4\pi}\int_\Sigma \varphi^*\left(|d\varphi|^2 + \langle\psi,\not{D}\psi\rangle + \tfrac{1}{4}Riem(\psi\otimes\psi\otimes\psi\otimes\psi)_0\right) \\ &= \frac{1}{4\pi}\int_\Sigma \Big(g_{ij}(\varphi)\left[\partial\varphi^i\cdot\partial\varphi^j + \bar\psi^i(\not\partial\psi^j + \Gamma^j_{km}(\not\partial\varphi^k)\psi^m)\right] \\ &\qquad\qquad + \tfrac{1}{6}R_{ijkm}(\bar\psi^i\psi^k)(\bar\psi^j\psi^m)\Big)\ . \end{aligned} \tag{5}$$

Where is the supersymmetry? It can be found as a symmetry of the action under the transformations

$$\delta\varphi^i \;\propto\; (\epsilon\otimes\psi^i)_0 = \bar\epsilon\psi^i \tag{6}$$

$$\delta\psi^i \;\propto\; (\not\partial\varphi^i + \Gamma^i_{jk}(\not\partial\varphi^j)\psi^k)\epsilon\,. \tag{7}$$

As explained above, these transformations can also be used to relate solutions to the extremal equations for φ and ψ, which are now the generalized Laplace and Dirac operators, respectively.

If M has a *complex structure* J, this construction can be generalized. In this case one can define more than one supersymmetry by using the complex structure

$$\delta\varphi^i \propto \varphi^*J[(\epsilon\otimes\psi)_0]^i = J^i{}_j(\bar\epsilon\psi^j)\,. \tag{8}$$

This leads to an interesting relation between the number of supersymmetries and the holonomy of M.

# SUSY's	dim(M)	Hol(M)	Type
0,1	n	$O(n)$	Riemann
2	n = 2m	$U(m)$	Rigid: Kähler Local: Hodge
3,4	n=4k	$Sp(k), Sp(1)\times Sp(k)$	Rigid: Hyperkähler Local: Quaternionic Kähler

TABLE 1. The relation between supersymmetries and holonomy.

Thus extra structure on M leads to extra supersymmetry on Σ. We will turn this around and use the supersymmetry on Σ to construct manifolds of special holonomy.

1.2. **Superspace.** Instead of using the fields φ^i and ψ^i separately, we can combine them into one object, a supermultiplet or a superfield. We do this by combining the spaces Σ and $S(\Sigma)$ into a single space, a so-called superspace. To describe this space we choose additional coordinates θ on $S(\Sigma)$ which are Grassman odd. Superfunctions are functions of these coordinates and since θ are odd, if we make a Taylor expansion in them, the expansion always ends after a finite number of terms, e.g.,

$$\Phi^i(\sigma,\theta) = \varphi^i(\sigma) + (\theta \otimes \psi^i)_0 + F^i(\theta \otimes \theta)_0 . \tag{9}$$

The number of spinor θ coordinates introduced is given by the number N of supersymmetries in the problem. Thus, when Σ is 3−dimensional, for $N = 1$ supersymmetry, there is one spinor θ coordinate with two independent Grassman odd components. Now a supertransformation can be written very compactly as

$$\delta\Phi^i = (\epsilon \otimes Q\Phi^i)_0 , \tag{10}$$

where Q is an differential operator defined by

$$Q \propto \frac{\partial}{\partial\theta} - \theta \otimes \frac{\partial}{\partial\sigma} . \tag{11}$$

This obeys the important relation

$$Q \otimes Q = -\not{\partial} , \tag{12}$$

and thus the supersymmetry generator Q is a "square root" of the translation generator. One may also construct an object which commutes with the SUSY transformation generator Q

$$D \propto \frac{\partial}{\partial\theta} + \theta \otimes \frac{\partial}{\partial\sigma} . \tag{13}$$

Since D anticommutes with Q $(DQ + QD = 0)$ we have that

$$D(\delta\Phi^i) = \delta(D\Phi^i) . \tag{14}$$

This allows us to use D when we construct supersymmetric actions. For example, in $N = 1$ superspace the action functional for a number of scalar multiplets can be written

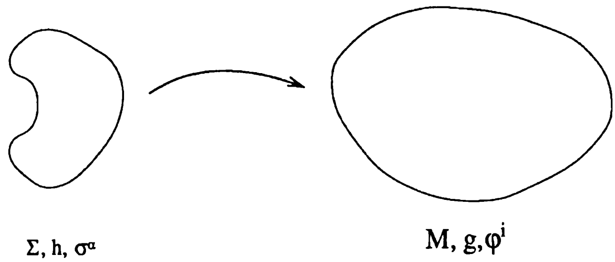
$$\int_\Sigma (D \otimes D)_0 [(\langle D\Phi, D\Phi\rangle)_0] . \tag{15}$$

For $N = 2$ supersymmetry, it is natural to combine the two linearly independent spinor derivatives D into complex combinations; then the action can be written suggestively as

$$\int_\Sigma (D \otimes D)_0 (\bar{D} \otimes \bar{D})_0 \left[K(\Phi^i, \bar{\Phi}^j)\right] . \tag{16}$$

2. LECTURE 2: QUOTIENTS

In this lecture we include a metric on both the source manifold Σ and on the target manifold M. The metric on M is called g and the metric on Σ is called h.



Then we can write an action for the immersion φ as

$$\int_\Sigma |d\varphi|^2_{g,h} = \int_\Sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha \varphi^i \partial_\beta \varphi^k g_{ik}(\varphi) . \tag{17}$$

Now consider a target space manifold M that has a Killing vector field X, that is, a vector field X such that

$$\mathcal{L}_X g = 0 . \tag{18}$$

(In general, the Lie derivative of g gives the change of the metric under an infinitesimal shift in the coordinates along a vector field; thus a Killing vector field preserves the metric). Now we would like to find the quotient of the space M by the action of the vector field X. The points on the quotient space are the orbits of the action of the vector field X. The isometry generated by X induces an invariance of the action (17). Thus we can define a $U(1)$ action (if the orbits are compact; otherwise, we have a real $GL(1)$ action) under which the sigma model (17) is invariant. To define a sigma model on the quotient manifold, *we gauge this action.* We can understand intuitively that this is the correct thing to do since gauging the action means that we can locally choose any representative of any orbit. Thus the action depends only on the orbits and must be well defined on the quotient. The gauging is performed by introducing a $U(1)$ connection 1-form on Σ

$$d\varphi^i \to d_A \varphi^i \equiv d\varphi^i + AX^i(\varphi) . \tag{19}$$

Under the shift $\varphi^i \to \varphi^i + \alpha X^i$ the connection 1-form transforms as $A \to A - d\alpha$, leaving the action invariant. The connection can be eliminated from the action by using its equation of motion, that is, by extremizing the action with respect to the connection. This gives a sigma model on the quotient manifold with the canonical induced metric. One might wonder why we did not include a kinetic term such as $|dA|^2$ for the gauge field. Such a term is unimportant for sufficiently slowly varying configurations, and leads to interactions that do not have the form (17) of a pure sigma model; it is interesting in some contexts, but is not relevant to the discussion of quotients.

We may introduce supersymmetry into the game as in the last lecture; the classification in table 1.1 stays intact, and thus supersymmetric quotients can be used to find explicit metrics on new manifolds of special holonomy.

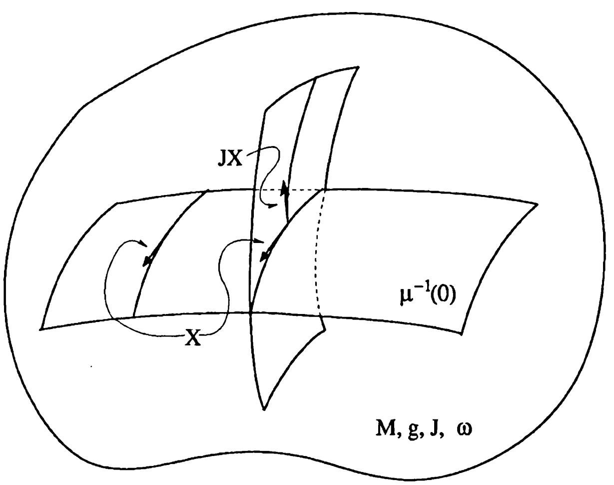
2.1. **Symplectic reduction.** We now focus on the case when M is a Kähler manifold with Kähler form ω, and the isometry preserves both the metric and ω: $\mathcal{L}_X g = \mathcal{L}_X \omega = 0$. Because the Kähler form is closed, we can define the moment map μ by

$$d\mu^X = i_X\omega\,, \tag{20}$$

where $i_X\omega$ is the contraction of the 2-form ω with the vector X, and is thus a 1-form. Now we define the Kähler quotient to be the quotient with respect to the isometry generated by the Killing field X of the submanifold $(\mu^X)^{-1}(0)$ defined by the zero set of the moment map. An alternative but equivalent way to define the Kähler quotient is as the quotient of (most of) M with respect to the action of the *complexified* vector field $\{X, JX\}$. Because of the definition of μ, the action of X always lies within $(\mu^X)^{-1}(0)$ whereas the action of JX takes us out of this submanifold. This can be written as

$$(\mu^X)^{-1}(0)/G \equiv \text{“}M\text{”}/G^*\,, \tag{21}$$

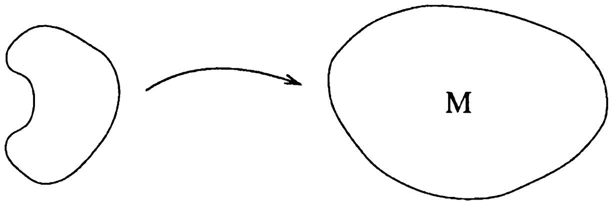
where G^* is the complexified gauge group and "M" denotes the stable submanifold of M, which consists of all the points in M that can be reached by the action of G^* on $(\mu^X)^{-1}(0)$. In the figure below, we see the orbits generated by X lying in $(\mu^X)^{-1}(0)$ as well as the complexified orbits generated by X and JX:



The definition of the Kähler quotient as the complexified quotient of the stable submanifold arises automatically in superspace. Consider the case when Σ is a three (bosonic) dimensional $N = 1$ superspace. As discussed at the end of the previous lecture, the coordinates on Σ are σ^α, $\alpha = 0, 1, 2$ and the grassman coordinates θ^A, $A = 1, 2$. Both transform in representations of $SO(1,2) \sim SL(2,R)$, σ in the vector representation and θ in the spinor representation (which is real in three dimensions). We also have the supercovariant derivative D_A which is Grassman odd and obeys

$$\{D_A, D_B\} = 2i\partial\!\!\!/_{AB} \tag{22}$$

where $\not\partial_{AB}$ is the Dirac operator on Σ (which is assumed to be flat). $N=1$ superfields on Σ are now functions $\Phi^i(\sigma,\theta)$ from the superspace Σ into M (which is still assumed to be an ordinary manifold).



Σ_s, σ^α, θ^A

If we Taylor expand the superfield we find

$$\Phi^i\big|_{\theta=0} = \varphi^i(\sigma) \tag{23}$$
$$D_A\Phi^i\big|_{\theta=0} = \psi^i_A(\sigma). \tag{24}$$

Here we recognize φ^i as the immersion we started with and ψ^i_A is a map from the spin bundle $S(\Sigma)$ on Σ to the tangent bundle $T(M)$ on M at the point φ^i. We can use the symplectic metric on the spin bundle ϵ^{AB} to build an action

$$\int_\Sigma D_C\epsilon^{CE}D_E\left[D_A\Phi^i\epsilon^{AB}D_B\Phi^k g_{ik}(\Phi)\right]. \tag{25}$$

For $N=2$ supersymmetry, we complexify the Grassman coordinate θ^A and add the conjugate $\bar\theta^A$ to our superspace Σ. There is consequently a second supercovariant derivative $\bar D_A$ and the algebra of derivatives is

$$\{D_A, D_B\} = 0 \tag{26}$$
$$\left\{\bar D_A, \bar D_B\right\} = 0 \tag{27}$$
$$\left\{D_A, \bar D_B\right\} = i\not\partial_{AB} \tag{28}$$

In this case the scalar superfield Φ^i is no longer an irreducible representation of the supersymmetry algebra. It can be decomposed into chiral and antichiral superfields defined by

$$\bar D\Phi = 0 \text{ chiral} \tag{29}$$
$$D\bar\Phi = 0 \text{ antichiral} \tag{30}$$

The superfields Φ^i and $\bar\Phi^i$ are now mapped into the complex coordinates on the manifold M, which is necessarily Kähler. The complex Grassman coordinates θ and $\bar\theta$ get tied to the complex structure on M:

$$\begin{array}{rclrcl} \Phi^i\big| &=& \varphi^i & \bar\Phi^i\big| &=& \bar\varphi^i \\ D_A\Phi^i\big| &=& \psi^i_A & \bar D_A\bar\Phi^i\big| &=& \bar\psi^i_A \\ \bar D_A\Phi^i\big| &=& 0 & D_A\bar\Phi^i\big| &=& 0 \end{array} \tag{31}$$

In the $N=4$ case we have four (real) Grassman coordinates, which naturally leads to quaternions and hyperkähler geometry on the target space.

To perform the superquotient we need a super version of a connection. To this end we consider another representation of supersymmetry, namely a real scalar superfield $V = \bar{V}$. Under gauge transformations V transforms as $V \to V - i\Lambda + i\bar{\Lambda}$ where $(\bar{\Lambda})$ Λ is some arbitrary (anti) chiral superfield. Using the gauge transformations some components of V can be gauged away. For instance, we can choose

$$V| = 0 \tag{32}$$
$$D_A V| = 0 \tag{33}$$
$$\bar{D}_A V| = 0 \tag{34}$$

and under the gauge transformation the component $[D_A, \bar{D}_B]\, V|$ transforms as

$$[D_A, \bar{D}_B]\, V \to [D_A, \bar{D}_B]\, (V - i(\Lambda + \bar{\Lambda})) = [D_A, \bar{D}_B]\, V + \not{\partial}_{AB}(\Lambda + \bar{\Lambda}) \tag{35}$$

i.e., it transforms as an ordinary gauge field (defining as usual $\lambda \equiv \Lambda|$, etc.). Thus we may define

$$[D_A, \bar{D}_B]\, V| = i\not{A}_{AB} \tag{36}$$

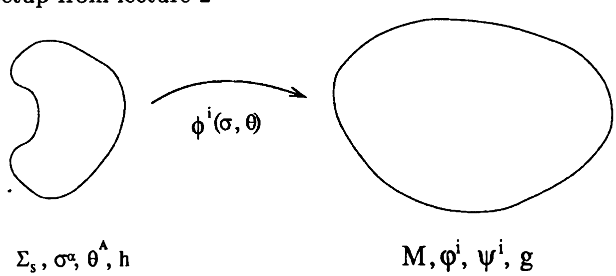
with

$$A \to A - i\, d(\lambda - \bar{\lambda})\,. \tag{37}$$

We continue this discussion in the next lecture.

3. Lecture 3

Recall our setup from lecture 2



In the $N = 2$ case the action takes the form

$$S = \int_\Sigma D^2 \bar{D}^2 K(\Phi, \bar{\Phi})\,. \tag{38}$$

For example, the supersymmetric sigma model for C^n is given by its Kähler potential $K = \sum_{i=1}^n \Phi^i \bar{\Phi}^{\bar{i}}$. We can get CP^{n-1} from C^n via a (symplectic) quotient. The complexified $U(1)$ action is given by

$$\Phi^i \to g\Phi^i \tag{39}$$
$$\bar{\Phi}^i \to g^\dagger \bar{\Phi}^i \tag{40}$$

Notice that g is the complexified group element so that $gg^\dagger \neq 1$. In fact, we can see that in order to preserve the chiral properties of Φ we have to choose g to also be chiral:

$\bar{D}g = 0$. Thus we can parametrize g as $g = e^{i\Lambda}$ where Λ is a chiral field: $\bar{D}\Lambda = 0$. The action as it stands is not gauge invariant since it transforms as

$$\sum_{i=1}^{n} \Phi^i \bar{\Phi}^i \rightarrow \sum_{i=1}^{n} \Phi^i \bar{\Phi}^i e^{i(\Lambda - \Lambda^\dagger)} . \tag{41}$$

To make it gauge invariant we have to introduce the gauge field V which transforms as $V \rightarrow V - i(\Lambda - \Lambda^\dagger)$ under gauge transformations. This gives the gauge invariant action

$$\int_\Sigma D^2 \bar{D}^2 \sum_{i=1}^{n} \Phi^i \bar{\Phi}^i e^V . \tag{42}$$

Also, since the measure in the action includes integration over all superspace coordinates, we could also include a gauge invariant term $\int_\Sigma D^2 \bar{D}^2 cV$ for some constant c. The action now depends only on the orbits of the action of the complexified gauge group, which is the quotient. The gauge field V can be removed by extremizing the action with respect to it. Variation with respect to V gives

$$\sum_{i=1}^{n} \Phi^i \bar{\Phi}^i e^V - c = 0 , \tag{43}$$

or

$$V = \ln \frac{c}{\sum \Phi \bar{\Phi}} . \tag{44}$$

This V gives the complexified gauge transformation that takes an arbitrary stable point in M down to the submanifold $\sum_{i=1}^{n} \Phi^i \bar{\Phi}^i - c$, namely $\mu^{-1}(0)$. Substituting back into the action gives

$$\int_\Sigma D^2 \bar{D}^2 \left(\sum_i \Phi^i \bar{\Phi}^i \frac{c}{\sum \Phi \bar{\Phi}} - c \ln \frac{c}{\sum \Phi \bar{\Phi}} \right) , \tag{45}$$

and thus to a Kähler potential for the quotient space CP^{n-1}

$$K = \text{const.} + c \ln \sum \Phi \bar{\Phi} \tag{46}$$

which we recognize as the Kähler potential for the Fubini-Study metric on CP^{n-1}.

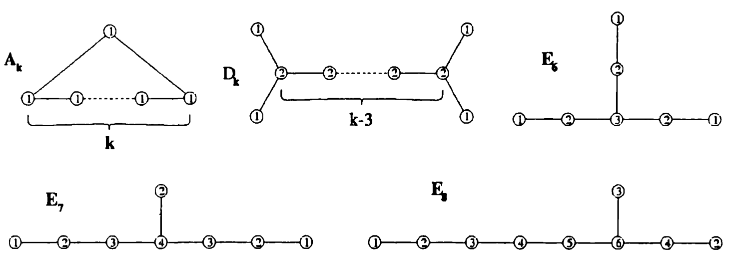
The algebraic geometric way to see the quotient is to define it as the solution of an *invariant* holomorphic polynomial equation. In $N = 4$ supersymmetry (without $N = 4$ superspace) this becomes relevant since there we have an S^2's worth of complex structures which means that we have an S^2's worth of moment maps. We choose the complex structure ω_3 so that $\omega_1 \pm i\omega_2$ is a holomorphic (antiholomorphic) two form. The basic supersymmetry representations are now given by hypermultiplets which in $N = 2$ language consist of two chiral superfields Φ_+ and Φ_-, and vector multiples which consist of a chiral superfield S and an $N = 1$ vector multiplet V. The action for a quotient of some flat quaternionic space described as a complex even-dimensional flat space is given by

$$\int_\Sigma D^2\bar{D}^2\left(\sum \Phi_+\bar{\Phi}_+e^V + \sum \Phi_-\bar{\Phi}_-e^{-V} - cV\right) \tag{47}$$
$$+ \int_\Sigma D^2\left(\sum \Phi_+\Phi_- - b\right) S + c.c.$$

(where *c.c.* stands for complex conjugate) and the variation with respect to S gives the holomorphic constraint

$$\sum \Phi_+\Phi_- = b\,. \tag{48}$$

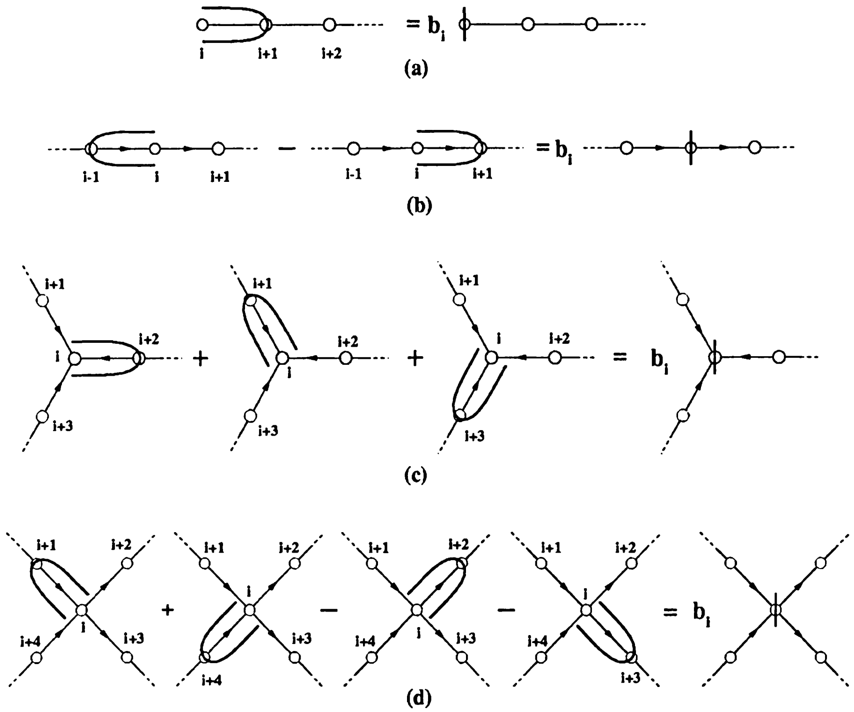
This is called the holomorphic moment map. This can be encoded in a so called "quiver diagram". A quiver diagram is essentially a labelled graph, for example



where each node (with label n_i) represents a complex vector space $V_i = C^{n_i}$ transforming under the gauge group $G_i = U(n_i)$. Links represent maps between the vector spaces. For instance a link between the node i and the node k represents a map $\Phi_+ \in \mathrm{Hom}V_i \to V_k$ and a map $\Phi_- \in \mathrm{Hom}V_k \to V_i$ which carry canonical G_i and G_k actions. The quiver diagram contains the gauge theory data that we use to set up the quotient construction. Therefore it is interesting to study if different quivers can give rise to the same quotient. This is in fact the case. The equivalent quivers are related by Seiberg duality which can be defined as an operation that takes quivers into other quivers that are equivalent in the sense that they give the same quotient. The operation acts on any node with label n_i in a diagram by replacing it with a node labelled by the number we get by summing over all n_j of the nodes connected to the node n_i and then subtracting n_i.

One can ask which diagrams are self dual? The answer turns out to be exactly the diagrams that correspond to the extended Dynkin diagrams of the ADE groups. Equivalently, this means that the only symmetric matrices with 2 along the diagonal, nonpositive integer coefficients off-diagonal, and a zero eigenvalue are the Cartan matrices of the affine ADE Lie algebras.

Invariant polynomials can be represented as closed contours on the quiver. The holomporphic moment map can also be described in this diagrammatic fashion. This is perhaps best done in a graph



The quotient space is given by the ring of contours on the quiver modulo the ideal generated by the holomorphic moment map relations.

It is clear the Seiberg duality gives a large class of diagrams (i.e. gauge theories) that define the same quotient. For instance, one may always add a node with index 0 and the perform Seiberg duality on that node. There are many interesting open questions here. Can one find a description that is manifestly invariant under Seiberg duality? Can one define a "minimal" quiver in any equivalence class under Seiberg duality?

Acknowledgement. I am happy to thank the organizers of the school, and Rikard von Unge for producing a first draft of these notes.

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