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ON THE CURVATURE OF TENSOR PRODUCT CONNECTIONS AND COVARIANT DIFFERENTIALS

JOSEF JANYŠKA

ABSTRACT. We give coordinate formula and geometric description of the curvature of the tensor product connection of linear connections on vector bundles with the same base manifold. We define the covariant differential of geometric fields of certain types with respect to a pair of a linear connection on a vector bundle and a linear symmetric connection on the base manifold. We prove the generalized Bianchi identity for linear connections and we prove that the antisymmetrization of the second order covariant differential is expressed via the curvature tensors of both connections.

INTRODUCTION

In the theory of linear symmetric (classical) connections on a manifold there are many very well known identities of the curvature tensor (see for instance [1, 4]). Some of these identities can be generalized for any linear connection on a vector bundle.

In this paper we give the coordinate formula for the curvature of the tensor product connection $K \otimes K'$ of two linear connections K or K' on vector bundles $\boldsymbol{E} \to \boldsymbol{M}$ or $\boldsymbol{E}' \to \boldsymbol{M}$, respectively, and we give also the geometric description of this curvature. We prove that the curvature of $K \otimes K'$ is determined by the curvatures of K and K'.

The above results are used in the case if one of linear connections is a classical (linear and symmetric) connection on the base manifold. We introduce the covariant differential of sections of tensor products (over the base manifold) of a vector bundle, its dual vector bundle, the tangent and the cotangent bundles of the base manifold. We prove that such (first order) covariant differential of the curvature tensor of a linear connection satisfies the generalized Bianchi identity and that the antisymmetrization of the second order covariant differential is expressed through the curvatures of linear and classical connections.

All manifolds and maps are supposed to be smooth.

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1. Linear connections on vector bundles

Let $p : \boldsymbol{E} \to \boldsymbol{M}$ be a vector bundle. Local linear fiber coordinate charts on $\boldsymbol{E}$ will be denoted by (x^λ, y^i). The corresponding base of local sections of $\boldsymbol{E}$ or $\boldsymbol{E}^*$ will be denoted by b_i or b^i, respectively.

Definition 1.1. We define a *linear connection* on $\boldsymbol{E}$ to be a linear splitting

$$K : \boldsymbol{E} \to J^1\boldsymbol{E}.$$

Proposition 1.2. *Considering the contact morphism $J^1\boldsymbol{E} \to T^*\boldsymbol{M} \otimes T\boldsymbol{E}$ over the identity of $T\boldsymbol{M}$, a linear connection can be regarded as a $T\boldsymbol{E}$-valued 1-form*

$$K : \boldsymbol{E} \to T^*\boldsymbol{M} \otimes T\boldsymbol{E}$$

projecting on the identity of $T\boldsymbol{M}$.

The coordinate expression of a linear connection K is of the form

$$K = d^\lambda \otimes \left(\partial_\lambda + K_j{}^i{}_\lambda \, y^j \, \partial_i\right), \qquad \textit{with} \qquad K_j{}^i{}_\lambda \in C^\infty(\boldsymbol{M}, I\!R).$$

Definition 1.3. The *covariant differential* of a section $\Phi : \boldsymbol{M} \to \boldsymbol{E}$ with respect to K is defined to be

$$\nabla^K \Phi = j^1\Phi - K \circ \Phi : \boldsymbol{M} \to \boldsymbol{E} \underset{\boldsymbol{M}}{\otimes} T^*\boldsymbol{M}.$$

Remark 1.4. From the affine structure of $\pi^1_0 : J^1\boldsymbol{E} \to \boldsymbol{E}$ we obtain that the difference $j^1\Phi - K \circ \Phi$ lies in the associated vector bundle $V\boldsymbol{E} \otimes T^*\boldsymbol{M}$. From $V\boldsymbol{E} = \boldsymbol{E} \underset{\boldsymbol{M}}{\times} \boldsymbol{E}$ we get the above Definition 1.3.

Let $\Phi = \phi^i \, \mathsf{b}_i$, then we have the coordinate expression

$$\nabla^K \Phi = (\partial_\lambda \phi^i - K_j{}^i{}_\lambda \phi^j) \, \mathsf{b}_i \otimes d^\lambda.$$

Definition 1.5. The *curvature* of a linear connection K on $\boldsymbol{E}$ turns out to be the vertical valued 2-form

$$R[K] = -[K, K] : \boldsymbol{E} \to V\boldsymbol{E} \otimes \Lambda^2 T^*\boldsymbol{M},$$

where $[,]$ is the Froelicher-Nijenhuis bracket.

The coordinate expression is

$$\begin{aligned} R[K] &= R[K]_j{}^i{}_{\lambda\mu} \, y^j \, \partial_i \otimes d^\lambda \wedge d^\mu \\ &= -2(\partial_\lambda K_j{}^i{}_\mu + K_j{}^p{}_\lambda K_p{}^i{}_\mu) \, y^j \, \partial_i \otimes d^\lambda \wedge d^\mu, \end{aligned}$$

i.e. the coefficients of the curvature are

$$R[K]_j{}^i{}_{\lambda\mu} = \partial_\mu K_j{}^i{}_\lambda - \partial_\lambda K_j{}^i{}_\mu + K_j{}^p{}_\mu K_p{}^i{}_\lambda - K_j{}^p{}_\lambda K_p{}^i{}_\mu.$$

If we consider the identification $V\boldsymbol{E} = \boldsymbol{E} \underset{\boldsymbol{M}}{\times} \boldsymbol{E}$ and linearity of $R[K]$, the curvature $R[K]$ can be considered as a tensor field (the curvature tensor field) $R[K] : \boldsymbol{M} \to \boldsymbol{E}^* \otimes \boldsymbol{E} \otimes \Lambda^2 T^*\boldsymbol{M}$.

Theorem 1.6. *We have the generalized Bianchi identity*

$$[K, R[K]] = 0.$$

Proof. It follows immediately from the graded Jacobi identity for the Froelicher-Nijenhuis bracket. □

We have, [3],

Proposition 1.7. *Let K be a linear connection on $\boldsymbol{E}$. Then, there is a unique linear connection $K^* : \boldsymbol{E}^* \to J^1\boldsymbol{E}^*$ on the dual vector bundle $\boldsymbol{E}^* \to \boldsymbol{M}$ such that the following diagram commutes*

$$\begin{array}{ccc} \boldsymbol{E} \underset{\boldsymbol{M}}{\times} \boldsymbol{E}^* & \xrightarrow{\langle,\rangle} & \boldsymbol{M} \times \mathbb{R} \\ {\scriptstyle K\times K^*}\downarrow & & \downarrow{\scriptstyle 0\times \mathrm{id}_{\mathbb{R}}} \\ J^1\boldsymbol{E} \underset{\boldsymbol{M}}{\times} J^1\boldsymbol{E}^* & \xrightarrow{J^1\langle,\rangle} & T^*\boldsymbol{M} \times \mathbb{R} \end{array}$$

Its coordinate expression is

$$K^* = d^\lambda \otimes \left(\partial_\lambda - K_i{}^j{}_\lambda \, y_j \, \partial^i\right), \qquad \textit{with} \qquad K_i{}^j{}_\lambda \in C^\infty(\boldsymbol{M}, \mathbb{R}),$$

where (x^λ, y_i) are the induced linear fiber coordinates on $\boldsymbol{E}^*$ and $\partial^i = \partial/\partial y_i$.

Definition 1.8. The connection K^* is said to be the *dual connection of K*.

Proposition 1.9. *We have $R[K^*] : \boldsymbol{M} \to \boldsymbol{E} \underset{\boldsymbol{M}}{\otimes} \boldsymbol{E}^* \underset{\boldsymbol{M}}{\otimes} \Lambda^2 T^*\boldsymbol{M}$ and*

$$R[K^*]^i{}_{j\lambda\mu} = -R[K]_j{}^i{}_{\lambda\mu}.$$

2. Tensor product linear connections

Let $p' : \boldsymbol{E}' \to \boldsymbol{M}$ be another vector bundle. Local linear fiber coordinate charts on $\boldsymbol{E}'$ will be denoted by (x^λ, z^a). The corresponding base of local sections of $\boldsymbol{E}'$ or $\boldsymbol{E}'^*$ will be denoted by $\mathfrak{b}'_a$ or $\mathfrak{b}'^a$, respectively.

Consider a linear connection K' on $\boldsymbol{E}'$ with coordinate expression

$$K' = d^\lambda \otimes \left(\partial_\lambda + K'_b{}^a{}_\lambda \, z^b \, \partial_a\right), \qquad \text{with} \qquad K'_b{}^a{}_\lambda \in C^\infty(\boldsymbol{M}, \mathbb{R}).$$

Let us consider the tensor product $\boldsymbol{E} \underset{\boldsymbol{M}}{\otimes} \boldsymbol{E}' \to \boldsymbol{M}$ with the induced fiber linear coordinate chart (x^λ, w^{ia}). We have, [3],

Proposition 2.1. *Let K be a linear connection on $\boldsymbol{E}$ and K' be a linear connection on $\boldsymbol{E}'$. Then, there is a unique linear connection $K \otimes K' : \boldsymbol{E} \underset{\boldsymbol{M}}{\otimes} \boldsymbol{E}' \to J^1(\boldsymbol{E} \underset{\boldsymbol{M}}{\otimes} \boldsymbol{E}')$ such that the following diagram commutes*

$$\begin{array}{ccc} \boldsymbol{E} \underset{\boldsymbol{M}}{\times} \boldsymbol{E}' & \xrightarrow{\otimes} & \boldsymbol{E} \underset{\boldsymbol{M}}{\otimes} \boldsymbol{E}' \\ {\scriptstyle K\times K'}\downarrow & & \downarrow{\scriptstyle K\otimes K'} \\ J^1\boldsymbol{E} \underset{\boldsymbol{M}}{\times} J^1\boldsymbol{E}' & \xrightarrow{J^1\otimes} & J^1(\boldsymbol{E} \underset{\boldsymbol{M}}{\otimes} \boldsymbol{E}') \end{array}$$

Its coordinate expression is

$$K \otimes K' = d^\lambda \otimes \left(\partial_\lambda + (K_j{}^i{}_\lambda w^{ja} + K'_b{}^a{}_\lambda w^{ib})\partial_{ia}\right).$$

Definition 2.2. The connection $K \otimes K'$ is said to be the *tensor product connection* of K and K'.

Remark 2.3. We remark that this concept was introduced in another way in [2], p. 381.

The tensor product connection is linear, so we can define its tensor product connection with another linear connection and we have by iteration

Proposition 2.4. *A linear connection K on $\boldsymbol{E}$ and a linear connection K' on $\boldsymbol{E}'$ induce the linear tensor product connection $K^p_q \otimes K'^r_s := \otimes^p K \otimes \otimes^q K^* \otimes \otimes^r K' \otimes \otimes^s K'^*$ on $\otimes^p \boldsymbol{E} \underset{\boldsymbol{M}}{\otimes} \otimes^q \boldsymbol{E}^* \underset{\boldsymbol{M}}{\otimes} \otimes^r \boldsymbol{E}' \underset{\boldsymbol{M}}{\otimes} \otimes^s \boldsymbol{E}'^*$ with coordinate expression*

$$\begin{aligned} K^p_q \otimes K'^r_s = d^\lambda \otimes \Big(\partial_\lambda + (&K_k{}^{i_1}{}_\lambda\, w^{ki_2\dots i_p a_1\dots a_r}_{j_1\dots j_q b_1\dots b_s} + \dots + K_k{}^{i_p}{}_\lambda\, w^{i_1\dots i_{p-1} k a_1\dots a_r}_{j_1\dots j_q b_1\dots b_s} \\ &- K_{j_1}{}^k{}_\lambda\, w^{i_1\dots i_p a_1\dots a_r}_{k j_2\dots j_q b_1\dots b_s} - \dots - K_{j_q}{}^k{}_\lambda\, w^{i_1\dots i_p a_1\dots a_r}_{j_1\dots j_{q-1} k b_1\dots b_s} \\ &+ K'_c{}^{a_1}{}_\lambda\, w^{i_1\dots i_p c a_2\dots a_r}_{j_1\dots j_q b_1\dots b_s} + \dots + K'_c{}^{a_r}{}_\lambda\, w^{i_1\dots i_p a_1\dots a_{r-1} c}_{j_1\dots j_q b_1\dots b_s} \\ &- K'_{b_1}{}^c{}_\lambda\, w^{i_1\dots i_p a_1\dots a_r}_{j_1\dots j_q c b_2\dots b_s} - \dots - K'_{b_s}{}^c{}_\lambda\, w^{i_1\dots i_p a_1\dots a_r}_{j_1\dots j_q b_1\dots b_{s-1} c})\, \partial^{j_1\dots j_q b_1\dots b_s}_{i_1\dots i_p a_1\dots a_r}\Big) \end{aligned}$$

where $(x^\lambda, w^{i_1\dots i_p a_1\dots a_r}_{j_1\dots j_q b_1\dots b_s})$ are the induced linear fiber coordinates on $\otimes^p \boldsymbol{E} \underset{\boldsymbol{M}}{\otimes} \otimes^q \boldsymbol{E}^* \underset{\boldsymbol{M}}{\otimes} \otimes^r \boldsymbol{E}' \underset{\boldsymbol{M}}{\otimes} \otimes^s \boldsymbol{E}'^*$.

The curvature of the linear tensor product connection $K \otimes K'$ on $\boldsymbol{E} \underset{\boldsymbol{M}}{\otimes} \boldsymbol{E}'$ turns out to be the vertical valued 2-form

$$R[K \otimes K'] = -[K \otimes K', K \otimes K'] : \boldsymbol{E} \underset{\boldsymbol{M}}{\otimes} \boldsymbol{E}' \to V(\boldsymbol{E} \underset{\boldsymbol{M}}{\otimes} \boldsymbol{E}') \otimes \Lambda^2 T^* \boldsymbol{M}.$$

Theorem 2.5. *The coordinate expression of $R[K \otimes K']$ is*

$$\begin{aligned} R[K \otimes K'] &= R[K \otimes K']_{jb}{}^{ia}{}_{\lambda\mu}\, w^{jb}\, \partial_{ia} \otimes d^\lambda \wedge d^\mu \\ &= \big(R[K]_j{}^i{}_{\lambda\mu}\, w^{ja} + R[K']_b{}^a{}_{\lambda\mu}\, w^{ib}\big)\partial_{ia} \otimes d^\lambda \wedge d^\mu, \end{aligned}$$

i.e. the coefficients of the curvature $R[K \otimes K']$ are

$$R[K \otimes K']_{jb}{}^{ia}{}_{\lambda\mu} = R[K]_j{}^i{}_{\lambda\mu}\delta^a_b + R[K']_b{}^a{}_{\lambda\mu}\delta^i_j.$$

Proof. This can be proved in coordinates. □

Theorem 2.5 implies that the curvature $R[K \otimes K']$ is determined by the curvatures $R[K]$ and $R[K']$. Now, we would like to find the geometric description of the curvature $R[K \otimes K']$. First we note that the curvatures of the above linear connections can be considered as bilinear morphisms, over $\boldsymbol{M}$,

$$\begin{aligned} R[K] &: \boldsymbol{E} \underset{\boldsymbol{M}}{\times} \boldsymbol{E}^* \to \Lambda^2 T^* \boldsymbol{M}, \\ R[K'] &: \boldsymbol{E}' \underset{\boldsymbol{M}}{\times} \boldsymbol{E}'^* \to \Lambda^2 T^* \boldsymbol{M}, \\ R[K \otimes K'] &: (\boldsymbol{E} \underset{\boldsymbol{M}}{\otimes} \boldsymbol{E}') \underset{\boldsymbol{M}}{\times} (\boldsymbol{E} \underset{\boldsymbol{M}}{\otimes} \boldsymbol{E}')^* \to \Lambda^2 T^* \boldsymbol{M}. \end{aligned}$$

Then we have

Theorem 2.6. *The curvature $R[K \otimes K']$ is a unique bilinear morphism such that the following diagram commutes*

$$\begin{array}{ccc} E \underset{M}{\times} E' \underset{M}{\times} E^* \underset{M}{\times} E'^* & \xrightarrow{\langle,\rangle' R[K] + \langle,\rangle R[K']} & \Lambda^2 T^* M \\ \Big\downarrow {\scriptstyle(\otimes,\otimes)} & & \Big\downarrow {\scriptstyle \mathrm{id}_{\Lambda^2 T^* M}} \\ (E \underset{M}{\otimes} E') \underset{M}{\times} (E^* \underset{M}{\otimes} E'^*) & \xrightarrow{R[K \otimes K']} & \Lambda^2 T^* M \end{array}$$

where $\langle , \rangle$ or $\langle , \rangle'$ are the evaluation morphisms on E or E', respectively.

Proof. Let us assume a bilinear morphism $R : (E \underset{M}{\otimes} E') \underset{M}{\times} (E \underset{M}{\otimes} E')^* \to \Lambda^2 T^* M$ and let us put $e = (e^i) \in E_x$, $e^* = (e_i) \in E^*_x$, $e' = (e'^a) \in E'_x$ and $e'^* = (e'_a) \in E'^*_x$. Then

$$\begin{aligned} \langle e', e'^* \rangle \, R[K](e, e^*) &= e'^a e'_a R[K]_j{}^i{}_{\lambda\mu} e^j e_i \, d^\lambda \wedge d^\mu \,, \\ \langle e, e^* \rangle \, R[K'](e', e'^*) &= e^i e_i R[K']_b{}^a{}_{\lambda\mu} e'^b e'_a \, d^\lambda \wedge d^\mu \,, \\ R(e \otimes e', e^* \otimes e'^*) &= R_{jb}{}^{ia}{}_{\lambda\mu} e^j e'^b e_i e'_a \, d^\lambda \wedge d^\mu \end{aligned}$$

and it is easy to see that $R(e \otimes e', e^* \otimes e'^*) = \langle e', e'^* \rangle \, R[K](e, e^*) + \langle e, e^* \rangle \, R[K'](e', e'^*)$ if and only if

$$R_{jb}{}^{ia}{}_{\lambda\mu} = R[K]_j{}^i{}_{\lambda\mu} \delta^a_b + R[K']_b{}^a{}_{\lambda\mu} \delta^i_j \,.$$

Now, Theorem 2.6 follows from Theorem 2.5. □

Proposition 2.7. *The curvature $R[K^p_q \otimes K'^r_s] := -[K^p_q \otimes K'^r_s, K^p_q \otimes K'^r_s]$ is determined by the curvatures $R[K]$ and $R[K']$. We have the coordinate expression*

$$\begin{aligned} R[K^p_q \otimes K'^r_s] = \Big(& R[K]_k{}^{i_1}{}_{\lambda\mu} w^{k i_2 \dots i_p a_1 \dots a_r}_{j_1 \dots j_q b_1 \dots b_s} + \dots + R[K]_k{}^{i_p}{}_{\lambda\mu} w^{i_1 \dots i_{p-1} k a_1 \dots a_r}_{j_1 \dots j_q b_1 \dots b_s} \\ & - R[K]_{j_1}{}^k{}_{\lambda\mu} w^{i_1 \dots i_p a_1 \dots a_r}_{k j_2 \dots j_q b_1 \dots b_s} - \dots - R[K]_{j_q}{}^k{}_{\lambda\mu} w^{i_1 \dots i_p a_1 \dots a_r}_{j_1 \dots j_{q-1} k b_1 \dots b_s} \\ & + R[K']_c{}^{a_1}{}_{\lambda\mu} w^{i_1 \dots i_p c a_2 \dots a_r}_{j_1 \dots j_q b_1 \dots b_s} + \dots + R[K']_c{}^{a_r}{}_{\lambda\mu} w^{i_1 \dots i_p a_1 \dots a_{r-1} c}_{j_1 \dots j_q b_1 \dots b_s} \\ & - R[K']_{b_1}{}^c{}_{\lambda\mu} w^{i_1 \dots i_p a_1 \dots a_r}_{j_1 \dots j_q c b_2 \dots b_s} - \dots - R[K']_{b_s}{}^c{}_{\lambda\mu} w^{i_1 \dots i_p a_1 \dots a_r}_{j_1 \dots j_q b_1 \dots b_{s-1} c} \Big) \\ & \mathfrak{b}_{i_1 \dots i_p} \otimes \mathfrak{b}^{j_1 \dots j_q} \otimes \mathfrak{b}_{a_1 \dots a_r} \otimes \mathfrak{b}^{b_1 \dots b_s} \otimes d^\lambda \wedge d^\mu \,, \end{aligned}$$

where we have put $\mathfrak{b}_{i_1 \dots i_p} = \mathfrak{b}_{i_1} \otimes \dots \otimes \mathfrak{b}_{i_p}$, $\mathfrak{b}^{j_1 \dots j_q} = \mathfrak{b}^{j_1} \otimes \dots \otimes \mathfrak{b}^{j_q}$, $\mathfrak{b}_{a_1 \dots a_r} = \mathfrak{b}_{a_1} \otimes \dots \otimes \mathfrak{b}_{a_r}$, $\mathfrak{b}^{b_1 \dots b_s} = \mathfrak{b}^{b_1} \otimes \dots \otimes \mathfrak{b}^{b_s}$.

Proof. This follows from the definition of the curvature, Proposition 1.9 and the iteration of Theorem 2.5. □

3. Classical connections

Let M be an m-dimensional manifold. Local coordinate charts on M will be denoted by (x^λ), $\lambda = 1, \dots, m$, the induced coordinate charts on TM or T^*M will be denoted by $(x^\lambda, \dot{x}^\lambda)$ or $(x^\lambda, \dot{x}_\lambda)$ and the induced local bases of sections of TM or T^*M are denoted by (∂_λ) or (d^λ), respectively.

A *classical connection* on M is defined to be a linear symmetric connection on $p_M : TM \to M$ with coordinate expression

$$\Gamma = d^\lambda \otimes \left(\partial_\lambda + \Gamma_\nu{}^\mu{}_\lambda \, \dot{x}^\nu \, \dot{\partial}_\mu\right), \quad \Gamma_\mu{}^\lambda{}_\nu \in C^\infty(M, \mathbb{R}), \quad \Gamma_\mu{}^\lambda{}_\nu = \Gamma_\nu{}^\lambda{}_\mu .$$

Remark 3.1. Let us recall the 1st and the 2nd Bianchi identities of classical connections expressed in coordinates by

$$R[\Gamma]_\nu{}^\rho{}_{\lambda\mu} + R[\Gamma]_\lambda{}^\rho{}_{\mu\nu} + R[\Gamma]_\mu{}^\rho{}_{\nu\lambda} = 0 ,$$
$$R[\Gamma]_\nu{}^\rho{}_{\lambda\mu;\sigma} + R[\Gamma]_\nu{}^\rho{}_{\mu\sigma;\lambda} + R[\Gamma]_\nu{}^\rho{}_{\sigma\lambda;\mu} = 0 ,$$

respectively, where ; denotes the covariant differential with respect to Γ.

Let us denote by $E^{p,r}_{q,s} := \otimes^p E \underset{M}{\otimes} \otimes^q E^* \underset{M}{\otimes} \otimes^r TM \underset{M}{\otimes} \otimes^s T^*M$. Then, as a direct consequence of Proposition 2.4, we have

Proposition 3.2. *A classical connection Γ on M and a linear connection K on E induce the linear tensor product connection $K^p_q \otimes \Gamma^r_s := \otimes^p K \otimes \otimes^q K^* \otimes \otimes^r \Gamma \otimes \otimes^s \Gamma^*$ on $E^{p,r}_{q,s}$*

$$K^p_q \otimes \Gamma^r_s : E^{p,r}_{q,s} \to T^*M \underset{M}{\otimes} TE^{p,r}_{q,s}$$

with coordinate expression

$$\begin{aligned} K^p_q \otimes \Gamma^r_s = d^\nu \otimes \Big(\partial_\nu + (&K_k{}^{i_1}{}_\nu \, y^{k i_2 \dots i_p \lambda_1 \dots \lambda_r}_{j_1 \dots j_q \mu_1 \dots \mu_s} + \dots + K_k{}^{i_p}{}_\nu \, y^{i_1 \dots i_{p-1} k \lambda_1 \dots \lambda_r}_{j_1 \dots j_q \mu_1 \dots \mu_s} \\ &- K_{j_1}{}^k{}_\nu \, y^{i_1 \dots i_p \lambda_1 \dots \lambda_r}_{k j_2 \dots j_q \mu_1 \dots \mu_s} - \dots - K_{j_q}{}^k{}_\nu \, y^{i_1 \dots i_p \lambda_1 \dots \lambda_r}_{j_1 \dots j_{q-1} k \mu_1 \dots \mu_s} \\ &+ \Gamma_\rho{}^{\lambda_1}{}_\nu \, y^{i_1 \dots i_p \rho \lambda_2 \dots \lambda_r}_{j_1 \dots j_q \mu_1 \dots \mu_s} + \dots + \Gamma_\rho{}^{\lambda_r}{}_\nu \, y^{i_1 \dots i_p \lambda_1 \dots \lambda_{r-1} \rho}_{j_1 \dots j_q \mu_1 \dots \mu_s} \\ &- \Gamma_{\mu_1}{}^\rho{}_\nu \, y^{i_1 \dots i_p \lambda_1 \dots \lambda_r}_{j_1 \dots j_q \rho \mu_2 \dots \mu_s} - \dots - \Gamma_{\mu_s}{}^\rho{}_\nu \, y^{i_1 \dots i_p \lambda_1 \dots \lambda_r}_{j_1 \dots j_q \mu_1 \dots \mu_{s-1} \rho}) \, \partial^{j_1 \dots j_q \mu_1 \dots \mu_s}_{i_1 \dots i_p \lambda_1 \dots \lambda_r} \Big) \end{aligned}$$

where $(x^\lambda, y^{i_1 \dots i_p \lambda_1 \dots \lambda_r}_{j_1 \dots j_q \mu_1 \dots \mu_s})$ are the induced linear fiber coordinates on $E^{p,r}_{q,s}$.

As a direct consequence of Proposition 2.7 we have

Proposition 3.3. *The curvature $R[K^p_q \otimes \Gamma^r_s]$ is determined by the curvatures $R[K]$ and $R[\Gamma]$. We have the coordinate expression*

$$\begin{aligned} R[K^p_q \otimes \Gamma^r_s] = \Big(& R[K]_k{}^{i_1}{}_{\nu_1\nu_2}\, y^{k i_2 \dots i_p \lambda_1 \dots \lambda_r}_{j_1 \dots j_q \mu_1 \dots \mu_s} + \dots + R[K]_k{}^{i_p}{}_{\nu_1\nu_2}\, y^{i_1 \dots i_{p-1} k \lambda_1 \dots \lambda_r}_{j_1 \dots j_q \mu_1 \dots \mu_s} \\ & - R[K]_{j_1}{}^{k}{}_{\nu_1\nu_2}\, y^{i_1 \dots i_p \lambda_1 \dots \lambda_r}_{k j_2 \dots j_q \mu_1 \dots \mu_s} - \dots - R[K]_{j_q}{}^{k}{}_{\nu_1\nu_2}\, y^{i_1 \dots i_p \lambda_1 \dots \lambda_r}_{j_1 \dots j_{q-1} k \mu_1 \dots \mu_s} \\ & + R[\Gamma]_\rho{}^{\lambda_1}{}_{\nu_1\nu_2}\, y^{i_1 \dots i_p \rho \lambda_2 \dots \lambda_r}_{j_1 \dots j_q \mu_1 \dots \mu_s} + \dots + R[\Gamma]_\rho{}^{\lambda_r}{}_{\nu_1\nu_2}\, y^{i_1 \dots i_p \lambda_1 \dots \lambda_{r-1} \rho}_{j_1 \dots j_q \mu_1 \dots \mu_s} \\ & - R[\Gamma]_{\mu_1}{}^{\rho}{}_{\nu_1\nu_2}\, y^{i_1 \dots i_p \lambda_1 \dots \lambda_r}_{j_1 \dots j_q \rho \mu_2 \dots \mu_s} - \dots - R[\Gamma]_{\mu_s}{}^{\rho}{}_{\nu_1\nu_2}\, y^{i_1 \dots i_p \lambda_1 \dots \lambda_r}_{j_1 \dots j_q \mu_1 \dots \mu_{s-1} \rho} \Big) \\ & \mathfrak{b}_{i_1 \dots i_p} \otimes \mathfrak{b}^{j_1 \dots j_q} \otimes \partial_{\lambda_1 \dots \lambda_r} \otimes d^{\mu_1 \dots \mu_s} \otimes d^{\nu_1} \wedge d^{\nu_2}, \end{aligned}$$

where we have put $\mathfrak{b}_{i_1 \dots i_p} = \mathfrak{b}_{i_1} \otimes \dots \otimes \mathfrak{b}_{i_p}$, $\mathfrak{b}^{j_1 \dots j_q} = \mathfrak{b}^{j_1} \otimes \dots \otimes \mathfrak{b}^{j_q}$, $\partial_{\lambda_1 \dots \lambda_r} = \partial_{\lambda_1} \otimes \dots \otimes \partial_{\lambda_r}$, $d^{\mu_1 \dots \mu_s} = d^{\mu_1} \otimes \dots \otimes d^{\mu_s}$.

4. Covariant differentials

Let us note that the tensor product connection $K^p_q \otimes \Gamma^r_s$ can be considered as a linear splitting

$$K^p_q \otimes \Gamma^r_s : \boldsymbol{E}^{p,r}_{q,s} \to J^1 \boldsymbol{E}^{p,r}_{q,s}.$$

Definition 4.1. Let $\Phi \in C^\infty(\boldsymbol{E}^{p,r}_{q,s})$. We define the *covariant differential of* Φ *with respect to a pair of connections* (K, Γ) as a section of $\boldsymbol{E}^{p,r}_{q,s} \otimes T^*\boldsymbol{M}$ defined by

$$\nabla^{(K,\Gamma)} \Phi = j^1 \Phi - (K^p_q \otimes \Gamma^r_s) \circ \Phi.$$

Remark 4.2. The covariant differential $\nabla^{(K,\Gamma)}\Phi$ is in fact the standard covariant differential (see Definition 1.3) $\nabla^{K^p_q \otimes \Gamma^r_s}\Phi$.

Proposition 4.3. *Let* $\Phi \in C^\infty(\boldsymbol{E}^{p,r}_{q,s})$, $\Phi = \phi^{i_1 \dots i_p \lambda_1 \dots \lambda_r}_{j_1 \dots j_q \mu_1 \dots \mu_s}\, \mathfrak{b}_{i_1 \dots i_p} \otimes \mathfrak{b}^{j_1 \dots j_q} \otimes \partial_{\lambda_1 \dots \lambda_r} \otimes d^{\mu_1 \dots \mu_s}$. *Then we have the coordinate expression*

$$\begin{aligned} \nabla^{(K,\Gamma)}\Phi &= \nabla^{(K,\Gamma)}_\nu \phi^{i_1 \dots i_p \lambda_1 \dots \lambda_r}_{j_1 \dots j_q \mu_1 \dots \mu_s}\, \mathfrak{b}_{i_1 \dots i_p} \otimes \mathfrak{b}^{j_1 \dots j_q} \otimes \partial_{\lambda_1 \dots \lambda_r} \otimes d^{\mu_1 \dots \mu_s} \otimes d^\nu \\ &= \Big(\partial_\nu \phi^{i_1 \dots i_p \lambda_1 \dots \lambda_r}_{j_1 \dots j_q \mu_1 \dots \mu_s} - K_k{}^{i_1}{}_\nu\, \phi^{k i_2 \dots i_p \lambda_1 \dots \lambda_r}_{j_1 \dots j_q \mu_1 \dots \mu_s} - \dots - K_k{}^{i_p}{}_\nu\, \phi^{i_1 \dots i_{p-1} k \lambda_1 \dots \lambda_r}_{j_1 \dots j_q \mu_1 \dots \mu_s} \\ &\quad + K_{j_1}{}^{k}{}_\nu\, \phi^{i_1 \dots i_p \lambda_1 \dots \lambda_r}_{k j_2 \dots j_q \mu_1 \dots \mu_s} + \dots + K_{j_q}{}^{k}{}_\nu\, \phi^{i_1 \dots i_p \lambda_1 \dots \lambda_r}_{j_1 \dots j_{q-1} k \mu_1 \dots \mu_s} \\ &\quad - \Gamma_\rho{}^{\lambda_1}{}_\nu\, \phi^{i_1 \dots i_p \rho \lambda_2 \dots \lambda_r}_{j_1 \dots j_q \mu_1 \dots \mu_s} - \dots - \Gamma_\rho{}^{\lambda_r}{}_\nu\, \phi^{i_1 \dots i_p \lambda_1 \dots \lambda_{r-1} \rho}_{j_1 \dots j_q \mu_1 \dots \mu_s} \\ &\quad + \Gamma_{\mu_1}{}^{\rho}{}_\nu\, \phi^{i_1 \dots i_p \lambda_1 \dots \lambda_r}_{j_1 \dots j_q \rho \mu_2 \dots \mu_s} + \dots + \Gamma_{\mu_s}{}^{\rho}{}_\nu\, \phi^{i_1 \dots i_p \lambda_1 \dots \lambda_r}_{j_1 \dots j_q \mu_1 \dots \mu_{s-1} \rho} \Big) \\ &\quad \mathfrak{b}_{i_1 \dots i_p} \otimes \mathfrak{b}^{j_1 \dots j_q} \otimes \partial_{\lambda_1 \dots \lambda_r} \otimes d^{\mu_1 \dots \mu_s} \otimes d^\nu. \end{aligned}$$

Proof. The proof follows immediately from Definition 4.1 and the coordinate expression (see Proposition 3.2) of the connection $K^p_q \otimes \Gamma^r_s$. □

In what follows we set $\nabla = \nabla^{(K,\Gamma)}$ and $\phi^{i_1 \dots i_p \lambda_1 \dots \lambda_r}_{j_1 \dots j_q \mu_1 \dots \mu_s; \nu} = \nabla_\nu \phi^{i_1 \dots i_p \lambda_1 \dots \lambda_r}_{j_1 \dots j_q \mu_1 \dots \mu_s}$.

Remark 4.4. If $p = q = 0$ the field Φ is a standard (r, s)-tensor field on $\boldsymbol{M}$ and $\nabla\Phi$ coincides with the standard covariant differential with respect to the classical connection Γ.

Corollary 4.5. *We have*

$$\begin{aligned}\nabla R[K] &= R[K]_j{}^i{}_{\lambda\mu;\nu}\, \mathfrak{b}^j \otimes \mathfrak{b}_i \otimes d^\lambda \wedge d^\mu \otimes d^\nu \\ &= \big(\partial_\nu R[K]_j{}^i{}_{\lambda\mu} - K_p{}^i{}_\nu\, R[K]_j{}^p{}_{\lambda\mu} + K_j{}^p{}_\nu\, R[K]_p{}^i{}_{\lambda\mu} \\ &\quad + \Gamma_\nu{}^\rho{}_\lambda\, R[K]_j{}^i{}_{\rho\mu} + \Gamma_\nu{}^\rho{}_\mu\, R[K]_j{}^i{}_{\lambda\rho}\big)\, \mathfrak{b}^j \otimes \mathfrak{b}_i \otimes d^\lambda \wedge d^\mu \otimes d^\nu\,.\end{aligned}$$

The generalized Bianchi identity can be expressed by covariant differentials as follows.

Theorem 4.6 (The generalized Bianchi identity). *We have*

$$R[K]_j{}^i{}_{\lambda\mu;\nu} + R[K]_j{}^i{}_{\mu\nu;\lambda} + R[K]_j{}^i{}_{\nu\lambda;\mu} = 0\,.$$

Proof. This can be proved easily in coordinates by using Corollary 4.5. □

Theorem 4.7. *Let* $\Phi \in C^\infty(\boldsymbol{E}^{p;r}_{q;s})$. *Then we have*

$$\operatorname{Alt} \nabla^2\Phi = -\frac{1}{2}\, R[\Gamma^p_q \otimes K^r_s] \circ \Phi \in C^\infty(\boldsymbol{E}^{p;r}_{q;s} \otimes \Lambda^2 T^*\boldsymbol{M})\,,$$

where Alt *is the antisymmetrization.*

Proof. This can be proved in coordinates by using Proposition 3.3 and Proposition 4.3. □

Example 4.8. Let $\Phi \in C^\infty(\boldsymbol{E})$, $\Phi = \phi^i \mathfrak{b}_i$. Then

$$\operatorname{Alt} \nabla^2\Phi = -\frac{1}{2} R[K] \circ \Phi : \boldsymbol{M} \to \boldsymbol{E} \otimes \Lambda^2 T^*\boldsymbol{M}\,,$$

i.e. in coordinates

$$\operatorname{Alt} \nabla^2\Phi = -\frac{1}{2}\, R[K]_j{}^i{}_{\lambda\mu}\, \phi^j\, \mathfrak{b}_i \otimes d^\lambda \wedge d^\mu\,.$$

Example 4.9. We have

$$\operatorname{Alt} \nabla^2 R[K] : \boldsymbol{M} \to \boldsymbol{E}^* \otimes \boldsymbol{E} \otimes \Lambda^2 T^*\boldsymbol{M} \otimes \Lambda^2 T^*\boldsymbol{M}\,,$$

expressed in coordinates by

$$\begin{aligned}\operatorname{Alt} \nabla^2 R[K] = -\frac{1}{2}\, &\big(R[K]_p{}^i{}_{\nu_1\nu_2}\, R[K]_j{}^p{}_{\lambda\mu} - R[K]_j{}^p{}_{\nu_1\nu_2}\, R[K]_p{}^i{}_{\lambda\mu} \\ &- R[\Gamma]_\lambda{}^\omega{}_{\nu_1\nu_2}\, R[K]_j{}^i{}_{\omega\mu} - R[\Gamma]_\mu{}^\omega{}_{\nu_1\nu_2}\, R[K]_j{}^i{}_{\lambda\omega}\big) \\ &\mathfrak{b}^j \otimes \mathfrak{b}_i \otimes d^\lambda \wedge d^\mu \otimes d^{\nu_1} \wedge d^{\nu_2}\,.\end{aligned}$$

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