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Persistent URL: http://dml.cz/dmlcz/701733

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A REMARK ON THE TOPOLOGY OF HIGHER ORDER FRAMES

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ABSTRACT. We study the problem whether 1st order flatness of a 2nd order G-structure implies its 2nd order flatness. We give a sufficient condition for the existence of 2nd order flat lifts. We indicate that certain characteristic classes can be defined which seem to be topological obstructions for the existence of 2nd order flat lifts. The problem whether these classes can be nontrivial requires further study.

Let $M$ be a differentiable manifold and $F^k$ be the set of $k$-jets of all local diffeomorphisms of $M$ with source at some fixed point $o \in M$ and target at any point of $M$. Projecting elements of $F^k$ to their targets, we obtain a right principal bundle $F^k \to M$, whose group $GL_k$ is the set of all $k$-jets with source and target at $o$ and $GL_k$ acts on $F^k$ by jet composition on the source. $F^k \to M$ can be identified with the $k$th order frame bundle of $M$. A $k$th order $G$-structure on $M$ may be viewed as a subbundle of $F^k \to M$. Now let $P^2 \to M$ be a 2nd order $G^2$-structure on $M$, $G^2 \subset GL_2$. $P^2 \to M$ defines a 1st order $G^1$-structure $P^1 \to M$, $G^1 \subset GL_1$. We have the projection homomorphism $G^2 \to G^1$ and also the bundle projection $P^2 \to P^1$. We will be interested here in two questions:

Q1: Is the bundle $P^2 \to P^1$ always trivial?

A connection on $P^2 \to M$ projects to a connection on $P^1 \to M$. Converse is also true: A connection on $P^1 \to M$ lifts to some connection on $P^2 \to M$. To see this, we start with a connection $\omega$ on $P^1 \to M$ and piece together the local lifts of $\omega$ by a partition of unity which recovers $\omega$ on $P^1 \to M$ and gives a lift of $\omega$ to $P^2 \to M$. We refer to [8] for technical details.

Q2: Let $\omega_1$ be a flat connection on $P^1 \to M$. Does $\omega_1$ always lift to some flat $\omega_2$ on $P^2 \to M$?

To give more substance to the above questions, we will first consider a rich source of examples supplied by homogeneous spaces $M = G/H$. Some $a \in G$ defines a diffeomorphism $\bar{a}$ of $M$ which maps $xH$ to $axH$ and the map $G \to Diff(M)$ sending $a$ to $\bar{a}$ is a continuous homomorphism with kernel $K$. If $K$ is trivial, the action of $G$
on $M$ is called effective and $(G, H)$ is called an effective pair. $K$ is the largest normal subgroup of $G$ contained in $H$. Let $j_k(\bar{a})(xH, axH)$ denote the $k$-jet of $\bar{a}$ with source at $xH$ and target $axH$. In this way we obtain a $k$th order transitive Lie groupoid $\Theta^k$ on $M$. Note that the elements of $\Theta^k$ are induced by global diffeomorphisms which may not be the case for an arbitrary Lie groupoid. Fixing the source of jets of $\Theta^k$ at $o = H$, we obtain a $k$th order $G$-structure $P^k \to M$. For a more explicit description of $P^k \to M$, first assume $k = 1$. Now $\bar{h}$ fixes $o$ and we obtain the 1st order isotropy representation $h \to j_1(\bar{h})(o, o)$ with kernel $K_1 \subset H$. Clearly $K \subset K_1$.

**Proposition 1.** The bundle $P^1 \to M$ coincides with the homogeneous bundle $G/K_1 \to G/H$.

Note that $G/L \to G/H$ is always a principal bundle with group $H/L$ if $L \subset H$. The new ingredient in Proposition 1 is that this bundle can be realized as a 1st order $G$-structure if $L = K_1$.

Proposition 1 is contained, for instance, in [6] but we will recall its proof here to see that it generalizes to the case $k > 1$ in a straightforward way. We first define a map $i : G/K_1 \to F^1$ by $i(aK_1) = j_1(\bar{a})(o, aH)$. Using definitions, it is easily checked that $i$ is well defined, injective and its image is clearly $P^1$ defined above. Further, $i$ injects $H/K_1$ isomorphically into $GL_1$ with image $G^1$. $H/K_1$ acts on $G/K_1$ on the right by $(aK_1)(hK_1) = ahK_1$. It is easily checked that this action is well defined and $i$ commutes with the actions of $H/K_1$ on $G/K_1$ and $GL_1$ on $F^1$ respectively. Finally, we check that $i$ commutes with projections $G/K_1 \to G/H$ and $F^1 \to M$ respectively and $i$ is continuous.

If $K_1$ is trivial, then all diffeomorphisms $\bar{a}$ are uniquely determined by their 1st order jets and the 1st order isotropy representation is faithful. This is the case, for instance, if i) $H$ is discrete ii) $H$ is compact iii) $G/H$ is reductive (see [3], pg. 190–199). However, for many important homogeneous spaces $K_1$ is not trivial as we will henceforth assume. Consider now the 2nd order isotropy representation $h \to j_2(\bar{h})(o, o)$ with kernel $K_2 \subset K_1$. Clearly $K \subset K_2$. As in the proof of Proposition 1, we define a map $j : G/K_2 \to F^2$ by $j(aK_2) = j_2(\bar{a})(o, aH)$. Repeating the steps in the proof of Proposition 1 with $i$ replaced by $j$, we see that $G/K_2 \to G/H$ is a 2nd order $H/K_2$-structure and we also have the projection of principal bundles $G/K_2 \to G/K_1$ with group $K_1/K_2$. Iterating this construction, we obtain a decreasing sequence of normal subgroups $H \supset K_1 \supset K_2 \supset \cdots \supset K$ and this sequence stabilizes at some $K_{m+1} = K$ (see [10], pg. 161–164). From now on we will assume that $(G, H)$ is an effective pair so that $K = \{e\}$. In this way we obtain the following sequence of principal bundles

\[
\begin{array}{cccccccc}
G & \rightarrow & G/K_m & \rightarrow & G/K_{m-1} & \rightarrow & \cdots & \rightarrow & G/K_1 & \rightarrow & G/H \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow id \\
F^{m+1} & \rightarrow & F^m & \rightarrow & F^{m-1} & \rightarrow & \cdots & \rightarrow & F^1 & \rightarrow & M
\end{array}
\]

where the vertical arrows are injections of principal bundles. In particular, $G \to G/H$ can be realized as a $(m + 1)$th order $H$-structure and a connection on $G \to G/H$ is a $(m + 1)$th order connection. We therefore see that even the most classical examples of principal bundles sometimes implicitly involve higher order jets. The natural question arises whether these higher order jets have any topological implication.
As for the groups of these principal bundles, we have the commutative diagram

\[
\begin{array}{c}
H & \rightarrow & H/K_m & \rightarrow & \ldots & \rightarrow & H/K_2 & \rightarrow & H/K_1 & \rightarrow & e \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
GL_{m+1} & \rightarrow & GL_m & \rightarrow & \ldots & \rightarrow & GL_2 & \rightarrow & GL_1 & \rightarrow & e
\end{array}
\]

where the vertical arrows are injective homomorphisms.

Let \( KL_i \) be the kernel of \( GL_i \rightarrow GL_{i-1}, i \geq 2 \). Then \( KL_i \) is a vector space and \( K_i/K_{i+1} \) is a closed subgroup of \( KL_i \). Therefore \( K_i/K_{i+1} \) is either a lattice, or a subspace or a direct sum of two such subgroups.

For simplicity, we now assume \( m = 1, i.e., K_2 = 0 \). We will consider two cases:

a) \( K_1 \) is a lattice,

b) \( K_1 \) is a subspace.

An effective pair satisfying a) (it is not difficult to give explicit examples) gives a remarkable subgroup \( H \subset GL_2 \): In this case (2) reduces to

\[
\begin{array}{c}
0 & \rightarrow & K_1 & \rightarrow & H & \rightarrow & H/K_1 & \rightarrow & e \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & KL_2 & \rightarrow & GL_2 & \rightarrow & GL_1 & \rightarrow & e
\end{array}
\]

and the principal bundle \( H \rightarrow H/K_1 \) is a covering map which is not trivial if \( H \) is connected. Thus we arrive at an example where 2\textsuperscript{nd} order jets carry nontrivial topology. Observe that the upper sequence in (3) can not split in this case for otherwise \( H/K_1 \) would be a subgroup of \( H \) and translating an element in the fiber by the subgroup \( H/K_1 \) would trivialize the covering map and disconnect \( H \). This problem of splitting will be more important below when we consider b).

As for the corresponding \( G \)-structures, (1) reduces to

\[
\begin{array}{c}
G & \rightarrow & G/K_1 & \rightarrow & G/H \\
\downarrow & & \downarrow & & \downarrow id \\
F^2 & \rightarrow & F^1 & \rightarrow & M
\end{array}
\]

and again \( G \rightarrow G/K_1 \) is a nontrivial covering if \( G \) is connected. It follows therefore that the answer to \( Q1 \) is negative. However, in this case, it is easy to see that the answer to \( Q2 \) is affirmative: In fact, all lifts of a flat \( \omega_1 \) are flat in this case.

We now assume b). Note that the fibers of \( G \rightarrow G/K_1 \) are contractible and the bundle admits a cross section in this case. We will first state

**Proposition 2.** Any flat connection on \( F^1 \rightarrow M \) lifts to a flat connection on \( F^2 \rightarrow M \).

\( M \) is not necessarily a homogeneous space in Proposition 2. To prove it, we observe that the lower sequence in (3) splits and \( GL_1 \) is a subgroup of \( GL_2 \). This splitting allows us to construct sections of \( F^2 \rightarrow F^1 \) which are special: They are invariant under the action of \( GL_1 \). Such sections are called \( \epsilon \)-connections in the literature (\( \epsilon \) standing for Ehresmann) and are in 1-1 correspondence with torsion free connections on \( F^1 \rightarrow M \) (see [1], [4], [7], [11], [5] pg. 105-115). Consequently, a choice of an
ε-connection imbeds \( F^1 \) in \( F^2 \) as a principal subbundle and Proposition 2 follows by extending the flat connection on \( F^1 \) to a flat connection on \( F^2 \).

Note that an arbitrary section of \( F^2 \rightarrow F^1 \) does not help to find a flat lift. To see this, suppose we fix such a section \( s \). Let \( f : U \rightarrow F^1 \) be a local section of \( F^1 \) such that \( f(U) \) is an integral manifold of the flat connection \( \omega_1 \) on \( F^1 \). If \( g : U \rightarrow F^1 \) is another such section, then \( f(x) = g(x)a \) for some \( a \in GL_1 \) which does not depend on \( x \in U \). It is now natural to consider the compositions of such sections with \( s \) and consider sections of \( F^2 \rightarrow F^1 \) of the form \( s \circ f \). Now \( s \circ f(x) = (s \circ g(x))a(x) \) for some \( a(x) \in GL_2 \), however there is no a priori reason why \( a(x) \) should be constant even though its projection in \( GL_1 \) is. Thus we see that the problem of lifting a flat \( \omega_1 \) involves more than the triviality of \( F^2 \rightarrow F^1 \).

The above argument gives also a proof of the following

**Proposition 3.** If the upper sequence in (3) splits, then the answer to Q2 is affirmative.

The proof of Proposition 3 works for general \( G \)-structures and not necessarily for homogeneous spaces. The construction of an \( \varepsilon \)-connection with the assumption of the existence of a splitting of \( 0 \rightarrow K^2 \rightarrow G^2 \rightarrow G^1 \rightarrow \varepsilon \) is straightforward ([8]). Note, however, that an \( \varepsilon \)-connection does not necessarily define a torsion free connection on \( P^1 \rightarrow M \) unless \( P^2 \) is contained in the 1st jet extension \( J^1P^1 \). In fact, \( P^1 \rightarrow M \) may not admit a torsion free connection. In this direction, we have

**Proposition 4.** If \( P^2 \subset J^1P^1 \), then the answer to Q2 is affirmative.

The proof of Proposition 4 consists in showing that the assumption \( P^2 \subset J^1P^1 \) forces the sequence \( 0 \rightarrow K^2 \rightarrow G^2 \rightarrow G^1 \rightarrow \varepsilon \) to split which is not difficult. In fact, if \( P^1 \rightarrow M \) is uniformly 1-flat (see [1], [8] for the definition of uniform 1-flatness), then we can define its prolongation \( prP^1 \subset J^1P^1 \) and any (flat) connection on \( P^1 \) uniquely prolongs to some (flat) connection on \( prP^1 \) ([8]). If \( P^2 \subset J^1P^1 \), then \( P^1 \) is uniformly 1-flat and \( P^2 \subset prP^1 \) by the definition of \( prP^1 \). Now the unique flat lift reduces to \( P^2 \).

However, note that the flat lifts given by \( \varepsilon \)-connections are very special: If \( \omega_2 \) lifts \( \omega_1 \), then the holonomy group \( Hol_1 \subset G^1 \) of \( \omega_1 \) is isomorphic to the holonomy group \( Hol_2 \subset G^2 \) because the fiber of \( Hol_2 \rightarrow Hol_1 \) consists of a single point, which need not be the case in general.

The above arguments suggest, we believe, that the answer to Q2 is not affirmative either. We will now fix a flat \( \omega_1 \) on \( P^1 \), lift it to some \( \omega_2 \) and define certain characteristic classes from the curvature \( R_2 \) of \( \omega_2 \) which do not depend on the lift. By construction, these classes vanish if there exists a flat lift. These classes seem to be richer than the ones given by the usual Chern-Weil construction by means of an arbitrary \( \omega_2 \) on \( P^2 \) as they will be defined only when \( \omega_1 \) is flat.

For this purpose, consider \( gkg^{-1} \) where \( k \in K^2 \) and \( g \in G^2 \). Then \( gkg^{-1} \in K^2 \) since \( K^2 \triangleleft G^2 \). Since \( K^2 \) is also abelian, \( gkg^{-1} \) depends only on the image \( \pi(g) \) of \( g \) where \( \pi : G^2 \rightarrow G^1 \) is the projection. Thus we have an action of \( G^1 \) on \( K^2 \) and therefore an action \( * \) of \( G^1 \) on the abelian Lie algebra \( k^2 \) of \( K^2 \). Now let \( \omega_1 \) be some fixed flat connection on \( P^1 \rightarrow M \) and \( \omega_2 \) be some arbitrary lift to \( P^2 \rightarrow M \). The curvature \( R_2 \) of \( \omega_2 \) takes values in the Lie algebra \( g^2 \) of \( G^2 \) and projects to the curvature \( R_1 \) of \( \omega_1 \). Since \( R_1 = 0 \), \( R_2 \) takes values actually in \( k^2 \). By the definition of \( * \), \( R_2 \) transforms
by \( \pi(g) \ast R_2 \) under the action of \( g \in G^2 \). Let \( I^m(G^1, k^2) \) be the space of symmetric \( m \)-linear maps \( k^2 \times \ldots \times k^2 \to \mathbb{R} \) which are invariant under \( \ast \). Now any \( Q \in I^m(G^1, k^2) \) defines a \( 2m \)-form \( Q(\omega_2) \) in a standard way. It is not difficult to check that \( Q(\omega_2) \) is horizontal and pulls back to closed form on \( M \) which is independent of the lift \( \omega_2 \).

Let \( I^m(G^2, g^2) \) be the space of symmetric \( m \)-linear maps \( g^2 \times \ldots \times g^2 \to \mathbb{R} \) which are invariant under the action of \( \text{Ad}(G^2) \). Note that there is a restriction map \( \theta : I^m(G^2, g^2) \to I^m(G^1, k^2) \) and the surjectivity of \( \theta \) implies that the above classes are nothing but the ones obtained by the Chern-Weil construction in terms of an arbitrary connection on \( P^2 \). In this case, it seems to us that the above classes vanish.

The following question therefore seems to be of significance:

**Q3:** Can the above classes be nontrivial?

More explicit constructions are possible in some special cases, for instance, when \( G^1 \) is contained in the orthogonal group \( O(n) \) (see [9]).

The above construction of characteristic classes becomes particularly transparent and elementary once formulated in terms of the gauge sequence worked out in [7], [8]. We will shortly recall this sequence here to formulate a more general form of **Q2** which has been our starting point.

Any transitive Lie groupoid of order \( k \) on \( M \) defines a \( k \)-th order \( G \)-structure \( P^k \to M \) by fixing the source (or target) of jets. All \( G \)-structures arise in this way. For any such principal bundle we have the exact sequence

\[
(5) \quad P^k \to CA^k \to \wedge^2(T^*) \otimes g^k
\]

called gauge sequence. The sequence (5) can be read as

\[
(6) \quad \text{bundle} \to \text{connection} \to \text{curvature}
\]

In fact, (5) can be taken as the definition of connection and curvature for such principal bundles. For the details for this sequence we refer to [7], [8] and the references therein. The curvature operator in (5) has components \( (R_1, \ldots, R_k) \) in local coordinates and \( R_1 \) is the classical curvature tensor. For simplicity, let \( k = 2 \). If 2\( \text{nd} \) order order jets contain any topology which is not contained in 1\( \text{st} \) order jets which can be deduced from a 2\( \text{nd} \) order connection, it is natural to expect that the new information is contained in \( R_2 \). However, \( R_2 \) has no invariant meaning but only \( (R_1, R_2) \) does. However, \( R_2 \) becomes an object by itself if \( R_1 = 0 \) which explains **Q2**. We therefore have

**Q4:** Can a 2\( \text{nd} \) order connection contain any topology which is not contained in a 1\( \text{st} \) order connection?

As we hinted above, it is not possible to formulate **Q4** within the framework of general principal bundles and connections because in this framework a connection is unable to detect the order of jets but treats them all together as a single object.

Finally, we will shortly indicate the relation of **Q2** to secondary characteristic classes (see, for instance, [2]). Let \( P \to M \) be an arbitrary principal bundle and \( Q \to M \) be a subbundle. A flat connection \( \omega \) on \( P \to M \) defines certain \( (P, Q) \)-characteristic classes which vanish if \( \omega \) reduces to \( Q \). Now, the flat connection \( \omega_1 \) which we assumed
on $P^1 \to M$ extends to a flat $\tilde{\omega}_1$ on $F^1 \to M$ which lifts to some $\tilde{\omega}_2$ on $F^2 \to M$ by Proposition 2. Q2 is therefore whether any such $\tilde{\omega}_2$ reduces to $P^2 \to M$.

REFERENCES