Jiří Vanžura

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# CHARACTERIZATION OF ONE TYPE OF MULTISYMPLECTIC 3-FORMS IN ODD DIMENSIONS

JIŘÍ VANŽURA

ABSTRACT. There is given an intrinsic characterization of one equivalence class of multisymplectic 3-forms on an odd dimensional vector space.

We consider an $n$-dimensional real vector space $V$. A $k$-form $\omega$ on $V$ is called multisymplectic if the homomorphism

$$V \to \Lambda^{k-1}V^*, \quad v \mapsto \iota_v\omega = \omega(v, \cdot, \dots, \cdot)$$

is injective. Let $\Lambda^k_{ms}V^* \subset \Lambda^k V^*$ denote the subset consisting of all multisymplectic forms. Obviously, the general linear group $GL(V)$ operates in the standard way on $\Lambda^k V^*$ preserving the subset $\Lambda^k_{ms}V^*$. We call two multisymplectic $k$-forms equivalent if they belong to the same orbit of $GL(V)$ in $\Lambda^k_{ms}V^*$. Let us set now $k = 3$, i. e. let us consider multisymplectic 3-forms. It is well known that the study of these forms is interesting starting from $\dim V \geq 6$, and that for $\dim V \leq 8$ there is in each dimension only a finite number of equivalence classes of multisymplectic 3-forms, while for $\dim V \geq 9$ there is in each dimension infinite number of such classes. (See e. g. [D].) The first interesting odd dimension is $\dim V = 7$. In this dimension we find 8 equivalence classes of multisymplectic 3-forms. The most simple class among them can be represented by a form $\omega$ defined in the following way. Let $\alpha_0, \alpha_1, \dots, \alpha_6$ be a basis of $V^*$. Then we set

$$\omega = \alpha_0 \wedge (\alpha_1 \wedge \alpha_2 + \alpha_3 \wedge \alpha_4 + \alpha_5 \wedge \alpha_6).$$

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(More information about this form can be found in [BV].) It is obvious that a form of this type can be defined on every odd dimensional vector space. If $\dim V = 2n+1$ and $\alpha_0, \alpha_1, \dots, \alpha_{2n}$ is a basis of $V^*$, we can set

$$\omega = \alpha_0 \wedge (\alpha_1 \wedge \alpha_2 + \cdots + \alpha_{2n-1} \wedge \alpha_{2n}) .$$

The aim of this paper is to present an intrinsic characterization of the equivalence class of multisymplectic 3-forms represented by the form of the above type.

We recall that a 2-form $\theta \in \Lambda^2 V^*$ is called decomposable if there exist two 1-forms $\beta_1, \beta_2 \in V^*$ such that $\theta = \beta_1 \wedge \beta_2$. It is well known that a 2-form $\theta$ is decomposable if and only if $\theta \wedge \theta = 0$. (In any vector space we denote by $[x, y, \dots]$ the subspace generated by the vectors $x, y, \dots$)

**1. Lemma.** *Let $\theta$ be a decomposable 2-form, and let $\beta_1, \beta_2, \gamma_1, \gamma_2$ be 1-forms such that $\theta = \beta_1 \wedge \beta_2 = \gamma_1 \wedge \gamma_2$. Then*

$$[\beta_1, \beta_2] = [\gamma_1, \gamma_2] .$$

**Proof.** We denote $K(\theta) = \ker \theta = \{v \in V; \iota_v \theta = 0\}$. It is easy to see that

$$[\beta_1, \beta_2] = \{\alpha \in V^*; \alpha | K(\theta) = 0\} = [\gamma_1, \gamma_2] . \qquad \square$$

This lemma shows that with each decomposable 2-form $\theta = \beta_1 \wedge \beta_2$ there is associated a 2-dimensional subspace

$$S(\theta) = [\beta_1, \beta_2] = \{\alpha \in V^*; \alpha | K(\theta) = 0\} .$$

The following lemma is obvious.

**2. Lemma.** *Let $\theta$ and $\theta'$ be two nonzero decomposable 2-forms. Then $\theta$ and $\theta'$ are linearly dependent if and only if $S(\theta) = S(\theta')$.*

**3. Lemma.** *Let $\theta, \theta' \in \Lambda^2 V^*$ be two linearly independent 2-forms such that the 2-dimensional subspace $[\theta, \theta']$ consists of decomposable forms. Then*

$$\dim(S(\theta) \cap S(\theta')) = 1 .$$

*There exist linearly independent 1-forms $\alpha_1, \alpha_2, \alpha_3$ such that*

$$\theta = \alpha_1 \wedge \alpha_2, \quad \theta' = \alpha_1 \wedge \alpha_3 .$$

**Proof.** Let us write $\theta = \beta_1 \wedge \beta_2$ and $\theta' = \gamma_1 \wedge \gamma_2$. We choose an $(n-2)$-dimensional subspace $B_{n-2} \subset V^*$ such that $[\beta_1] + [\beta_2] + B_{n-2} = V^*$. Then we have

$$\begin{aligned} \gamma_1 &= c_{11}\beta_1 + c_{12}\beta_2 + b_1 , \\ \gamma_2 &= c_{21}\beta_1 + c_{22}\beta_2 + b_2 , \end{aligned}$$

where $b_1, b_2 \in B_{n-2}$. Because $\beta_1 \wedge \beta_2 + \gamma_1 \wedge \gamma_2$ is decomposable, we have $\beta_1 \wedge \beta_2 \wedge \gamma_1 \wedge \gamma_2 = 0$. Consequently, we get

$$0 = \beta_1 \wedge \beta_2 \wedge \gamma_1 \wedge \gamma_2 = \beta_1 \wedge \beta_2 \wedge b_1 \wedge b_2 \,,$$

which implies $b_1 \wedge b_2 = 0$. At least one of the forms $b_1$ and $b_2$ must be non-zero. Let us assume it is the form $b_2$, and let us write $b_1 = ab_2$. Then we have $\theta' = \delta_1 \wedge \delta_2$, with

$$\delta_1 = d_{11}\beta_1 + d_{12}\beta_2$$
$$\delta_2 = d_{21}\beta_1 + d_{22}\beta_2 + b_2$$

where $d_{11} = c_{11} - ac_{21}$, $d_{12} \doteq c_{12} - ac_{22}$, $d_{21} = c_{21}$, and $d_{22} = c_{22}$. Now the first part of the assertion is obvious. Let us choose a generator $\alpha_1 \in S(\theta) \cap S(\theta')$. Choosing conveniently $\alpha_2 \in S(\theta)$ and $\alpha_3 \in S(\theta')$, we get $\theta = \alpha_1 \wedge \alpha_2$ and $\theta' = \alpha_1 \wedge \alpha_3$. ☐

**4. Lemma.** *Let $A_3 \subset \Lambda^2 V^*$ be a 3-dimensional subspace consisting of decomposable 2-forms. Then either*

(i) *there exist linearly independent 1-forms $\alpha_1, \alpha_2, \alpha_3$ such that*

$$\alpha_1 \wedge \alpha_2, \alpha_1 \wedge \alpha_3, \alpha_2 \wedge \alpha_3$$

*is a basis of $A_3$, or*

(ii) *there exist linearly independent 1-forms $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that*

$$\alpha_1 \wedge \alpha_2, \alpha_1 \wedge \alpha_3, \alpha_1 \wedge \alpha_4$$

*is a basis of $A_3$.*

**Proof.** Let us choose a basis $\theta, \theta', \theta''$ in $A_3$. We shall consider 1-dimensional subspaces

$$S(\theta) \cap S(\theta') \subset S(\theta) \,, \quad S(\theta) \cap S(\theta'') \subset S(\theta) \,.$$

Either they have trivial intersection, or they coincide. Let us start with the first case. We choose generators

$$\beta_1 \in S(\theta) \cap S(\theta') \,, \quad \beta_2 \in S(\theta) \cap S(\theta'') \,, \quad \beta_3 \in S(\theta') \cap S(\theta'') \,,$$

and we have

$$\theta = c\beta_1 \wedge \beta_2 \,, \quad \theta' = c'\beta_1 \wedge \beta_3 \,, \quad \theta'' = c''\beta_2 \wedge \beta_3 \,.$$

If $cc'c'' < 0$ we change the basis of $A_3$ for the basis $-\theta, \theta', \theta''$. Now it is easy to see that with conveniently chosen $a_1, a_2, a_3$, setting $\alpha_1 = a_1\beta_1, \alpha_2 = a_2\beta_2, \alpha_3 = a_3\beta_3$, we get

$$\theta = \alpha_1 \wedge \alpha_2 \,, \quad \theta' = \alpha_1 \wedge \alpha_3 \,, \quad \theta'' = \alpha_2 \wedge \alpha_3 \,.$$

It remains to consider the case when

$$S(\theta) \cap S(\theta') = S(\theta) \cap S(\theta'') \,.$$

We take a generator $\alpha_1 \in S(\theta) \cap S(\theta') = S(\theta) \cap S(\theta'')$. Then we can choose $\alpha_2 \in S(\theta)$ (resp. $\alpha_3 \in S(\theta')$, resp. $\alpha_4 \in S(\theta'')$) in such a way that

$$\theta = \alpha_1 \wedge \alpha_2 \,, \quad \theta' = \alpha_1 \wedge \alpha_3 \,, \quad \theta'' = \alpha_1 \wedge \alpha_4 \,.$$ ☐

Let us consider now subspaces $A \subset \Lambda^2 V^*$ consisting of decomposable 2-forms. We shall be interested in such subspaces of maximal possible dimensions.

**5. Proposition.** *Let $A_3 \subset \Lambda^2 V^*$ be a 3-dimensional subspace having a basis of the form*

$$\theta = \alpha_1 \wedge \alpha_2\,, \quad \theta' = \alpha_1 \wedge \alpha_3\,, \quad \theta'' = \alpha_2 \wedge \alpha_3\,.$$

*Then $A_3$ is a maximal subspace consisting of decomposable elements.*

**Proof.** Let us assume that there exist a subspace $A \subset \Lambda^2 V^*$, $\dim A \geq 4$ consisting of decomposable elements, and such that $A_3 \subset A$. Then we can choose an element $\lambda \in A - A_3$. It is obvious that

$$S(\theta) \cap S(\lambda)\,, \quad S(\theta') \cap S(\lambda)\,, \quad S(\theta'') \cap S(\lambda)$$

are 1-dimensional subspaces, and consequently

$$S(\lambda) \subset S(\theta) + S(\theta') + S(\theta'')\,,$$

which is a contradiction. □

**6. Proposition.** *Let $A \subset \Lambda^2 V^*$ be a subpace consisting of decomposable elements, $\dim A = k \geq 4$. Then there exist linearly independent 1-forms $\alpha_0, \dots, \alpha_k$ such that*

$$\alpha_0 \wedge \alpha_1, \dots, \alpha_0 \wedge \alpha_k$$

*is a basis of $A$. If $\dim V = n$, then a maximal subspace with the above property has dimension $n-1$.*

**Proof.** Let us choose a 3-dimensional subspace $A_3 \subset A$. Because $A_3$ is not maximal, we can find linearly independent 1-forms $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ such that

$$\theta_1 = \alpha_0 \wedge \alpha_1\,, \quad \theta_2 = \alpha_0 \wedge \alpha_2\,, \quad \theta_3 = \alpha_0 \wedge \alpha_3$$

is a basis of $A_3$. Moreover, we choose $\theta_4 \in A - A_3$. The subspaces $S(\theta_1) \cap S(\theta_4)$ and $S(\theta_2) \cap S(\theta_4)$ are 1-dimensional. They must coincide because otherwise $[\theta_1, \theta_2, \theta_4]$ would be a maximal subspace, which is a contradiction. In this way we can easily see that

$$S(\theta_1) \cap S(\theta_4) = S(\theta_2) \cap S(\theta_4) = S(\theta_3) \cap S(\theta_4) = [\alpha_0]\,.$$

Obviously, we can find $\alpha_4 \in S(\theta_4)$ such that $\theta_4 = \alpha_0 \wedge \alpha_4$. Proceeding in this way, we find easily the desired result. Moreover, we can see that the subspace $A$ is contained in the subspace $A_{n-1} \subset \Lambda^2 V^*$ with the basis

$$\alpha_0 \wedge \alpha_1, \dots, \alpha_0 \wedge \alpha_{n-1}\,,$$

where $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ is a basis of $V$. It is clear that this subspace is a maximal subspace consisting of decomposable 2-forms. Moreover, this subspace is uniquely determined. Namely, for any two linearly independent 2-forms $\theta, \theta' \in A$ we have $S(\theta) \cap S(\theta') = [\alpha_0]$. Denoting $B_0 = [\alpha_0]$, we get

$$A_{n-1} = B_0 \wedge V^*.$$

□

Before proceeding further, let us recall now that with every 3-form $\omega$ on $V$ we associate a subset $\Delta^2(\omega)$ defined by

$$\Delta^2(\omega) = \{v \in V; (\iota_v\omega) \wedge (\iota_v\omega) = 0\}.$$

In other words, $\Delta^2(\omega)$ is the subset of all $v \in V$ such that $\iota_v\omega$ is a decomposable 2-form.

We shall consider now a $(2n+1)$-dimensional real vector space $V$. Let us choose a basis $e_0, \dots, e_{2n}$ of $V$, and let $\alpha_0, \dots, \alpha_{2n}$ be the dual basis. We shall consider a multisymplectic 3-form

$$\omega = \alpha_0 \wedge (\alpha_1 \wedge \alpha_2 + \cdots + \alpha_{2n-1} \wedge \alpha_{2n}).$$

We find easily that $\Delta^2(\omega) = V_{n-1}$, where

$$V_{n-1} = \{v \in V; \alpha_0(v) = 0\} = [e_1, \dots, e_{2n}].$$

Moreover, we can see that the injective homomorphism defined by $v \mapsto \iota_v\omega$ maps $V_{n-1}$ isomorphically onto $B_0 \wedge V^*$, where we denote again $B_0 = [\alpha_0]$.

Our final task is to consider a multisymplectic 3-form $\omega$ on $V$, $\dim V \geq 5$, such that $\Delta^2(\omega) = V_{2n}$ is a $2n$-dimensional subspace of $V$. The mapping

$$V_{2n} \to \Lambda^2 V^*, \quad v \mapsto \iota_v\omega = \omega(v, \cdot, \cdot)$$

is injective, and its image $A_{2n}$ is a $2n$-dimensional subspace of $\Lambda^2 V^*$ consisting of decomposable 2-forms. According to Proposition 6 there exists a form $\alpha_0$ such that $\alpha_0 \wedge A_{2n} = 0$. This means that for every $v \in V_{2n}$ we have

$$\begin{gathered} \alpha_0 \wedge (\iota_v\omega) = 0 \\ -\iota_v(\alpha_0 \wedge \omega) + \alpha_0(v)\omega = 0. \end{gathered}$$

Applying $\iota_v$ to the last equality, we get

$$\alpha_0(v)\iota_v\omega = 0,$$

which omplies that $\alpha_0|V_{2n} = 0$.

We complete now $\alpha_0$ to a basis $\alpha_0, \beta_1, \dots, \beta_{2n}$ of $V^*$. Let us write

$$\omega = \alpha_0 \wedge \theta + \zeta,$$

where $\theta \in \Lambda^2[\beta_1, \dots, \beta_{2n}]$ and $\zeta \in \Lambda^3[\beta_1, \dots, \beta_{2n}]$. For any $v \in V_{2n}$ we have

$$0 = \alpha_0 \wedge (\iota_v\omega) = \alpha_0 \wedge (-\alpha_0 \wedge (\iota_v\theta) + \iota_v\zeta) = \alpha_0 \wedge \iota_v\zeta,$$

which shows that $\iota_v\zeta = 0$ for every $v \in V_{2n}$, and consequently $\zeta = 0$. We have thus proved that

$$\omega = \alpha_0 \wedge \theta, \quad \text{where} \quad \theta \in \Lambda^2[\beta_1, \dots, \beta_{2n}].$$

We take now the dual basis $e_0, e_1, \dots, e_{2n}$ to the basis $\alpha_0, \beta_1, \dots, \beta_{2n}$. For $v \in V_{2n}$, $v \neq 0$ we have $\alpha_0 \wedge \iota_v\omega = 0$, and therefore there exists a nonzero form $\gamma_v$ such that $\iota_v\omega = \alpha_0 \wedge \gamma_v$. Now we can compute

$$\iota_v\theta = \iota_v\iota_{e_0}(\alpha_0 \wedge \theta) = \iota_v\iota_{e_0}\omega. = -\iota_{e_0}\iota_v\omega = -\iota_{e_0}(\alpha_0 \wedge \gamma_v) = -\gamma_v\,,$$

which shows that $\iota_v\theta \neq 0$. This implies that the 2-form $\theta|V_{2n}$ is regular. Therefore we can find forms $\alpha_1, \dots, \alpha_{2n}$ such that

$$[\alpha_1, \dots, \alpha_{2n}] = [\beta_1, \dots, \beta_{2n}]\,, \quad \text{and}$$
$$\theta = \alpha_1 \wedge \alpha_2 + \dots + \alpha_{2n-1} \wedge \alpha_{2n}\,.$$

Finally, we get

$$\omega = \alpha_0 \wedge (\alpha_1 \wedge \alpha_2 + \dots + \alpha_{2n-1} \wedge \alpha_{2n})\,.$$

We have thus proved the following proposition.

**7. Proposition.** *Let $\omega$ be a multisymplectic 3-form on a $(2n+1)$-dimensional vector space $V$, $n \geq 2$. Then there exists a basis $\alpha_0, \alpha_1, \dots, \alpha_{2n}$ of $V^*$ such that*

$$\omega = \alpha_0 \wedge (\alpha_1 \wedge \alpha_2 + \dots + \alpha_{2n-1} \wedge \alpha_{2n})$$

*if and only if $\Delta^2(\omega)$ is a $2n$-dimensional subspace of $V$. If this is the case, we have $\Delta^2(\omega) = \{v \in V; \alpha_0(v) = 0\}$.*

Let us consider now a 3-form $\omega$ on $V$, $\dim V = 2n+1$, such that $\Delta^2(\omega)$ is a subspace $V_{2n}(\omega)$ of dimension $2n$. Using the explicite form of $\omega$ described in Proposition 7, we find easily that the mapping $V \to \Lambda^2 V_{2n}^*(\omega)$, $v \mapsto (\iota_v\omega)|V_{2n}(\omega)$ has kernel $V_{2n}(\omega)$, and consequently we obtain an injective homomorphism

$$\kappa(\omega) : V/V_{2n}(\omega) \to \Lambda^2 V_{2n}^*(\omega)\,.$$

It is obvious that the image of $\kappa(\omega)$ is a 1-dimensional subspace each nonzero element of which is a symplectic form on $V_{2n}(\omega)$. These data characterize completely the form $\omega$. Namely, we have the following proposition.

**8. Proposition.** *Let us assume that the following data are given:*

(i) *$2n$-dimensional subspace of $V_{2n} \subset V$,*
(ii) *1-dimensional subspace $A_1 \subset \Lambda^2 V_{2n}^*$ each nonzero element of which is a symplectic form,*
(iii) *an isomorphism $\kappa : V/V_{2n} \to A_1$.*

*Then there is a unique 3-form $\omega \in \Lambda^3 V^*$ such that $V_{2n}(\omega) = V_{2n}$, $\operatorname{im}\kappa(\omega) = A_1$, and $\kappa(\omega) = \kappa$.*

**Proof.** Let us take a nonzero 1-form $\alpha_0$ on $V$ such that $\alpha_0|V_{2n} = 0$, and a nonzero symplectic form $\sigma \in A_1$. Next, let us choose a 2-form $\hat{\sigma}$ on $V$ such that $\hat{\sigma}|V_{2n} = \sigma$. It is easy to see that the 3-form $\alpha_0 \wedge \hat{\sigma}$ does not depend on the choice of $\hat{\sigma}$. Now, it suffices to take $\omega = c\alpha_0 \wedge \hat{\sigma}$ with conveniently choosen $c \neq 0$. The unicity is obvious. □

The last proposition makes easier the construction of 3-forms $\omega$ of the type under consideration on odd dimensional vector bundles.

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MATHEMATICAL INSTITUTE, ACADEMY OF SCIENCES OF THE CZECH REPUBLIC
ŽIZKOVA 22, 616 62 BRNO, CZECH REPUBLIC
E-MAIL: `vanzura@drs.ipm.cz`