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SPECIAL CONNECTIONS ON SYMPLECTIC MANIFOLDS

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ABSTRACT. On a given symplectic manifold, there are many symplectic connections, i.e. torsion free connections w.r.t. which the symplectic form is parallel. We call such a connection special if it is either the Levi-Civita connection of a Bochner-Kähler metric of arbitrary signature, a Bochner-bi-Lagrangian connection, a connection of Ricci type or a connection with special symplectic holonomy.

We link these special connections to parabolic contact geometry, showing that the symplectic reduction of (an open cell of) a parabolic contact manifold by a symmetry vector field is special symplectic in a canonical way. Moreover, we show that any special symplectic manifold or orbifold is locally equivalent to one of these symplectic reductions.

As a consequence, we are able to prove a number of rigidity results and other global properties.

1. INTRODUCTION

Torsion free connections on a differentiable manifold M which preserve a given geometric structure are among the basic objects of interest in differential geometry. For example, if M carries a Riemannian metric, then there is a unique torsion free connection which is compatible with this metric, called the Levi-Civita connection. Thus, every feature of the connection reflects a property of the metric structure.

In contrast, for a symplectic manifold (M, ω), there are many symplectic connections, where we call a connection on M symplectic if it is torsion free and ω is parallel. Thus, in order to investigate 'meaningful' symplectic connections, we have to impose further conditions.

In [CS], the notion of a *special symplectic connection* was established. Special symplectic connections are defined as symplectic connections on a manifold of dimension at least 4 which belong to one of the following seemingly unrelated classes.

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1. **Bochner-Kähler and Bochner-bi-Lagrangian connections**
   If the symplectic form is the Kähler form of a (pseudo-)Kähler metric, then its curvature decomposes into the Ricci curvature and the Bochner curvature ([Bo]). If the latter vanishes, then (the Levi-Civita connection of) this metric is called Bochner-Kähler.
   Similarly, if the manifold is equipped with a bi-Lagrangian structure, i.e. two complementary Lagrangian distributions, then the curvature of a symplectic connection for which both distributions are parallel decomposes into the Ricci curvature and the Bochner curvature. Such a connection is called Bochner-bi-Lagrangian if its Bochner curvature vanishes.
   For results on Bochner-Kähler and Bochner-bi-Lagrangian connections, see [Br2] and [K] and the references cited therein. We shall give a brief summary of these structures in section 2.2.
2. **Connections of Ricci type**
   Under the action of the symplectic group, the curvature of a symplectic connection decomposes into two irreducible summands, namely the Ricci curvature and a Ricci flat component. If the latter component vanishes, then the connection is said to be of Ricci type.
   Connections of Ricci type are critical points of a certain functional on the moduli space of symplectic connections ([BC1]). Furthermore, the canonical almost complex structure on the twistor space induced by a symplectic connection is integrable iff the connection is of Ricci type ([BR], [V]). For further properties see also [CGR], [CGHR], [BC2], [CGS]. We shall treat these connections in section 2.1.
3. **Connections with special symplectic holonomy**
   A symplectic connection is said to have *special symplectic holonomy* if its holonomy is contained in a proper absolutely irreducible subgroup of the symplectic group.
   The special symplectic holonomies have been classified in [MS] and further investigated in [Br1], [CMS], [S1], [S2], [S3]. These connections shall be discussed in section 2.3.

At first, it may seem unmotivated to collect all these structures in one definition, but we shall provide ample justification for doing so. Indeed, our main results show that there is a beautiful link between special symplectic connections and parabolic contact geometry.

For this, consider a simple Lie group G with Lie algebra $\mathfrak{g}$. We say that $\mathfrak{g}$ is 2-gradable, if $\mathfrak{g}$ contains the root space of a long root. In this case, the projectivization of the adjoint orbit of a maximal root vector $\mathcal{C} \subset \mathbb{P}^o(\mathfrak{g})$ carries a canonical G-invariant contact structure. Here, $\mathbb{P}^o(V)$ denotes the set of oriented lines through 0 of a vector space V, so that $\mathbb{P}^o(V)$ is diffeomorphic to a sphere. Each $a \in \mathfrak{g}$ induces an action field a^* on $\mathcal{C}$ with flow $\mathrm{T}_a := \exp(\mathbb{R}a) \subset \mathrm{G}$, which hence preserves the contact structure on $\mathcal{C}$. Let $\mathcal{C}_a \subset \mathcal{C}$ be the open subset on which a^* is positively transversal to the contact distribution. We can cover $\mathcal{C}_a$ by open sets U such that the local quotient $M_U := \mathrm{T}_a^{\mathrm{loc}} \backslash U$, i.e. the quotient of U by a sufficiently small neighborhood of the identity in T_a, is a manifold. Then M_U inherits a canonical symplectic structure. Our first main result is the following (cf. Theorem 3.11).

Theorem A [CS]. *Let $\mathfrak{g}$ be a simple 2-gradable Lie algebra with $\dim \mathfrak{g} \geq 14$, and let $\mathcal{C} \subset \mathbb{P}^o(\mathfrak{g})$ be the projectivization of the adjoint orbit of a maximal root vector. Let $a \in \mathfrak{g}$ be such that $\mathcal{C}_a \subset \mathcal{C}$ is nonempty, and let $\mathrm{T}_a = \exp(\mathbb{R}a) \subset \mathrm{G}$. If for an open subset $U \subset \mathcal{C}_a$ the local quotient $M_U = \mathrm{T}_a^{\mathrm{loc}} \backslash U$ is a manifold, then M_U carries a special symplectic connection.*

The dimension restriction on $\mathfrak{g}$ guarantees that $\dim M_U \geq 4$ and rules out the Lie algebras of type A_1, A_2 and B_2.

The type of special symplectic connection on M_U is determined by the Lie algebra $\mathfrak{g}$. In fact, there is a one-to-one correspondence between the various conditions for special symplectic connections and simple 2-gradable Lie algebras. More specifically, if the Lie algebra $\mathfrak{g}$ is of type A_n, then the connections in Theorem A are Bochner-Kähler of signature (p, q) if $\mathfrak{g} = \mathfrak{su}(p+1, q+1)$ or Bochner-bi-Lagrangian if $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$; if $\mathfrak{g}$ is of type C_n, then $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$ and these connections are of Ricci type; if $\mathfrak{g}$ is a 2-gradable Lie algebra of one of the remaining types, then the holonomy of M_U is contained in one of the special symplectic holonomy groups. Also, for two elements $a, a' \in \mathfrak{g}$ for which $\mathcal{C}_a, \mathcal{C}_{a'} \subset \mathcal{C}$ are nonempty, the corresponding connections from Theorem A are equivalent iff a' is G-conjugate to a positive multiple of a.

Surprisingly, the connections from Theorem A exhaust *all* special symplectic connections, at least locally. Namely we have the following

Theorem B [CS]. *Let (M, ω) be a symplectic manifold with a special symplectic connection of class C^4, and let $\mathfrak{g}$ be the Lie algebra associated to the special symplectic condition as above.*

1. *Then there is a principal $\hat{\mathrm{T}}$-bundle $\hat{M} \to M$, where $\hat{\mathrm{T}}$ is a one dimensional Lie group which is not necessarily connected, and this bundle carries a principal connection with curvature ω.*
2. *Let $\mathrm{T} \subset \hat{\mathrm{T}}$ be the identity component. Then there is an $a \in \mathfrak{g}$ such that $\mathrm{T} \cong \mathrm{T}_a \subset \mathrm{G}$, and a T_a-equivariant local diffeomorphism $\hat{\imath} : \hat{M} \to \mathcal{C}_a$ which for each sufficiently small open subset $V \subset \hat{M}$ induces a connection preserving diffeomorphism $\imath : \mathrm{T}^{\mathrm{loc}} \backslash V \to \mathrm{T}_a^{\mathrm{loc}} \backslash U = M_U$, where $U := \hat{\imath}(V) \subset \mathcal{C}_a$ and M_U carries the connection from Theorem A.*

The situation in Theorem B can be illustrated by the following commutative diagram, where the vertical maps are quotients by the indicated Lie groups, and $\mathrm{T}\backslash\hat{M} \to M$ is a regular covering.

$$\begin{tikzcd} & \hat{M} \arrow[r, "\hat{\imath}"] \arrow[dl, "\hat{\mathrm{T}}"'] \arrow[d, "\mathrm{T}"] & \mathcal{C}_a \arrow[d, "\mathrm{T}_a"] \\ M & \mathrm{T}\backslash\hat{M} \arrow[l] \arrow[r, "\imath"] & \mathrm{T}_a\backslash\mathcal{C}_a \end{tikzcd}$$

In fact, one might be tempted to summarize Theorems A and B by saying that for each $a \in \mathfrak{g}$, the quotient $\mathrm{T}_a\backslash\mathcal{C}_a$ carries a canonical special symplectic connection, and the map $\imath : \mathrm{T}\backslash\hat{M} \to \mathrm{T}_a\backslash\mathcal{C}_a$ is a connection preserving local diffeomorphism. If $\mathrm{T}_a\backslash\mathcal{C}_a$ is a manifold or an orbifold, then this is indeed correct. In general, however, $\mathrm{T}_a\backslash\mathcal{C}_a$ may be neither Hausdorff nor locally Euclidean, hence one has to formulate these results more carefully.

As consequences, we obtain the following

Corollary C [CS]. *All special symplectic connections of C^4-regularity are analytic, and the local moduli space of these connections is finite dimensional, in the sense that the germ of the connection at one point up to 3rd order determines the connection entirely. In fact, the generic special symplectic connection associated to the Lie algebra $\mathfrak{g}$ depends on $(\mathrm{rk}(\mathfrak{g})-1)$ parameters.*

Moreover, the Lie algebra $\mathfrak{s}$ of vector fields on M whose flow preserves the connection is isomorphic to $\mathrm{stab}\,(a)/(\mathbb{R}a)$ with $a\in\mathfrak{g}$ from Theorem B, where $\mathrm{stab}\,(a)=\{x\in\mathfrak{g}\mid [x,a]=0\}$. In particular, $\dim\mathfrak{s}\geq\mathrm{rk}(\mathfrak{g})-1$ with equality implying that $\mathfrak{s}$ is abelian.

When counting the parameters in the above corollary, we regard homothetic special symplectic connections as equal, i.e. (M,ω,∇) is considered equivalent to $(M,e^{t_0}\omega,\nabla)$ for all $t_0\in\mathbb{R}$.

We can generalize Theorem B and Corollary C easily to orbifolds. Indeed, if M is an orbifold with a special symplectic connection, then we can write $M=\hat{\mathrm{T}}\backslash\hat{M}$ where $\hat{M}$ is a manifold and $\hat{\mathrm{T}}$ is a one dimensional Lie group acting properly and locally freely on $\hat{M}$, and there is a local diffeomorphism $\hat{\imath}:\hat{M}\to\mathcal{C}_a$ with the properties stated in Theorem B.

We also address the question of the existence of compact manifolds with special symplectic connections. In the simply connected case, compactness already implies that the connection is Hermitean symmetric. More specifically, we have the following

Theorem D [CS]. *Let M be a compact simply connected manifold with a special symplectic connection of class C^4. Then M is equivalent to one of the following Hermitean symmetric spaces.*

1. *$M\cong(\mathbb{CP}^p\times\mathbb{CP}^q,((q+1)g_0,-(p+1)g_0))$, where g_0 is the Fubini-Study metric. These are Bochner-Kähler metrics of signature (p,q). Moreover, $M\cong(\mathbb{CP}^n,g_0)$ is also of Ricci type.*
2. *$M\cong\mathrm{SO}(n+2)/(\mathrm{SO}(2)\cdot\mathrm{SO}(n))$, whose holonomy is contained in the special symplectic holonomy group $\mathrm{SL}(2,\mathbb{R})\cdot\mathrm{SO}(n)\subset\mathrm{Aut}(\mathbb{R}^2\otimes\mathbb{R}^n)$.*
3. *$M\cong\mathrm{SU}(2n+2)/\mathrm{S}(\mathrm{U}(2)\cdot\mathrm{U}(2n))$, whose holonomy is contained in the special symplectic holonomy group $\mathrm{Sp}(1)\cdot\mathrm{SO}(n,\mathbb{H})\subset\mathrm{Aut}(\mathbb{H}^n)$.*
4. *$M\cong\mathrm{SO}(10)/\mathrm{U}(5)$, whose holonomy is contained in the special symplectic holonomy group $\mathrm{SU}(1,5)\subset\mathrm{GL}(20,\mathbb{R})$.*
5. *$M\cong\mathrm{E}_6/(\mathrm{U}(1)\cdot\mathrm{Spin}(10))$, whose holonomy is contained in the special symplectic holonomy group $\mathrm{Spin}(2,10)\subset\mathrm{GL}(32,\mathbb{R})$.*

In particular, there are no compact simply connected manifolds with any of the remaining types of special symplectic connections, i.e. M can be neither Bochner-bi-Lagrangian, nor can the holonomy of M be contained in any of the remaining special symplectic holonomies.

This paper is structured as follows. Following this introduction, we first describe the various types of special symplectic connections and review some of their geometric features. We then show how on a formal level, we can link these conditions to the algebraic formalism of 2-gradable simple Lie algebras. In section 3, we describe 2-gradable Lie algebras and perform the symplectic reduction, showing that the local quotients $T_a^{loc}\backslash\mathcal{C}_a$ carry special symplectic connections in a canonical way, and hence prove Theorem A. In section 4, we investigate the structure equations of special symplectic connections and derive results which culminate in Theorem B. Finally, in section 5 we show the existence of connection preserving vector fields and Corollary C, and the rigidity result from Theorem D.

This report is closely related to the reference [CS]. In fact, the main results are shown in that paper, and we shall refer to it for many of the proofs. In this report, we emphasize the more concrete description of special symplectic connections and their various geometrical features as opposed to the more abstract construction from [CS], and we describe the link to parabolic contact geometry in more detail.

2. Symplectic connections

Let (M, ω) be a symplectic manifold. A *symplectic connection* is a connection on the (tangent bundle of) M which is torsion free and for which ω is parallel.

It is not hard to see that symplectic connections exist on any symplectic manifold (M, ω). Namely, by Darboux's theorem, around each point in M we can find a coordinate system $(x^1, \dots, x^n)$ in which

$$\omega = \omega_{ij}\, dx^i \wedge dx^j\,,$$

where $(\omega)_{i,j}$ is a constant skew symmetric non-degenerate matrix. Then a connection is symplectic iff the Christoffel symbols Γ_{ij}^k have the property that the tensor

$$\sigma_{ijk} := \sum_l \omega_{il}\Gamma_{jk}^l$$

is totally symmetric. Thus, there are many symplectic connections. In coordinate free notation, this is seen by the observation that for two symplectic connections ∇ and ∇' the tensor

$$\sigma(X, Y, Z) := \omega(\nabla'_X Y - \nabla_X Y, Z) \tag{1}$$

is totally symmetric, i.e. $\sigma \in \Gamma(S^3(TM))$. Conversely, given a symplectic connection ∇ and $\sigma \in \Gamma(S^3(TM))$, then (1) determines the symplectic connection ∇'. Thus, the space of symplectic connections is an affine space whose linear part is given by the sections of $S^3(TM)$. In particular, this space is infinite dimensional, even if we take the quotient by the action of the symplectomorphism group.

This may be one of the reasons why symplectic connections in their full generality are difficult to be utilized in order to obtain information on the underlying symplectic manifold. Thus, it is natural to put certain 'reasonable' restrictions on their curvature and thereby single out symplectic connections with special behaviour.

In order to make this more precise, we recall the following terminology. For a given Lie subalgebra $\mathfrak{h} \subset \mathrm{End}(V)$ we define the space of *formal curvature maps* as

$$K(\mathfrak{h}) := \{R \in \Lambda^2 V^* \otimes \mathfrak{h} \mid R(x,y)z + R(y,z)x + R(z,x)y = 0 \text{ for all } x, y, z \in V\}\,.$$

This terminology is due to the fact that the curvature map of a torsion free connection always satisfies the first Bianchi identity. Thus, $R_p \in K(\mathfrak{h}_p)$ where $\mathfrak{h}_p \subset \mathrm{End}(T_pM)$ is a subalgebra which contains the holonomy algebra of the connection. Note that $K(\mathfrak{h})$ is an $\mathfrak{h}$-module in an obvious way, with the action given by the formula

$$(h \cdot R)(x,y) := [h, R(x,y)] - R(hx,y) - R(x,hy) \quad \text{for all} \quad h \in \mathfrak{h} \quad \text{and} \quad x,y \in V.$$

There is an $\mathfrak{h}$-equivariant map $\mathrm{Ric} : K(\mathfrak{h}) \to V^* \otimes V^*$, given by

$$\mathrm{Ric}\,(R)(x,y) := tr(R(\_,x)y).$$

The Bianchi identity implies that $\mathrm{Ric}\,(R)(x,y) - \mathrm{Ric}\,(R)(y,x) = -trR(x,y)$, hence the image of Ric lies in $S^2(V^*) \subset V^* \otimes V^*$ if $\mathfrak{h} \subset \mathrm{End}(V)$ consists of trace free endomorphisms.

2.1. **Connections of Ricci type.** In general, given a symplectic vector space (V,ω), i.e. $\omega \in \Lambda^2 V^*$ is non-degenerate, we define the *symplectic group* $\mathrm{Sp}(V,\omega)$ and the *symplectic Lie algebra* $\mathfrak{sp}(V,\omega)$ by

$$\mathrm{Sp}(V,\omega) := \{g \in \mathrm{Aut}(V) \mid \omega(gx,gy) = \omega(x,y) \text{ for all } x,y \in V\},$$
$$\mathfrak{sp}(V,\omega) := \{h \in \mathrm{End}(V) \mid \omega(hx,y) + \omega(x,hy) = 0 \text{ for all } x,y \in V\}.$$

Then $\mathrm{Sp}(V,\omega)$ is a Lie group with Lie algebra $\mathfrak{sp}(V,\omega)$.

If ∇ is a *symplectic* connection, then $\omega_p \in \Lambda^2 T_pM$ is invariant under the holonomy group, hence $Hol_p \subset \mathrm{Sp}(T_pM,\omega_p)$ so that at each point $p \in M$ the curvature $R_p \in K(\mathfrak{sp}(T_pM,\omega_p))$.

Note that $\mathfrak{sp}(V,\omega)$ consists of trace free endomorphisms, so that we have the $\mathfrak{sp}(V,\omega)$-equivariant map

$$\mathrm{Ric} : K(\mathfrak{sp}(V,\omega)) \longrightarrow S^2(V^*).$$

More explicitly, this map is given by the following

Lemma 2.1. *Let* $R \subset K(\mathfrak{sp}(V,\omega))$. *Then* $\mathrm{Ric}\,(R)(x,y) = \omega(R(\omega^{-1})x,y)$.

Proof. Let (e_i, f_i) be a basis of V such that, using the summation convention, $\omega^{-1} = e_i \wedge f_i$. Thus,

$$\begin{aligned}\mathrm{Ric}\,(R)(x,y) &= tr(R(\_,x)y) = \omega(R(e_i,x)y,f_i) - \omega(R(f_i,x)y,e_i)\\ &= \omega(R(e_i,x)f_i,y) + \omega(R(x,f_i)e_i,y) = \omega(R(e_i,f_i)x,y). \qquad \square\end{aligned}$$

Note that as a $\mathrm{Sp}(V,\omega)$-module, we have $V \cong V^*$ and $\mathfrak{sp}(V,\omega) \cong S^2V^* \cong S^2V$, with an isomorphism for the latter being given by the equivariant map

(2) $\quad \circ : S^2V \longrightarrow \mathfrak{sp}(V,\omega), \quad (x \circ y)z := \omega(x,z)y + \omega(y,z)x$ for all $x,y,z \in V$.

In fact, by virtue of Lemma 2.1 we may reinterpret the map Ric as

$$\mathrm{Ric} : K(\mathfrak{sp}(V,\omega)) \longrightarrow \mathfrak{sp}(V,\omega), \qquad R \longmapsto R(\omega^{-1}).$$

Now we make the following

Definition 2.2. A Lie subalgebra $\mathfrak{h} \subset \mathfrak{sp}(V,\omega)$ is called *special symplectic* if there is an $\mathfrak{h}$-equivariant linear map $\circ : S^2(V) \to \mathfrak{h}$ satisfying

(3) $\quad (x \circ y)z - (x \circ z)y = 2\,\omega(y,z)x - \omega(x,y)z + \omega(x,z)y \quad$ for all $x,y,z \in V$.

By (2), it is evident that $\mathfrak{sp}(V,\omega)$ is special symplectic. Moreover, it is straightforward to verify the following important fact.

Proposition 2.3. *Let $\mathfrak{h} \subset \mathfrak{sp}(V,\omega)$ be a special symplectic subalgebra. Then for each $\rho \in \mathfrak{h}$, the map*

$$R_\rho : \Lambda^2 V \longrightarrow \mathfrak{sp}(V,\omega)\,, \quad R_\rho(x,y) := 2\omega(x,y)\rho + x \circ \rho y - y \circ \rho x \tag{4}$$

satisfies the first Bianchi identity, i.e. $R_\rho \in K(\mathfrak{h})$.

From this, we now obtain the following

Proposition 2.4. *For $\rho \in \mathfrak{sp}(V,\omega)$, we have* $\mathrm{Ric}\,(R_\rho)(x,y) = (\dim V + 2)\omega(\rho x, y)$. *Moreover, as a* $\mathrm{Sp}(V,\omega)$*-module, we have the decomposition*

$$K(\mathfrak{sp}(V,\omega)) = \mathcal{R} \oplus \mathcal{W}\,,$$

where

$$\mathcal{R} := \{R_\rho \mid \rho \in \mathfrak{sp}(V,\omega)\} \cong \mathfrak{sp}(V,\omega)\,, \quad \text{and} \quad \mathcal{W} := \ker(\mathrm{Ric})\,.$$

Thus, $\mathcal{R}$ is irreducible. Moreover, $\dim \mathcal{W} = \frac{1}{8}(n-2)n(n+1)(n+3)$ *where* $n := \dim V$. *In particular, $\mathcal{W} \neq 0$ iff $\dim V \geq 4$, and in this case, $\mathcal{W}$ is irreducible as well.*

Proof. Since $\mathfrak{sp}(V,\omega)$ is special, it follows that $R_\rho \in K(\mathfrak{sp}(V,\omega))$ by Proposition 2.3. Moreover, it is straightforward to verify that $R_\rho(\omega^{-1}) = (n+2)\rho$ which implies the second assertion. In particular, Ric : $K(\mathfrak{sp}(V,\omega)) \to S^2V^*$ is surjective, so that the asserted decomposition follows. $\mathcal{R}$ is irreducible as $\mathfrak{sp}(V,\omega)$ is simple. Now consider the sequence of $\mathrm{End}(V)$-equivariant linear maps

$$0 \longrightarrow S^4V^* \longrightarrow S^3V^* \otimes V^* \longrightarrow S^2V^* \otimes \Lambda^2V^* \longrightarrow V^* \otimes \Lambda^3V^* \longrightarrow \Lambda^4V^* \longrightarrow 0\,,$$

which are given by skew symmetrization. Since we can regard these maps as the exterior differentiation of differential forms on V with polynomial coefficients, it follows easily that this sequence is exact.

Let us now consider the map $B : \Lambda^2V^* \otimes \mathfrak{sp}(V,\omega) \to \Lambda^3V^* \otimes V$ which is given by $B(R)(x,y,z) := R(x,y)z + R(y,z)x + R(z,x)y$. Thus, $K(\mathfrak{sp}(V,\omega)) = \ker(B)$.

Using the isomorphism $\mathfrak{sp}(V,\omega) \cong S^2V^*$ from (2), it is easy to verify that $B : \Lambda^2V^* \otimes S^2V^* \to \Lambda^3V^* \otimes V^*$ coincides up to a multiple with the differential in the above exact sequence, hence $K(\mathfrak{sp}(V,\omega)) \cong (S^3V^* \otimes V^*)/S^4V^*$, and from this, the assertions follow from a dimension count and by standard representation theoretical arguments. □

By virtue of this proposition, we can now decompose

$$K(\mathfrak{sp}(T_pM,\omega_p)) \cong \mathcal{R}_p \oplus \mathcal{W}_p\,. \tag{5}$$

for each $p \in M$, and we make the following

Definition 2.5. Let (M,ω,∇) be a symplectic manifold with a symplectic connection. We say that ∇ is of *Ricci type* if its curvature satisfies $R^\nabla_p \in \mathcal{R}_p$ for all $p \in M$ with the decomposition (5).

If ∇ is of Ricci type, then there is a unique section ρ of endomorphisms of the tangent spaces of M for which the curvature of ∇ at each point has the form (4).

We shall now give a more geometric interpretation of connections of Ricci type. For this, consider a symplectic manifold (M,ω), and define its *twistor space* as

$$\mathcal{Z} := \left\{J_p \in \mathfrak{sp}(T_pM,\omega_p) \mid J_p^2 = -\mathrm{Id}\,,\ \omega_p(J_p\_,\_) \text{ is positive definite.}\right\}$$

The fibers of the canonical fibration $\pi : \mathcal{Z} \to M$ can be identified with the Hermitean symmetric space $\mathrm{Sp}(n, \mathbb{R})/\mathrm{U}(n)$ and hence carry a canonical complex structure. Also, if ∇ is a connection on M, then ∇ induces a connection on $\mathrm{End}(TM) = T^*M \otimes TM$, and evidently, $\mathcal{Z} \subset \mathrm{End}(TM)$ is parallel w.r.t. ∇. Thus, ∇ induces a decomposition of the tangent space

$$T_{J_p}\mathcal{Z} = \mathcal{V}_{J_p} \oplus \mathcal{H}_{J_p},$$

where $\mathcal{V}_{J_p} = \ker(d\pi_{J_p})$ is the vertical space. We now define the almost complex structure J on $\mathcal{Z}$ by the requirement that

1. $J(\mathcal{H}_{J_p}) = \mathcal{H}_{J_p}$ and $J(\mathcal{V}_{J_p}) = \mathcal{V}_{J_p}$ for all $J_p \in \mathcal{Z}$,
2. $J|_{\mathcal{V}_{J_p}}$ coincides with the complex structure of $\pi^{-1}(p) \cong \mathrm{Sp}(n, \mathbb{R})/\mathrm{U}(n)$,
3. $d\pi : (\mathcal{H}_{J_p}, J) \to (T_pM, J_p)$ is complex linear.

Then the following is known.

Proposition 2.6. [BR] [V] *Let ∇ be a symplectic conection on (M, ω). Then the almost complex structure on the twistor space $\mathcal{Z} \to M$ which is induced by ∇ is integrable iff ∇ is of Ricci type.*

2.2. **Bochner-Kähler and Bochner-bi-Lagrangian connections.** Suppose that ∇ is the Levi-Civita connection of a Kähler metric g on M with Kähler form ω. Here, we use the term Kähler metric for a metric of arbitrary signature unless explicitly stated otherwise. Thus, there is a complex structure J on M with $g(x, y) = \omega(Jx, y)$ for all $x, y \in TM$, and J is parallel w.r.t. ∇. It follows that the curvature of ∇ takes values in the Lie algebra $\mathfrak{u}(p, q)$, where (p, q) is the complex signature of the Kähler metric.

A *bi-Lagrangian structure* on a symplectic manifold (M, ω) is a splitting of the tangent bundle $TM = L_1 \oplus L_2$ where L_i are Lagrangian distributions. A symplectic connection ∇ on such an M is called *bi-Lagrangian* if both distributions are parallel. We can define a section J of the endomorphism bundle by $J|_{L_i} = (-1)^i \,\mathrm{Id}_{L_i}$. Evidently, $J_p \in \mathfrak{sp}(T_pM, \omega_p)$ for all $p \in M$, and $J^2 = \mathrm{Id}$. Conversely, given a section J with $J_p \in \mathfrak{sp}(T_pM, \omega_p)$ for all $p \in M$ and $J^2 = \mathrm{Id}$ as above, we get a splitting of TM in to the Eigenspaces of J, and it follows that both of them must be isotropic and hence Lagrangian. Evidently, J is parallel w.r.t. any bi-Lagrangian connection ∇, hence the curvature of ∇ takes values in the Lie algebra $\mathfrak{u}'(m) \subset \mathfrak{sp}(m, \mathbb{R})$ which is defined as the stabilizer of the endomorphism $J \in \mathfrak{sp}(m, \mathbb{R})$ with $J^2 = \mathrm{Id}$.

Therefore, in order to determine the curvature spaces $K(\mathfrak{u}(p, q))$ and $K(\mathfrak{u}'(m))$ simultaneously, we observe that they are all of the form $\mathfrak{h}_J := \{x \in \mathfrak{sp}(m, \mathbb{R}) \mid [x, J] = 0\}$, where in the case of $\mathfrak{u}(p, q)$, $J \in \mathfrak{sp}(m, \mathbb{R})$ is the invariant complex structure so that $J^2 = -\mathrm{Id}$, and in the case of $\mathfrak{u}'(m)$, $J \in \mathfrak{sp}(m, \mathbb{R})$ is the endomorphism with $J^2 = \mathrm{Id}$.

We define the circle product

$$\circ : S^2(V) \longrightarrow \mathfrak{h}_J,$$

$$(x \circ y)z := \omega(x, z)y + \omega(y, z)x - \varepsilon(\omega(Jx, z)Jy + \omega(Jy, z)Jx + \omega(Jx, y)Jz), \tag{6}$$

where $\varepsilon \in \{\pm 1\}$ is given by $J^2 = \varepsilon \,\mathrm{Id}$. Indeed, it is straightforward to verify that $\omega((x \circ y)z, w) + \omega(z, (x \circ y)w) = 0$ and $(x \circ y)Jz = J(x \circ y)z$ for all $x, y, z, w \in V$, so that $x \circ y \in \mathfrak{h}_J$ in either case. Moreover, $\circ$ also satisfies (3), so that $\mathfrak{h}_J \subset \mathfrak{sp}(V, \omega)$ is a special symplectic subalgebra in the sense of Definition 2.2, hence by Proposition 2.3 the map $R_\rho : \Lambda^2 V \to \mathfrak{h}_J$ from (4) is an element of $K(\mathfrak{h}_J)$ for all $\rho \in \mathfrak{h}_J$.

Proposition 2.7. *Let $\mathfrak{h}_J \subset \mathfrak{sp}(V, \omega)$ be one of the special symplectic subalgebras $\mathfrak{u}(p,q)$ or $\mathfrak{u}'(m)$ with the product $\circ : S^2(V) \to \mathfrak{h}_J$ from (6), and let $J \in \mathfrak{sp}(V,\omega)$ be the endomorphism with $J^2 = \varepsilon\ \mathrm{Id}_V$, $\varepsilon = \pm 1$, which is stabilized by $\mathfrak{h}_J$.*

Then for all $\rho \in \mathfrak{h}_J$, the map $R_\rho : \Lambda^2 V \to \mathfrak{h}_J$ from (4) satisfies $\mathrm{Ric}\,(R_\rho)(x,y) = 4\omega(\rho x, y) + \varepsilon\, tr(J\rho)\omega(Jx, y)$ for all $x, y \in V$, and as a $\mathfrak{h}_J$-module, we have the deomponsition

$$K(\mathfrak{h}_J) = \mathcal{R} \oplus \mathcal{W},$$

where

$$\mathcal{R} := \{R_\rho \mid \rho \in \mathfrak{h}_J\} \cong \mathfrak{h}_J, \quad \text{and} \quad \mathcal{W} := \ker(\mathrm{Ric}).$$

Thus, $\mathcal{R}$ decomposes into two irreducible summands, one of which is trivial and spanned by R_J. Moreover, $\dim \mathcal{W} = \frac{1}{64}n^2(n-2)(n+6)$ where $n := \dim_{\mathbb{R}} V$. In particular, $\mathcal{W} \neq 0$ iff $\dim V \geq 4$, and in this case, $\mathcal{W}$ is irreducible as well.

Proof. It is straightforward to calculate that $R_\rho(\omega^{-1}) = 4\rho + \varepsilon\, tr(J\rho)J$ form which the asserted formula for the Ricci curvature follows by Lemma 2.1. In particular, $\mathrm{Ric}\,(R_\rho) \neq 0$ for all $0 \neq \rho \in \mathfrak{h}_J$ which shows the asserted decomposition.

In order to calculate the dimension of $\mathcal{W}$ and show that it is irreducible, observe that all the Lie algebras $\mathfrak{h}_J$ have the same complexification, so it suffices to treat the Lie subalgebra $\mathfrak{h}_{\mathbb{C}} := \mathfrak{gl}(n, \mathbb{C})$ acting on the vector space $V := W \oplus W^*$, where $W := \mathbb{C}^n$ is the standard representation.

Let $x, y \in W$ and $\overline{z}, \overline{w} \in W^*$. Then for any $R \in K(\mathfrak{h}_{\mathbb{C}})$ we have $R(\overline{z}, x)y - R(\overline{z}, y)x = -R(x,y)\overline{z}$, and since the left hand side lies in W while the right hand side lies in W^*, it follows that both sides vanish.

The vanishing of the right hand side implies that $R(W, W) = 0$ since $x, y \in W$ and $\overline{z} \in W^*$ are arbitrary. Analogously, $R(W^*, W^*) = 0$. Moreover, the vanishing of the left hand side implies that $R(\overline{z}, x)y = R(\overline{z}, y)x$ and, analogously, $R(x, \overline{z})\overline{w} = R(x, \overline{w})\overline{z}$. Thus, if we define the tensor $\sigma_R \in W \otimes W \otimes W^* \otimes W^*$ by

(7) $\sigma_R(x, y, \overline{z}, \overline{w}) := \overline{w}(R(\overline{z}, x)y) = -(R(\overline{z}, x)\overline{w})y$ for all $x, y \in W$ and $\overline{z}, \overline{w} \in W^*$,

then σ_R is symmetric in x and y and in $\overline{z}$ and $\overline{w}$, i.e. $\sigma_R \in S^2(W) \otimes S^2(W^*)$.

Conversely, given $\sigma \in S^2(W) \otimes S^2(W^*)$, we verify that the map $R_\sigma : \Lambda^2(V) \to \mathfrak{h}$ determined by $R(W, W) = R(W^*, W^*) = 0$ and (7) lies in $K(\mathfrak{h}_{\mathbb{C}})$, showing that $K(\mathfrak{h}_C) \cong S^2(W) \otimes S^2(W^*)$, and the decomposition of this space as a $\mathfrak{h}_{\mathbb{C}}$-module as well as the dimension of this space follow from standard representation theoretical arguments. □

By virtue of this proposition, we have for each of the Lie algebras $\mathfrak{h}_J = \mathfrak{u}(p,q)$ or $\mathfrak{h}_J = \mathfrak{u}'(m)$

$$K(\mathfrak{h}_J) \cong \mathcal{R}_p \oplus \mathcal{W}_p. \tag{8}$$

for each $p \in M$, and we make the following

Definition 2.8. Let (M, ω, J) be a symplectic manifold with a section J of the endomorphism bundle of M such that $J_p \in \mathfrak{sp}(T_pM, \omega_p)$ and $J_p^2 = \varepsilon\ \mathrm{Id}_{T_pM}$ for all $p \in M$, where $\varepsilon = \pm 1$.

Consider a symplectic connection ∇ for which J is parallel which thus is bi-Lagrangian if $\varepsilon = 1$, or the Levi-Civita connection of the Kähler metric $g := \omega(J\_, \_)$ if $\varepsilon = -1$.

If the curvature of ∇ satisfies $R_p^\nabla \in \mathcal{R}_p$ for all $p \in M$ with the decomposition (8), then we call ∇ *Bochner-bi-Lagrangian* in the first and *Bochner-Kähler* in the second case.

If ∇ is Bochner-Kähler or Bochner-bi-Lagrangian, then there is a unique section ρ of endomorphisms of the tangent spaces of M for which the curvature of ∇ at each point has the form (4).

For Kähler metrics, the decomposition (8) has first been achieved by Bochner ([Bo]), and for this reason, the Ricci flat component $\mathcal{W}$ of the curvature is called the *Bochner curvature* of the Kähler metric. The same terminology is also adapted for bi-Lagrangian connections.

2.3. **Connections with special symplectic holonomy.** Let (M, ω) be a symplectic manifold which in this section we assume to be simply connected. If ∇ is a symplectic connection on M, then evidently, ω is invariant under parallel translation, so that the holonomy group of ∇ is contained in the symplectic group $\mathrm{Sp}(V, \omega)$. We say that ∇ has *special symplectic holonomy* if the holonomy group of ∇ is contained in an absolutely irreducible proper subgroup of $\mathrm{Sp}(V, \omega)$. Here, recall that a subgroup $\mathrm{H} \subset \mathrm{Aut}(V)$ is called *absolutely irreducible* if H acts irreducibly on V, and the complexified group $\mathrm{H}_{\mathbb{C}}$ acts irreducibly on $V_{\mathbb{C}} := V \otimes \mathbb{C}$ under the complexified representation.

The absolutely irreducible proper subgroups $\mathrm{H} \subset \mathrm{Sp}(V, \omega)$ which can occur as the holonomy of a torsion free connection have been classified. The list of these connections is the following (cf. [MS], [S3]).

Table 1: List of Real Special Symplectic Holonomies

Group H	Representation space	Group H	Representation space
$\mathrm{SL}(2,\mathbb{R})$	$\mathbb{R}^4 \simeq S^3(\mathbb{R}^2)$	E_7^5	$\mathbb{R}^{56}$
$\mathrm{SL}(2,\mathbb{R}) \cdot \mathrm{SO}(p,q)$	$\mathbb{R}^{2(p+q)}$, $(p+q) \geq 3$	E_7^7	$\mathbb{R}^{56}$
$\mathrm{Sp}(1)\mathrm{SO}(n,\mathbb{H})$	$\mathbb{H}^n \simeq \mathbb{R}^{4n}$, $n \geq 2$	$\mathrm{Spin}(2,10)$	$\mathbb{R}^{32}$
$\mathrm{SL}(6,\mathbb{R})$	$\mathbb{R}^{20} \simeq \Lambda^3\mathbb{R}^6$	$\mathrm{Spin}(6,6)$	$\mathbb{R}^{32}$
$\mathrm{SU}(1,5)$	$\mathbb{R}^{20} \subset \Lambda^3\mathbb{C}^6$	$\mathrm{Spin}(6,\mathbb{H})$	$\mathbb{R}^{32}$
$\mathrm{SU}(3,3)$	$\mathbb{R}^{20} \subset \Lambda^3\mathbb{C}^6$	$\mathrm{Sp}(3,\mathbb{R})$	$\mathbb{R}^{14} \subset \Lambda^3\mathbb{C}^6$

For these representations, the following is known (cf. Proposition 3.3 below).

Proposition 2.9. [MS], [S3] *Let* $\mathrm{H} \subset \mathrm{Sp}(V, \omega)$ *be a special symplectic holonomy group with Lie algebra* $\mathfrak{h} \subset \mathfrak{sp}(V, \omega)$. *Then* $\mathfrak{h}$ *is a special symplectic subalgebra, i.e. there is a linear map* $\circ : S^2(V) \to \mathfrak{h}$ *which satisfies (3).*

Thus, by Proposition 2.3, for each $\rho \in \mathfrak{h}$, the map $R_\rho : \Lambda^2 V \to \mathfrak{h}$ defined in (4) is contained in $K(\mathfrak{h})$. Moreover, we have the following

Proposition 2.10. [MS], [S3] *For each special symplectic holonomy group* H *with Lie algebra* $\mathfrak{h}$, *we have* $\mathrm{Ric}\,(R_\rho) \neq 0$ *for all* $\rho \neq 0$, *and moreover,* $K(\mathfrak{h}) = \{R_\rho \mid \rho \in \mathfrak{h}\}$. *Thus,* $K(\mathfrak{h}) \cong \mathfrak{h}$ *as an* H*-module.*

This means that for each symplectic connection ∇ on (M, ω) with special symplectic holonomy, there is a unique section ρ of endomorphisms of the tangent spaces of M for which the curvature of ∇ at each point has the form (4).

2.4. **Special symplectic connections.** The various conditions for symplectic connections which we presented in the preceding sections have at first glance nothing in common. On the other hand, there is a striking formal similarity in the presentation of the curvature which turns out to be of tremendous significance. Therefore, we make the following

Key Definition 2.11. Let (M, ω) be a symplectic manifold of dimension at least 4, equipped with a symplectic connection ∇, i.e. a torsion free connection for which ω is parallel. We say that ∇ is a *special symplectic connection with structure group* $\mathrm{H} \subset \mathrm{Sp}(V, \omega)$ *and structure Lie algebra* $\mathfrak{h} \subset \mathfrak{sp}(V, \omega)$ if

1. ∇ is of Ricci type in the sense of Definition 2.5 In this case, $\mathrm{H} = \mathrm{Sp}(V, \omega)$ and $\mathfrak{h} = \mathfrak{sp}(V, \omega)$.
2. ∇ is Bochner-Kähler in the sense of Definition 2.8 for a Kähler metric of signature (p, q). In this case, $\mathrm{H} = \mathrm{U}(p, q)$ and $\mathfrak{h} = \mathfrak{u}(p, q)$.
3. ∇ is Bochner-bi-Lagrangian in the sense of Definition 2.8. In this case, $\mathrm{H} \cong \mathrm{GL}(m, \mathbb{R}) \subset \mathrm{Aut}((\mathbb{R}^m) \oplus (\mathbb{R}^m)^*)$ and $\mathfrak{h} = \mathfrak{u}'(m) \subset \mathrm{End}((\mathbb{R}^m) \oplus (\mathbb{R}^m)^*)$.
4. The holonomy of ∇ is contained in the special symplectic holonomy group $\mathrm{H} \subset \mathrm{Sp}(V, \omega)$ with Lie algebra $\mathfrak{h} \subset \mathfrak{sp}(V, \omega)$.

Note that in all cases, the structure Lie algebra $\mathfrak{h} \subset \mathfrak{sp}(V, \omega)$ is special symplectic in the sense of Definition 2.2, and there is a section ρ of the endomorphism bundle for which the curvature is given by R_ρ as in (4).

We are now interested in the covariant derivative of the curvature of a special symplectic connection. For this, we consider one of the structure Lie algebras $\mathfrak{h} \subset \mathfrak{sp}(V, \omega)$, and define the space of *covariant $\mathcal{R}$-derivations* by

$$\mathcal{R}_{\mathfrak{h}}^{(1)} := \left\{ \psi \in V^* \otimes \mathfrak{h} \mid R_{\psi(x)}(y, z) + R_{\psi(y)}(z, x) + R_{\psi(z)}(x, y) = 0 \ \text{ for all } x, y, z \in V \right\}.$$

Again, $\mathcal{R}_{\mathfrak{h}}^{(1)}$ is an H-module in an obvious way. The significance of this space is due to the fact that for any torsion free connection the *second Bianchi identity* holds:

$$(\nabla_x R)(y, z) + (\nabla_y R)(z, x) + (\nabla_z R)(x, y) = 0 \qquad \text{for all } x, y, z \in T_pM \text{ and } p \in M.$$

Thus, if ∇ is *special symplectic* so that at each point $p \in M$ the curvature is of the form R_ρ for some $\rho \in \mathfrak{h}$, then $\nabla_x R$ is also of this form, so that the correspondence $x \mapsto \nabla_x R$ yields a linear map $\psi : V \to \mathfrak{h}$ so that $\nabla_x R = R_{\psi(x)}$. The second Bianchi identity then implies that $\psi \in \mathcal{R}_{\mathfrak{h}}^{(1)}$.

From (3) it is straightforward to verify that for each $u \in V$, the map $\psi_u : V \to \mathfrak{h}$, $\psi_u(x) := u \circ x$ is contained in $\mathcal{R}_{\mathfrak{h}}^{(1)}$. Evidently, the correspondence $u \mapsto \psi_u$ is equivariant and injective, so that $\mathcal{R}_{\mathfrak{h}}^{(1)}$ contains a submodule isomorphic to V. Indeed, this exhausts *all* of $\mathcal{R}_{\mathfrak{h}}^{(1)}$. Namely, we have the

Proposition 2.12 ([CS]). *If* $\dim V \geq 4$ *and* $\mathfrak{h} \subset \mathfrak{sp}(V, \omega)$ *is one of the subalgebras associated to a special symplectic connection as in Definition 2.11, then* $\mathcal{R}_{\mathfrak{h}}^{(1)} \cong V$ *as* *an* H*-module, i.e.*

$$\mathcal{R}_{\mathfrak{h}}^{(1)} = \{\psi_u : V \to \mathfrak{h} \mid u \in V\}, \quad \text{where} \quad \psi_u(x) := u \circ x \text{ for all } x \in V.$$

Now we are ready to investigate the structure equations of special symplectic connections.

Proposition 2.13 ([CS]). *Let (M, ω, ∇) be a simply connected symplectic manifold of dimension at least 4 with a special symplectic connection of regularity C^4 with the associated Lie subgroup $\mathrm{H} \subset \mathrm{Sp}(V, \omega)$ and special symplectic Lie subalgebra $\mathfrak{h} \subset \mathfrak{sp}(V, \omega)$ from Definition 2.11. Then there is an associated* H*-structure $\pi : B \to M$ on M which is compatible with ∇, and there are maps $\rho : B \to \mathfrak{h}$, $u : B \to V$ and $f : B \to \mathbb{R}$, such that the tautological form $\theta \in \Omega^1(B) \otimes V$, the connection form $\eta \in \Omega^1(B) \otimes \mathfrak{h}$ and the functions ρ, u and f satisfy the structure equations*

$$\begin{aligned} & d\theta + \eta \wedge \theta = 0, \\ & d\eta + \tfrac{1}{2}[\eta, \eta] = R_\rho(\theta \wedge \theta), \end{aligned} \qquad \begin{aligned} d\rho + [\eta, \rho] &= u \circ \theta \\ du + \eta \cdot u &= (\rho^2 + f) \cdot \theta \\ df + d(\rho, \rho) &= 0. \end{aligned} \tag{9}$$

The assumption of simply connectedness of M is imposed only to ensure that the H-structure $B \to M$ exists which slightly simplifies the proof. However, our result also hold if M is *not* simply connected.

For clarification, we reformulate the structure equations (9) as follows. If for $h \in \mathfrak{h}$ and $x \in V$ we let $\xi_h, \xi_x \in \mathfrak{X}(B)$ be the vector fields characterized by

$$\theta(\xi_h) \equiv 0, \quad \eta(\xi_h) \equiv h \quad \text{and} \quad \theta(\xi_x) \equiv x, \quad \eta(\xi_x) \equiv 0,$$

then (9) holds iff for all $h, l \in \mathfrak{h}$ and $x, y \in V$,

$$\begin{aligned} &[\xi_h, \xi_l] = \xi_{[h,l]}, && [\xi_h, \xi_x] = \xi_{hx}, && [\xi_x, \xi_y] = -2\omega(x, y)\xi_\rho - \xi_{x \circ \rho y} + \xi_{y \circ \rho x} \\ &\xi_h(\rho) = -[h, \rho], && \xi_h(u) = -hu, && \xi_h(f) = 0, \\ &\xi_x(\rho) = u \circ x, && \xi_x(u) = (\rho^2 + f)x, && \xi_x(f) = -2\omega(\rho u, x) \end{aligned}$$

Proof. Let F be the H-structure on the manifold M, and denote the tautological and the connection 1-form on F by θ and η, respectively. Since by hypothesis, the curvature maps are all contained in $\mathcal{R}_{\mathfrak{h}}$, it follows that there is an H-equivariant map $\rho : B \to \mathfrak{h}$ such that the curvature at each point is given by R_ρ with the notation from (4). Thus, we have the structure equations

$$\begin{aligned} d\theta + \eta \wedge \theta &= 0 \\ d\eta + \tfrac{1}{2}[\eta, \eta] &= R_\rho(\theta \wedge \theta), \end{aligned} \tag{10}$$

The H-equivariance of ρ yields that $\xi_h(\rho) = -[h, \rho]$ for all $h \in \mathfrak{h}$. Moreover, since the covariant derivative of the curvature is represented by $\xi_x(\rho)$ for all $x \in V$ and this must lie in $\mathcal{R}_{\mathfrak{h}}^{(1)}$, it follows by Proposition 2.12 that $\xi_x(\rho) = u \circ \rho$ for some H-equivariant map $u : B \to V$, which shows the asserted formula

$$d\rho + [\eta, \rho] = u \circ \theta.$$

Since u is H-equivariant, it follows that $\xi_h(u) = -hu$ for all $h \in \mathfrak{h}$. Also, elaborating the equation $\xi_x\xi_y\rho - \xi_y\xi_x\rho = [\xi_x, \xi_y]\rho$ yields that for all $x, y \in V$

$$(\xi_x u - \rho^2 x) \circ y = (\xi_y u - \rho^2 y) \circ x . \tag{11}$$

Now we need the following lemma whose proof we will postpone.

Lemma 2.14. *Let $\mathfrak{h} \subset \mathfrak{sp}(V, \omega)$ be a special symplectic subalgebra, $\dim V \geq 4$, and let $\varphi : V \to V$ be a linear map such that*

$$\varphi(x) \circ y = \varphi(y) \circ x \quad \text{for all } x, y \in V. \tag{12}$$

Then φ is a multiple of the identity.

Applying the lemma to the function $x \mapsto \xi_x u - \rho^2 x$, (11) implies that there is a smooth function $f : B \to \mathbb{R}$ for which $\xi_x u - \rho^2 x = f x$ for all $x \in V$ so that

$$du + \eta \cdot u = (\rho^2 + f)\theta .$$

Finally, elaborating the equation $\xi_x\xi_y u - \xi_y\xi_x u = [\xi_x, \xi_y]u$ yields that $df + d(\rho, \rho) = 0$. □

Proof of Lemma 2.14. By (3) we have

$$(\varphi(x) \circ y)z - (\varphi(x) \circ z)y = 2\omega(y, z)\varphi(x) + \omega(\varphi(x), z)y - \omega(\varphi(x), y)z .$$

But (12) now implies that the cyclic sum in x, y, z of the left hand side vanishes, hence so does the cyclic sum of the right hand side, i.e.

$$\begin{aligned} 2(\omega(x,y)\varphi(z) + \omega(y,z)\varphi(x) + \omega(z,x)\varphi(y)) &= (\omega(\varphi(y), z) - \omega(\varphi(z), y))x \\ &\quad + (\omega(\varphi(z), x) - \omega(\varphi(x), z))y + (\omega(\varphi(x), y) - \omega(\varphi(y), x))z . \end{aligned} \tag{13}$$

For each $x \in V$, we may choose vectors $y, z \in V$ with $\omega(x, y) = \omega(x, z) = 0$ and $\omega(y, z) \neq 0$ since $\dim V \geq 4$. Then (13) implies that $\varphi(x) \in span(x, y, z)$ so that $\omega(\varphi(x), x) = 0$. Polarization then implies that $\omega(\varphi(x), y) + \omega(\varphi(y), x) = 0$ for all $x, y \in V$.

Next, we take the symplectic form of (13) with x, and together with the preceding identity this yields

$$\omega(x, y)\omega(\varphi(x), z) = \omega(x, z)\omega(\varphi(x), y) \quad \text{for all} \quad x, y, z \in V.$$

Thus, $\omega(x, y)\varphi(x) = \omega(\varphi(x), y)x$ for all $x, y \in V$, and since for $0 \neq x \in V$ we can pick $y \in V$ such that $\omega(x, y) \neq 0$, this implies that $\varphi(x)$ is a scalar multiple of x for all $x \in V$, whence φ is a multiple of the identity. □

It is now our aim to interpret the structure equations (9) in a way that links them to parabolic contact geometry. This shall be pursued in the following section.

3. Parabolic contact structures

3.1. Two-gradable Lie algebras.

Definition 3.1. *A real simple Lie algebra $\mathfrak{g}$ is called* 2-gradable *if there exists a decomposition*

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{1} \oplus \mathfrak{g}^{2} \tag{14}$$

with $\dim \mathfrak{g}^{\pm 2} = 1$, *and an element* $H_0 \in [\mathfrak{g}^2, \mathfrak{g}^{-2}]$ *such that* $\mathrm{ad}_{H_0}|_{\mathfrak{g}^i} = i\ \mathrm{Id}_{\mathfrak{g}^i}$ *for all* i.

Since $\mathfrak{g}^i$ is an Eigenspace of ad_{H_0}, it follows that $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$ by the Jacobi identity. Moreover, note that $\mathfrak{sl}_0 := \mathfrak{g}^2 \oplus \mathfrak{g}^{-2} \oplus \mathbb{R}H_0$ is a Lie subalgebra of $\mathfrak{g}$ which is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

Recall that the complexification $\mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes \mathbb{C}$ of $\mathfrak{g}$ is a complex simple Lie algebra for which we can choose the Cartan decomposition

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha ,$$

where $\mathfrak{t} \subset \mathfrak{g}_\mathbb{C}$ is a maximal abelian self-normalizing subalgebra, and $\Delta \subset \mathfrak{t}^* \backslash 0$ is the *root system* of $\mathfrak{g}_\mathbb{C}$ such that $\mathrm{ad}_t|_{\mathfrak{g}_\alpha} = \alpha(t) \, \mathrm{Id}_{\mathfrak{g}_\alpha}$ for all $t \in \mathfrak{t}$ and $\alpha \in \Delta$. Moreover,

$$\mathfrak{g} = (\mathfrak{g} \cap \mathfrak{t}) \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) .$$

If $\mathfrak{g}$ is two-gradable then $H_0 \in \mathfrak{g}$ is diagonalizable, so that we can choose the Cartan decomposition of $\mathfrak{g}_\mathbb{C}$ such that $H_0 \in \mathfrak{t} \cap \mathfrak{g}$, hence $\alpha(H_0) \in \{0, \pm 1, \pm 2\}$ for all $\alpha \in \Delta$, and there is exactly one $\alpha_0 \in \Delta$ such that $\mathfrak{g}^{\pm 2} = \mathfrak{g} \cap \mathfrak{g}_{\pm \alpha_0}$ and $H_0 \in [\mathfrak{g}_{\alpha_0}, \mathfrak{g}_{-\alpha_0}]$ is uniquely determined by $\alpha_0(H_0) = 2$. Thus, we have

$$\mathfrak{g}^i = \bigoplus_{\{\beta \in \Delta | \langle \beta, \alpha_0 \rangle = i\}} \mathfrak{g}_\beta \quad \text{for} \quad i \neq 0, \quad \text{and} \quad \mathfrak{g}^0 = \mathfrak{t} \oplus \bigoplus_{\{\beta \in \Delta | \langle \beta, \alpha_0 \rangle = 0\}} \mathfrak{g}_\beta .$$

Here, $\langle \beta, \alpha \rangle$ denotes the *Cartan number.* In particular, $\mathfrak{g}^{\pm 2} = \mathfrak{g}_{\pm \alpha_0}$, and if Δ has roots of different length, then α_0 must be a long root, as $| \langle \beta, \alpha_0 \rangle | \leq 1$ for all $\beta \neq \pm \alpha_0$.

We can decompose $\mathfrak{g}^0 = \mathbb{R}H_{\alpha_0} \oplus \mathfrak{h}$, where the Lie algebra $\mathfrak{h}$ is characterized by $[\mathfrak{h}, \mathfrak{sl}_0] = 0$. Observe that $\mathfrak{g}^0$ and hence $\mathfrak{h}$ are reductive. Thus, as a Lie algebra,

$$\mathfrak{g}^{ev} := \mathfrak{g}^{-2} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^2 \cong \mathfrak{sl}_0 \oplus \mathfrak{h} \quad \text{and} \quad \mathfrak{g}^{odd} := \mathfrak{g}^{-1} \oplus \mathfrak{g}^1 \cong \mathbb{R}^2 \otimes V \quad \text{as a} \quad \mathfrak{g}^{ev}\text{-module},$$

where $\mathfrak{h}$ acts effectively on V. Identifying $\mathfrak{h}$ with its image under this representation, we may regard it as a subalgebra $\mathfrak{h} \subset \mathrm{End}(V)$, and hence we have the decomposition

$$\mathfrak{g} = \mathfrak{g}^{ev} \oplus \mathfrak{g}^{odd} \cong (\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{h}) \oplus (\mathbb{R}^2 \otimes V) , \tag{15}$$

where this notation indicates the representation $\mathrm{ad} : \mathfrak{g}^{ev} \to \mathrm{End}(\mathfrak{g}^{odd})$.

We fix a non-zero $\mathbb{R}$-bilinear area form $a \in \Lambda^2(\mathbb{R}^2)^*$. There is a canonical $\mathfrak{sl}(2, \mathbb{R})$-equivariant isomorphim

$$S^2(\mathbb{R}^2) \longrightarrow \mathfrak{sl}(2, \mathbb{R}) , \quad (ef) \cdot g := a(e, g) f + a(f, g) e \quad \text{for all} \quad e, f, g \in \mathbb{R}^2 , \tag{16}$$

and under this isomorphism, the Lie bracket on $\mathfrak{sl}(2, \mathbb{R})$ is given by

$$[ef, gh] = a(e, g) fh + a(e, h) fg + a(f, g) eh + a(f, h) eg .$$

Thus, if we fix a basis $e_+, e_- \in \mathbb{R}^2$ with $a(e_+, e_-) = 1$, then we have the identifications

$$H_0 = -e_+ e_- , \quad \mathfrak{g}^{\pm 2} = \mathbb{R}e_\pm^2 , \quad \mathfrak{g}^{\pm 1} = e_\pm \otimes V .$$

Now one can show the following

Proposition 3.2 ([CS]). *Let $\mathfrak{g}$ be a 2-gradable simple Lie algebra, and consider the decompositions* (14) *and* (15). *Then there is an $\mathfrak{h}$-invariant symplectic form $\omega \in \Lambda^2 V^*$ and an $\mathfrak{h}$-equivariant product $\circ : S^2(V) \to \mathfrak{h}$ such that*

$$[\ ,\ ] : \Lambda^2(\mathfrak{g}^{odd}) \longrightarrow \mathfrak{g}^{ev} \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{h}$$

is given as

(17) $$[e \otimes x, f \otimes y] = \omega(x,y)ef + a(e,f)x \circ y \quad \textit{for} \quad e,f \in \mathbb{R}^2 \textit{ and } x,y \in V,$$

using the identification $S^2(\mathbb{R}^2) \cong \mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{g}^{ev}$ *from* (16). *Moreover, there is a multiple* $(\ ,\ )$ *of the Killing form on* $\mathfrak{g}$ *which satisfies the following:*

1. $(\mathfrak{g}^i, \mathfrak{g}^j) = 0$ *if* $i + j \neq 0$,
2. *For all* $x, y \in V$ *and* $h \in \mathfrak{h}$, *we have* $(h, x \circ y) = \omega(hx, y) = \omega(hy, x)$
3. *For all* $x, y, z \in V$, *(3) holds, so that* $\mathfrak{h} \subset \mathfrak{sp}(V,\omega)$ *is a special symplectic subalgebra.*

Thus, each 2-gradable simple Lie algebra yields a special symplectic subalgebra $\mathfrak{h} \subset \mathfrak{sp}(V,\omega)$. The converse is also true. Namely, we have

Proposition 3.3 ([CS]). *Let* (V,ω) *be a symplectic vector space, and let* $\mathfrak{h} \subset \mathfrak{sp}(V,\omega)$ *be a special symplectic subalgebra with product* $\circ : S^2(V) \to \mathfrak{h}$. *Then there is a unique 2-gradable simple Lie algebra* $\mathfrak{g}$ *which admits the decompositions* (14) *and* (15), *and the Lie bracket of* $\mathfrak{g}$ *is given by* (17).

We shall call $\mathfrak{g}$ the *simple Lie algebra associated to the special symplectic subalgebra* $\mathfrak{h} \subset \mathfrak{sp}(V,\omega)$.

From this proposition, we obtain a complete classification of special symplectic subalgebras by considering all 2-gradable real simple Lie algebras ([OV]). Namely, a simple Lie algebra $\mathfrak{g}$ is 2-gradable iff we can choose the Cartan decomposition of $\mathfrak{g}_{\mathbb{C}}$ such that $\mathfrak{g} \cap \mathfrak{g}_{\pm\alpha_0} \neq 0$ for some long root $\alpha_0 \in \Delta$.

Corollary 3.4. *Table 2 yields the complete list of special symplectic subgroups* $\mathrm{H} \subset \mathrm{Sp}(V,\omega)$.

From Table 2, we now observe the link between 2-gradable simple Lie algebras and the Lie groups $\mathrm{H} \subset \mathrm{Sp}(V,\omega)$ which are associated to special symplectic connections in the sense of Definition 2.11. Namely, note that the Lie groups $\mathrm{H} \subset \mathrm{Sp}(V,\omega)$ corresponding to entries (i) and (ii) are precisely the groups associated to Bochner-bi-Lagrangian and Bochner Kähler connections; the H of entry (iii) is associated to the connections of Ricci type, whereas a comparison with Table 1 yields that the H's of entries (iv) – (xv) are the special symplectic holonomy groups.

This shall be the conceptual background of our construction of special symplectic connections out of 2-gradable simple Lie algebras.

3.2. **Contact manifolds.** We shall now recall some well known facts about contact manifolds and their symplectic reductions.

Definition 3.5. A *contact structure* on a manifold $\mathcal{C}$ is a smooth distribution $\mathcal{D} \subset T\mathcal{C}$ of codimension one such that the Lie bracket induces a non-degenerate map

$$\mathcal{D} \times \mathcal{D} \longrightarrow T\mathcal{C}/\mathcal{D} =: L.$$

The line bundle $L \to \mathcal{C}$ is called the *contact line bundle*, and its dual can be embedded as

(18) $$L^* = \{\lambda \in T^*\mathcal{C} \mid \lambda(\mathcal{D}) = 0\} \subset T^*\mathcal{C}.$$

Table 2: Real 2-gradable Lie groups

	Type of Δ	G	H	V
(i)	$\mathrm{A}_k,\ k \geq 2$	$\mathrm{SL}(n+2,\mathbb{R}),\ n \geq 1$	$\mathrm{GL}(n,\mathbb{R})$	$W \oplus W^*$ with $W \cong \mathbb{R}^n$
(ii)		$\mathrm{SU}(p+1,q+1),\ p+q \geq 1$	$\mathrm{U}(p,q)$	$\mathbb{C}^{p+q}$
(iii)	$\mathrm{C}_k,\ k \geq 2$	$\mathrm{Sp}(n+1,\mathbb{R})$	$\mathrm{Sp}(n,\mathbb{R})$	$\mathbb{R}^{2n}$
(iv)	$\mathrm{B}_k,\ k \geq 3$	$\mathrm{SO}(p+2,q+2),\ p+q \geq 3$	$\mathrm{SL}(2,\mathbb{R}) \cdot \mathrm{SO}(p,q)$	$\cdot\, \mathbb{R}^2 \otimes \mathbb{R}^{p+q}$
(v)	$\mathrm{D}_k,\ k \geq 4$	$\mathrm{SO}(n+2,\mathbb{H}),\ n \geq 2$	$\mathrm{Sp}(1) \cdot \mathrm{SO}(n,\mathbb{H})$	$\mathbb{H}^n$
(vi)	G_2	G_2'	$\mathrm{SL}(2,\mathbb{R})$	$S^3(\mathbb{R}^2)$
(vii)	F_4	$\mathrm{F}_4^{(1)}$	$\mathrm{Sp}(3,\mathbb{R})$	$\mathbb{R}^{14} \subset \Lambda^3\mathbb{R}^6$
(viii)	E_6	$\mathrm{E}_6^{\mathbb{R}}$	$\mathrm{SL}(6,\mathbb{R})$	$\Lambda^3\mathbb{R}^6$
(ix)		$\mathrm{E}_6^{(2)}$	$\mathrm{SU}(1,5)$	$\mathbb{R}^{20} \subset \Lambda^3\mathbb{C}^6$
(x)		$\mathrm{E}_6^{(3)}$	$\mathrm{SU}(3,3)$	$\mathbb{R}^{20} \subset \Lambda^3\mathbb{C}^6$
(xi)	E_7	$\mathrm{E}_7^{(5)}$	$\mathrm{Spin}(6,6)$	$\mathbb{R}^{32} \subset \Delta^{\mathbb{C}}$
(xii)		$\mathrm{E}_7^{(6)}$	$\mathrm{Spin}(6,\mathbb{H})$	$\mathbb{R}^{32} \subset \Delta^{\mathbb{C}}$
(xiii)		$\mathrm{E}_7^{(7)}$	$\mathrm{Spin}(2,10)$	$\mathbb{R}^{32} \subset \Delta^{\mathbb{C}}$
(xiv)	E_8	$\mathrm{E}_8^{(8)}$	$\mathrm{E}_7^{(5)}$	$\mathbb{R}^{56}$
(xv)		$\mathrm{E}_8^{(9)}$	$\mathrm{E}_7^{(7)}$	$\mathbb{R}^{56}$

Notice that we can define the line bundles $L \to C$ and $L^* \to C$ for an arbitrary distribution $\mathcal{D} \subset TC$ of codimension one. It is well known that such a distribution $\mathcal{D}$ yields a contact structure iff the restriction of the canonical symplectic form Ω on T^*C to $L^*\backslash 0$ is non-degenerate, so that in this case $L^*\backslash 0$ is a symplectic manifold in a canonical way.

We regard $p : L^*\backslash 0 \to C$ as a principal $(\mathbb{R}\backslash 0)$-bundle. We call the contact structure *orientable* if $L^*\backslash 0$ has two components each of which is a principal $\mathbb{R}^+$-bundle. In fact, if the contact structure is *not* orientable, then there is a double cover $\tilde{C}$ of C such that the induced contact structure $\tilde{\mathcal{D}}$ on $\tilde{C}$ *is* orientable. Thus, for an orientable contact structure, we get the principal $\mathbb{R}^+$-bundle

$$p : \hat{C} \longrightarrow C,$$

where $\hat{C} \subset L^*\backslash 0$ is a connected component. The choice of connected component is called an *orientation* of the contact structure.

The vector field $E_0 \in \mathfrak{X}(\hat{C})$ which generates the principal action is called *Euler field*, so that the flow along E_0 is fiberwise scalar multiplication in $\hat{C} \subset L^* \subset T^*C$. Thus, the *Liouville form* on T^*C is given as $\lambda := E_0 \lrcorner \Omega$, and hence $\mathfrak{L}_{E_0}(\Omega) = \Omega$ and $\Omega = d\lambda$. This process can be reverted. Namely, we have the following

Proposition 3.6. *Let $p : \hat{C} \to C$ be a principal $\mathbb{R}^+$-bundle with a symplectic form Ω on $\hat{C}$ such that $\mathfrak{L}_{E_0}\Omega = \Omega$ where $E_0 \in \mathfrak{X}(\hat{C})$ generates the principal action. Then there is a unique contact structure $\mathcal{D}$ on C and an equivariant imbedding $\imath : \hat{C} \hookrightarrow L^*\backslash 0 \subset T^*C$ with L^* from (18) such that Ω is the pullback of the canonical symplectic form on T^*C to $\hat{C}$.*

Proof. By hypothesis, $\Omega = d\lambda$ where $\lambda := (E_0 \lrcorner \Omega)$. Since $\lambda(E_0) = 0$, there is for each $x \in \hat{C}$ a unique $\underline{\lambda}_x \in T^*_{p(x)}C$ satisfying $p^*(\underline{\lambda}_x) = \lambda_x$. Moreover, $\mathfrak{L}_{E_0}(\lambda) = \lambda$, hence $\underline{\lambda}_{e^t x} = e^t \underline{\lambda}_x$ for all $t \in \mathbb{R}$, so that the codimension one distribution $\mathcal{D} := dp(\ker(\lambda)) \subset TC$ is well defined, and the correspondence $x \mapsto \underline{\lambda}_x$ yields an equivariant imbedding $\hat{C} \hookrightarrow L^*\backslash 0$ whose image is thus a connected component of $L^*\backslash 0$. Moreover, by construction, λ is the restriction of the Liouville form to $\hat{C} \subset L^*\backslash 0 \subset T^*C$. Since $\Omega = d\lambda$ is non-degenerate on $\hat{C}$ by assumption, it follows that $\mathcal{D}$ is a contact structure. □

Next, we define the fiber bundle

$$\mathfrak{R} := \{(\lambda, \hat{\xi}) \in \hat{C} \times T\hat{C} \subset T^*C \times T\hat{C} \mid \lambda(dp(\hat{\xi})) = 1\}.$$

Projection onto the first factor yields a fibration $\mathfrak{R} \to \hat{C}$ whose fiber is an affine space.

Definition 3.7. *Let C be a contact manifold. We call a vector field ξ on C a* contact symmetry *if $\mathfrak{L}_\xi(\mathcal{D}) \subset \mathcal{D}$. This means that the flow along ξ preserves the contact structure $\mathcal{D}$.*

We call ξ a transversal *contact symmetry if in addition $\xi \notin \mathcal{D}$ at all points. If C is oriented with orientation $\hat{C} \subset L^*\backslash 0$, then ξ is called* positively transversal *if $\lambda(\xi) > 0$ for all $\lambda \in \hat{C}$.*

For each contact symmetry ξ on C, there is a unique vector field $\hat{\xi} \in \mathfrak{X}(\hat{C})$, called the *Hamiltonian lift of* ξ, satisfying $dp(\hat{\xi}) = \xi$ and $\mathfrak{L}_{\hat{\xi}}\lambda = 0$, so that $\mathfrak{L}_{\hat{\xi}}\Omega = 0$.

Given a positively transversal contact symmetry ξ with Hamiltonian lift $\hat{\xi}$, there is a unique section λ of the bundle $p : \hat{C} \to C$ such that $\lambda(\xi) \equiv 1$, and hence we obtain a section of the bundle $\mathfrak{R} \to \hat{C} \to C$

$$\sigma_\xi : C \longrightarrow \mathfrak{R}, \quad \sigma_\xi := (\lambda, \hat{\xi}) \in \mathfrak{R}. \tag{19}$$

We call an open subset $U \subset C$ *regular* w.r.t. the transversal contact symmetry ξ if there is a submersion $\pi_U : U \to M_U$ onto some manifold M_U whose fibers are connected lines tangent to ξ. Evidently, since ξ is pointwise non-vanishing, C can be covered by regular open subsets.

Since ξ is a contact symmetry, it follows that $\xi \lrcorner d\lambda = 0$ and $\mathfrak{L}_\xi\lambda = 0$. Thus, on each M_U there is a unique symplectic form ω such that

$$\pi_U^*\omega = -2d\lambda, \tag{20}$$

where the factor -2 only occurs to make this form coincide with one we shall construct later on.

3.3. **Parabolic contact structures.** To link all of this to our situation, let $\mathfrak{g}$ be a 2-gradable simple real Lie algebra and let G be the corresponding connected Lie group with trivial center $Z(\mathrm{G}) = \{1\}$. Recall the decomposition

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2 \cong \mathbb{R}e_-^2 \oplus e_- \otimes V \oplus (\mathbb{R}e_+e_- \oplus \mathfrak{h}) \oplus e_+ \otimes V \oplus \mathbb{R}e_+^2$$

from (14). We let $\mu := g^{-1}dg$ be the left invariant Maurer-Cartan form on G, which we can decompose as

$$(21) \qquad \mu = \sum_{i=-2}^{2} \mu_i\,, \quad \mu_0 = \mu_{\mathfrak{h}} + \nu_0 e_+ e_-$$

where $\mu_i \in \Omega^1(\mathrm{G}) \otimes \mathfrak{g}^i$, $\mu_{\mathfrak{h}} \in \Omega^1(\mathrm{G}) \otimes \mathfrak{h}$ and $\nu_0 \in \Omega^1(\mathrm{G})$. Furthermore, we define the subalgebras

$$\mathfrak{p} := \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2\,, \quad \text{and} \quad \mathfrak{p}_0 := \mathfrak{h} \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2\,,$$

and we let $\mathrm{P}, \mathrm{P}_0 \subset \mathrm{G}$ be the corresponding connected subgroups. Using the bilinear form $(\ ,\ )$ from Proposition 3.2, we identify $\mathfrak{g}$ and $\mathfrak{g}^*$. Now we define the *root cone* and its (oriented) projectivization

$$\hat{\mathcal{C}} := \mathrm{G} \cdot e_+^2 \subset \mathfrak{g} \cong \mathfrak{g}^*\,, \quad \mathcal{C} := p(\hat{\mathcal{C}}) \subset \mathbb{P}^o(\mathfrak{g}) \cong \mathbb{P}^o(\mathfrak{g}^*)\,,$$

where $\mathbb{P}^o(\mathfrak{g})$ is the set of *oriented* lines in $\mathfrak{g}$, i.e. $\mathbb{P}^o \cong S^{\dim \mathfrak{g}-1}$, where $p : \mathfrak{g}\backslash 0 \to \mathbb{P}^o(\mathfrak{g})$ is the principal $\mathbb{R}^+$-bundle defined by the canonical projection. Thus, the restriction $p : \hat{\mathcal{C}} \to \mathcal{C}$ is a principal bundle as well.

Being a coadjoint orbit, $\hat{\mathcal{C}}$ carries a canonical G-invariant symplectic structure Ω. Moreover, the *Euler vector field* defined by

$$E_0 \in \mathfrak{X}(\hat{\mathcal{C}})\,, \quad (E_0)_v := v$$

generates the principal action of p and satisfies $\mathfrak{L}_{E_0}(\Omega) = \Omega$, so that the distribution $\mathcal{D} = dp(E_0^{\perp_\Omega}) \subset T\mathcal{C}$ yields a G-invariant contact distribution on $\mathcal{C}$ by Proposition 3.6. Now one can show the following

Lemma 3.8 ([CS]). *As homogeneous spaces, we have $\mathcal{C} = \mathrm{G}/\mathrm{P}$, $\hat{\mathcal{C}} = \mathrm{G}/\mathrm{P}_0$ and $\mathfrak{R} = \mathrm{G}/\mathrm{H}$. Moreover, the fiber bundles $\mathfrak{R} \to \hat{\mathcal{C}} \to \mathcal{C}$ from before are equivalent to the corresponding homogeneous fibrations.*

For each $a \in \mathfrak{g}$ we define the vector fields $a^* \in \mathfrak{X}(\mathcal{C})$ and $\hat{a}^* \in \mathfrak{X}(\hat{\mathcal{C}})$ corresponding to the infinitesimal action of a, i.e.

$$(22) \qquad (a^*)_{[v]} := \left.\frac{d}{dt}\right|_{t=0} (\exp(ta) \cdot [v]) \quad \text{and} \quad (\hat{a}^*)_v := \left.\frac{d}{dt}\right|_{t=0} (\exp(ta) \cdot v)\,.$$

Note that a^* is a contact symmetry and $\hat{a}^*$ is its Hamiltonian lift. Let

$$(23) \qquad \hat{\mathcal{C}}_a := \{\lambda \in \hat{\mathcal{C}} \mid \lambda(a^*) > 0\} \quad \text{and} \quad \mathcal{C}_a := p(\hat{\mathcal{C}}_a) \subset \mathcal{C}\,,$$

so that $p : \hat{\mathcal{C}}_a \to \mathcal{C}_a$ is a principal $\mathbb{R}^+$-bundle and the restriction of a^* to $\mathcal{C}_a$ is a positively transversal contact symmetry. Therefore, we obtain the section $\sigma_a : \mathcal{C}_a \to \mathfrak{R} = \mathrm{G}/\mathrm{H}$ from (19).

Let $\pi : \mathrm{G} \to \mathrm{G}/\mathrm{H} = \mathfrak{R}$ be the canonical projection, and let $\Gamma_a := \pi^{-1}(\sigma_a(\mathcal{C}_a)) \subset \mathrm{G}$. Then evidently, the restriction $\pi : \Gamma_a \to \sigma_a(\mathcal{C}_a) \cong \mathcal{C}_a$ is a (right) principal H-bundle.

Theorem 3.9 ([CS]). *Let $a \in \mathfrak{g}$ be such that $\mathcal{C}_a \subset \mathcal{C}$ from (23) is non-empty, define $a^* \in \mathfrak{X}(\mathcal{C})$ and $\hat{a}^* \in \mathfrak{X}(\hat{\mathcal{C}})$ as in (22), and let $\pi : \Gamma_a \to \mathcal{C}_a$ with $\Gamma_a \subset \mathrm{G}$ be the principal*

H-bundle from above. Then there are functions $\rho : \Gamma_a \to \mathfrak{h}$, $u : \Gamma_a \to V$, $f : \Gamma_a \to \mathbb{R}$ such that

$$\mathrm{Ad}_{g^{-1}}(a) = \frac{1}{2}e_-^2 + \rho + e_+ \otimes u + \frac{1}{2}fe_+^2 \tag{24}$$

for all $g \in \Gamma_a$. Moreover, the restriction of the components $\mu_{\mathfrak{h}} + \mu_{-1} + \mu_{-2}$ of the Maurer-Cartan form (21) to Γ_a yields a pointwise linear isomorphism $T\Gamma_a \to \mathfrak{h} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{-2}$, and if we decompose this coframe as

$$\begin{gathered} \mu_{\mathfrak{h}} + \mu_{-1} + \mu_{-2} = -2\kappa \ \left(\tfrac{1}{2}e_-^2 + \rho\right) + e_- \otimes \theta + \eta\,, \\ \text{where} \quad \kappa \in \Omega^1(\Gamma_a)\,, \quad \theta \in \Omega^1(\Gamma_a) \otimes V\,, \quad \eta \in \Omega^1(\Gamma_a) \otimes \mathfrak{h}\,, \end{gathered} \tag{25}$$

then $\kappa = -\frac{1}{2}\pi^*(\lambda)$ where $\lambda \in \Omega^1(\mathcal{C}_a)$ is the contact form for which $\sigma_a = (\lambda, \hat{a}^*)$. Moreover,

$$d\kappa = \frac{1}{2}\omega(\theta \wedge \theta)\,, \tag{26}$$

and θ, η, ρ, u, f satisfy the structure equations (9).

Proof. According to the above identifications, we have $g \in \Gamma_a$ iff $(g \cdot e_+^2, g \cdot (\frac{1}{2}e_-^2 + \mathfrak{p}_0)) = \sigma_a([g \cdot e_+^2])$ iff $g \cdot (\frac{1}{2}e_-^2 + \mathfrak{p}_0) = (\hat{a}^*)_{g \cdot e_+^2}$ iff $(\mathrm{Ad}_{g^{-1}}(\hat{a}^*))_{e_+^2} = \frac{1}{2}e_-^2 \mod \mathfrak{p}_0$ iff $\mathrm{Ad}_{g^{-1}}(a) = \frac{1}{2}e_-^2 \mod \mathfrak{p}_0$, i.e.

$$\begin{gathered} \Gamma_a = \{g \in \mathrm{G} \mid \mathrm{Ad}_{g^{-1}}(a) \in Q\}\,, \\ \text{where} \\ Q := \tfrac{1}{2}e_-^2 + \mathfrak{p}_0 = \left\{\tfrac{1}{2}e_-^2 + \rho + e_+ \otimes u + \tfrac{1}{2}fe_+^2 \mid \rho \in \mathfrak{h}, u \in V, f \in \mathbb{R}\right\}\,, \end{gathered} \tag{27}$$

and from this (24) follows. Thus, if $dL_g v \in T_g\Gamma_a$ with $v \in \mathfrak{g}$, then we must have

$$\mathfrak{p}_0 \ni \left.\frac{d}{dt}\right|_{t=0} \left(\mathrm{Ad}_{(g\exp(tv))^{-1}}(a)\right) = -[v, \mathrm{Ad}_{g^{-1}}(a)] = -\left[v, \frac{1}{2}e_-^2 + \rho + e_+ \otimes u + \frac{1}{2}fe_+^2\right]\,,$$

and from here it follows by a straightforward calculation that v must be contained in the space

$$\mathbb{R}\mathrm{Ad}_{g^{-1}}a \oplus \left\{\, e_- \otimes x + e_+ \otimes \rho x + \frac{1}{2}\omega(u,x)e_+^2 \;\middle|\; x \in V \right\} \oplus \mathfrak{h}\,, \tag{28}$$

and since v was arbitrary, it follows that $\mu(T_g\Gamma_a)$ is contained in (28). In fact, a dimension count yields that $\dim(\mu(T_g\Gamma_a)) = \dim \Gamma_a = \dim \mathcal{C}_a + \dim \mathrm{H}$ coincides with the dimension of (28), hence (28) equals $\mu(T_g\Gamma)$, i.e. $\mu_{\mathfrak{h}} + \mu_{-1} + \mu_{-2} : T\Gamma_a \to \mathfrak{h} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{-2}$ yields a pointwise isomorphism. From there, the structure equations (26) and (9) follow by a straightforward calculation.

With these equations, it follows that κ is H-invariant and vanishes along the principal fibers, hence $\kappa = -\frac{1}{2}\pi^*(\lambda)$ for some $\lambda \in \Omega^1(\mathcal{C}_a)$. Since $\kappa|_{\mu^{-1}(\mathfrak{g}^{-1})} = 0$, it follows that λ is a contact form. Moreover, if we let $\tilde{a}^*$ denote the *right* invariant vector field on G characterized by $\mu(\tilde{a}^*) = \mathrm{Ad}_{g^{-1}}(a)$, then $dp(\tilde{a}^*) = \hat{a}^*$, where $p : \Gamma_a \to \hat{\mathcal{C}}$ is the canonical projection, and from (24) it follows that $\lambda(a^*) = -2\kappa(\tilde{a}^*) \equiv 1$, so that $(\lambda, \hat{a}^*) \in \mathfrak{R}$ which shows the final assertion. □

Consider the principal T_a-bundle $\Gamma_a \to \mathrm{T}_a\backslash\Gamma_a =: B_a$ whose fundamental vector field we denote by ξ_a. That is, ξ_a is the restriction of the right invariant vector field on G corresponding to $a \in \mathfrak{g}$ to $\Gamma_a \subset$ G. Thus, $\mu(\xi_a) = \mathrm{Ad}_{g^{-1}} a$, and the flow along ξ_a preserves μ. Therefore, by (21), (24) and (25), it follows that

$$\kappa(\xi_a) \equiv -\frac{1}{2}, \quad \eta(\xi_a) \equiv 0, \quad \theta(\xi_a) \equiv 0,$$

and we obtain the following

Corollary 3.10. 1. *The differential form $\kappa \in \Omega^1(\Gamma_a)$ defined above yields a connection on the principal T_a-bundle $p : \Gamma_a \to B_a$ from above.*

2. *The functions $\rho : \Gamma_a \to \mathfrak{h}$, $u : \Gamma_a \to V$ and $f : \Gamma_a \to \mathbb{R}$ defined above are constant along the fibers of $p : \Gamma_a \to B_a$, hence they induce functions on B_a which by abuse of notation shall also be denoted by ρ, u and f, respectively.*
3. *There are differential forms on B_a whose pull back under p equals θ and η, respectively. By abuse of notation, these forms will also be denoted by θ and η, respectively.*
4. $\theta + \eta \in \Omega^1(B_a) \otimes (V \oplus \mathfrak{h})$ *is a* coframing, *i.e. yields a pointwise isomorphism of TB_a with $V \oplus \mathfrak{h}$, and θ, η, ρ, u and f satisfy the structure equations* (9).
5. *There is a differential form on $\mathcal{C}_a = \Gamma_a/$H whose pull back equals κ, and again, we shall denote this form also by κ.*

Thus, if we let $X_a := \mathrm{T}_a \backslash \Gamma_a/\mathrm{H}$, then we obtain the following commutative diagram, where the labeled maps are principal bundles with the indicated structure groups.

(29)
$$\begin{tikzcd} \Gamma_a \arrow[r, "\mathrm{T}_a"] \arrow[d, "\mathrm{H}"'] & B_a \arrow[d] \\ \mathcal{C}_a \arrow[r] & X_a \end{tikzcd}$$

Let us assume for the moment that X_a is a manifold. Then the maps $B_a \to X_a$ and $\mathcal{C}_a \to X_a$ are principal bundles with structure group H and T_a, respectively. Moreover, κ yields a connection on the T_a-bundle $\mathcal{C}_a \to X_a$, and θ induces an embedding of B_a into the coframe bundle of X_a such that θ is the pull back of the tautological form. Hence we may regard $B_a \to X_a$ as an H-structure on X_a, and η yields a connection on this structure which satisfies (9) by Corollary 3.10 and hence is special symplectic. In particular, X_a carries a symplectic structure ω which by (26) is the curvature of the connection κ on the line bundle $\mathcal{C}_a \to X_a$. Therefore, this line bundle is a *quantization bundle* of X_a.

If $\mathrm{T}_a \cong S^1$, then X_a is an *orbifold*, and the statements in the preceding paragraph are still valid in this sense, i.e. X_a carries a special symplectic *orbifold connection* and $\mathcal{C}_a \to X_a$ is the *orbifold quantization bundle.*

However, for general $a \in \mathfrak{g}$, X_a will be neither a manifold nor an orbifold. In fact, there are plenty of choices where X_a is neither Hausdorff nor locally Euclidean, so the statement that X_a should carry a special symplectic connection is somewhat delicate. Therefore, in order to get a precise statement in the general case, we have to "localize" the structure.

For this, note that the action of T_a on $\mathcal{C}_a$ whose quotient equals X_a is *locally free.* Thus, we can cover $\mathcal{C}_a = \Gamma_a/\mathrm{H}$ by open subsets $U \subset \mathcal{C}_a$ with the property that the *local quotient* $M_U := \mathrm{T}_a^{\mathrm{loc}} \backslash U$, i.e. the set of connected components of the intersections of T_a-orbits with U, is a manifold. We let $\Gamma_U := p^{-1}(U) \subset \Gamma_a$ where $p : \Gamma_a \to C_a$ is the principal H-bundle from above. Thus, restricting the maps of (29), we get the following commutative diagram

$$\begin{tikzcd} \Gamma_U \arrow[r, "\mathrm{T}_a^{\mathrm{loc}}"] \arrow[d, "\mathrm{H}"'] & B_U^{\mathrm{loc}} \arrow[d, "\mathrm{H}"] \\ U \arrow[r, "\mathrm{T}_a^{\mathrm{loc}}"'] & M_U \end{tikzcd}$$

where $B_U^{\mathrm{loc}} := \mathrm{T}_a^{\mathrm{loc}} \backslash \Gamma_U$. Now we argue again that the differential forms θ and η on $\Gamma_U \subset \Gamma_a$ as well as the functions ρ, u and f fator through to differential forms and functions on B_U^{loc}, respectively, which satisfy (9). Thus, we may regard $B_U^{\mathrm{loc}} \to M_U$ as an H-structure, and η yields a special symplectic connection on M_U. Moreover, we can extend the local principal bundle $U \to M_U$ to a principal T_a-bundle $\overline{U} \to M_U$ and we can extend $\kappa \in \Omega^1(U)$ to a connection form on $\overline{U}$ for which (26) holds. Thus, $\overline{U} \to M_U$ is a quantization bundle of M_U. That is, we have the following result from which Theorem A from the introduction follows immediately.

Theorem 3.11. *Let $\mathfrak{g}$ be a 2-gradable real simple Lie algebra, and let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie subalgebra from (15), i.e. $\mathfrak{h} \subset \mathfrak{sp}(V,\omega)$ is special symplectic. Let $a \in \mathfrak{g}$ and $C_a \subset C$ as before. Let $U \subset C_a$ be an open subset for which the local quotient $M_U := \mathrm{T}_a^{\mathrm{loc}} \backslash U$ is a manifold, where*

$$\mathrm{T}_a := \exp(\mathbb{R}a) \subset \mathrm{G}\,.$$

Let $\omega \in \Omega^2(M_U)$ be the symplectic form from (20). Then M_U carries a canonical special symplectic connection associated to $\mathfrak{h}$, and the (local) principal T_a-bundle $\pi : U \to M_U$ admits a connection $\kappa \in \Omega^1(U)$ whose curvature is given by $d\kappa = \pi^*(\omega)$.

Remark 3.12. If we replace a by $a' := \mathrm{Ad}_{g_0}(a)$, then it is clear that in the above construction we have $\Gamma_{a'} = L_{g_0}\Gamma_a$. Thus, identifying Γ_a and $\Gamma_{a'}$ via L_{g_0}, the functions $\rho + \mu + f$ and the forms $\kappa + \theta + \omega$ will be canonically identified and hence both satisfy (9). Therefore, the connections from the preceding theorem only depend on the adjoint orbit of a.

Also, since $C_a = C_{e^{t_0}a}$ and $\mathrm{T}_a = \mathrm{T}_{e^{t_0}a}$ for all $t_0 \in \mathbb{R}$, the above construction yields equivalent connections when replacing a by $e^{t_0}a$. In this case, however, the symplectic form ω on the quotient will be replaced by $e^{-t_0}\omega$.

4. The developing map

In this section, we shall revert the process of the preceding section, showing that any special symplectic connection is equivalent to one of those given in Theorem 3.11 in a sense which is to be made precise. Namely, recall that by Proposition 2.13 each special symplectic connection of regularity C^4 associated to the special symplectic Lie algebra $\mathfrak{h} \subset \mathfrak{sp}(V,\omega)$ on a symplectic manifold (M,ω) of dimension at least 4 induces functions $\rho : B \to \mathfrak{h}$, $u : B \to V$ and $f : B \to \mathbb{R}$ where $\pi : B \to M$ is the associated

H-structure on M, such that for the tautological one form $\theta \in \Omega^1(B) \otimes V$ and the connection form $\eta \in \Omega^1(B) \otimes \mathfrak{h}$, the structure equations (9) hold.

It is now our aim to construct the equivalent to the principal line bundle $\Gamma \to B$ from the preceding section. Namely, we let $\mathfrak{g}$ be the 2-gradable simple Lie algebra associated to $\mathfrak{h}$ by Proposition 3.3. Motivated by (27) and (28), we define the following function A and one form σ

$$(30)\qquad \begin{aligned} &A : B \longrightarrow Q \subset \mathfrak{g}, && A := \tfrac{1}{2}e_-^2 + \rho + e_+ \otimes u + \tfrac{1}{2} f e_+^2, \\ &\sigma \in \Omega(B) \otimes \mathfrak{g}, && \sigma := e_- \otimes \theta + \eta + e_+ \otimes (\rho\theta) + \tfrac{1}{2}\omega(u,\theta)e_+^2, \end{aligned}$$

where $Q := \frac{1}{2}e_-^2 + \mathfrak{p}_0 \subset \mathfrak{g}$ is the affine hyperplane from (27). It is then straightforward to verify that (9) is equivalent to

$$(31)\qquad dA = -[\sigma, A] \quad \text{and} \quad d\sigma + \frac{1}{2}[\sigma,\sigma] = 2\pi^*(\omega)A.$$

Let us now enlarge the principal H-bundle $B \to M$ to the principal G-bundle

$$\mathbf{B} := B \times_{\mathrm{H}} \mathrm{G} \longrightarrow M,$$

where H acts on $B \times \mathrm{G}$ from the right by $(b,g)\cdot h := (b \cdot h, h^{-1}g)$, using the principal H-action on B in the first component. Evidently, the inclusion $B \times \mathrm{H} \hookrightarrow B \times \mathrm{G}$ induces an embedding $B \hookrightarrow \mathbf{B}$.

Proposition 4.1. *The function* $\mathbf{A}$ *and the one form* α *defined by*

$$\begin{aligned} &\mathbf{A} : \mathbf{B} \longrightarrow \mathfrak{g}, && \mathbf{A}([b,g]) := \mathrm{Ad}_{g^{-1}}(A(b)), \\ &\alpha \in \Omega^1(\mathbf{B}) \otimes \mathfrak{g}, && \alpha_{[(b,g)]} := \mathrm{Ad}_{g^{-1}}\sigma_b + \mu, \end{aligned}$$

on $\mathbf{B}$ *are well defined, where* $\mu = g^{-1}dg \in \Omega^1(\mathrm{G}) \otimes \mathfrak{g}$ *is the left invariant Maurer-Cartan form on* G, *and the restriction of* $\mathbf{A}$ *to* $B \subset \mathbf{B}$ *coincides with* A. *Moreover,* α *yields a connection on the principal* G*-bundle* $\mathbf{B} \to M$ *which satisfies*

$$(32)\qquad d\mathbf{A} = -[\alpha, \mathbf{A}] \quad \textit{and} \quad d\alpha + \frac{1}{2}[\alpha,\alpha] = 2\pi^*(\omega)\mathbf{A}.$$

Proof. First, note that $A : B \to \mathrm{H}$ and $\sigma \in \Omega^1(B) \otimes \mathfrak{g}$ are H-equivariant, i.e. $R_h^*A = \mathrm{Ad}_{h^{-1}}A$ and $R_h^*\sigma = \mathrm{Ad}_{h^{-1}}\sigma$. Thus, if we define the function $\hat{\mathbf{A}}$ and the one form $\hat{\alpha}$ by

$$\begin{aligned} &\hat{\mathbf{A}} := \mathrm{Ad}_{g^{-1}}(A) : \quad B \times \mathrm{G} \longrightarrow \mathfrak{g} \\ &\hat{\alpha} := \mathrm{Ad}_{g^{-1}}\sigma + \mu \quad \in \Omega^1(B \times \mathrm{G}) \otimes \mathfrak{g}, \end{aligned}$$

then $\hat{\mathbf{A}}(bh, h^{-1}g) = \hat{\mathbf{A}}(b,g)$, so that $\hat{\mathbf{A}}$ is the pull back of a well defined function $\mathbf{A} : \mathbf{B} \to \mathfrak{g}$. Also, $\hat{\alpha}$ is invariant under the right H-action from above, and for $h \in \mathfrak{h}$ we have

$$\hat{\alpha}((\xi_h)_b, dR_g(-h)) = \mathrm{Ad}_{g^{-1}}(\sigma_b(\xi_h)) - \mu(dR_g(h)) = \mathrm{Ad}_{g^{-1}}(h) - \mathrm{Ad}_{g^{-1}}(h) = 0,$$

so that $\hat{\alpha}$ is indeed the pull back of a well defined form $\alpha \in \Omega^1(\mathbf{B}) \otimes \mathfrak{g}$. Moreover, $R_g^*(\hat{\alpha}) = \mathrm{Ad}_g^{-1}\hat{\alpha}$ is easily verified, and since $\hat{\alpha}$ coincides with μ on the fibers of the projection $B \times \mathrm{G} \to B$, it follows that the value of $\hat{\alpha}$ on each left invariant vector field on G is constant. Since the left invariant vector fields generate the principal right action of the bundle $B \times \mathrm{G} \to B$, it follows that $\hat{\alpha}$ is a connection on this bundle, hence so is α on the quotient $\mathbf{B} \to M$.

Finally, to show (32) it suffices to show the corresponding equations for $\hat{\alpha}$ and $\hat{\mathrm{A}}$. We have

$$\begin{aligned} d\hat{\mathrm{A}} &= -[\mu, \mathrm{Ad}_{g^{-1}}(A)] + \mathrm{Ad}_{g^{-1}}(dA) = -[\mu, \hat{\mathrm{A}}] - \mathrm{Ad}_{g^{-1}}([\sigma, A]) \\ &= -[\mu, \hat{\mathrm{A}}] - [\mathrm{Ad}_{g^{-1}}\sigma, \hat{\mathrm{A}}] = -[\hat{\alpha}, \hat{\mathrm{A}}] \end{aligned}$$

by (31), and

$$\begin{aligned} d\hat{\alpha} + \frac{1}{2}[\hat{\alpha}, \hat{\alpha}] &= (-[\mu, \mathrm{Ad}_{g^{-1}}\sigma] + \mathrm{Ad}_{g^{-1}}d\sigma + d\mu) + \frac{1}{2}(\mathrm{Ad}_{g^{-1}}[\sigma, \sigma] + 2[\mu, \mathrm{Ad}_{g^{-1}}\sigma] + [\mu, \mu]) \\ &= \mathrm{Ad}_{g^{-1}}(d\sigma + \frac{1}{2}[\sigma, \sigma]) + d\mu + \frac{1}{2}[\mu, \mu] \\ &= \mathrm{Ad}_{g^{-1}}(2\pi^*(\omega)A) = 2\pi^*(\omega)\hat{\mathrm{A}}, \end{aligned}$$

where the second to last equation follows from the Maurer-Cartan equation and (31).

□

Let $\hat{M} \subset \mathbf{B}$ be a holonomy reduction of α, and let $\hat{\mathrm{T}} \subset \mathrm{G}$ be the holonomy group, so that the restriction $\hat{M} \to M$ becomes a principal $\hat{\mathrm{T}}$-bundle. By the first equation of (32), it follows that $\hat{M} \subset \mathrm{A}^{-1}(a)$ for some $a \in \mathfrak{g}$, and by choosing the holonomy reduction such that it contains an element of $B \subset \mathbf{B}$, we may assume w.l.o.g. that $a \in Q$. We let

$$\hat{\mathrm{S}} := \mathrm{Stab}\,(a) = \{g \in \mathrm{G} \mid \mathrm{Ad}_g a = a\} \subset \mathrm{G} \quad \text{and} \quad \hat{\mathfrak{s}} := \mathfrak{z}(a) = \{x \in \mathfrak{g} \mid [x, a] = 0\},$$

so that $\hat{\mathrm{S}} \subset \mathrm{G}$ is a closed Lie subgroup whose Lie algebra equals $\hat{\mathfrak{s}}$. Observe that the restriction $\mathrm{A}^{-1}(a) \to M$ is a principal $\hat{\mathrm{S}}$-bundle, hence we conclude that $\hat{\mathrm{T}} \subset \hat{\mathrm{S}}$. Moreover, on $\hat{M}$, we have

$$\hat{\alpha} = 2\kappa a$$

for some $\kappa \in \Omega^1(\hat{M})$ which by (32) satisfies $d\kappa = \pi^*(\omega)$. In particular, the *Ambrose-Singer Holonomy theorem* implies that $\mathrm{T}_a = \exp(\mathbb{R}a) \subset \mathrm{G}$ is the identity component of $\hat{\mathrm{T}}$ which is thus a one dimensional (possibly non-regular) subgroup of $\hat{\mathrm{S}}$, and κ yields the desired connection form on the principal $\hat{\mathrm{T}}$-bundle $\hat{M} \to M$.

Define $\mathcal{C}_a \subset \mathcal{C}$ as in (23) and $\Gamma_a \subset \mathrm{G}$ and $Q \subset \mathfrak{g}$ as in (27), and let

$$\hat{B} := p^{-1}(\hat{M}) \subset B \times \mathrm{G},$$

where $p : B \times \mathrm{G} \to B \times_{\mathrm{H}} \mathrm{G} = \mathbf{B}$ is the canonical projection. Then the restriction of the map

$$\bar{\imath} : B \times \mathrm{G} \longrightarrow \mathrm{G}, \qquad \bar{\imath}(b, g) := g^{-1}$$

satisfies $\bar{\imath}(\hat{B}) \subset \Gamma_a$; indeed, since $\mathbf{A}(\hat{M}) \equiv a$, it follows that $\mathrm{Ad}_{g^{-1}}A(b) = a$ for all $(b, g) \in \hat{B}$ and hence $\mathrm{Ad}_g a = A(b) \in Q$, so that $g^{-1} \in \Gamma_a$. Since $2\kappa a = \hat{\alpha} = \mathrm{Ad}_{g^{-1}}\sigma + \mu$, it follows by (30) that

$$\bar{\imath}^*(\mu) = -\mathrm{Ad}_g\mu = -2\kappa\mathrm{Ad}_g a + \sigma = -2\kappa A + e_- \otimes \theta + \eta + e_+ \otimes (\rho\theta) + \frac{1}{2}\omega(u,\theta)e_+^2\,,$$

and hence

$$\bar{\imath}^*(\mu) = -2\kappa\left(\frac{1}{2}e_-^2 + \rho\right) + e_- \otimes \theta + \eta \qquad \mathrm{mod}\ \mathfrak{g}^1 \oplus \mathfrak{g}^2\,.$$

Comparing this equation with the structure equations in Theorem 3.9, it follows that the induced map $\hat{\imath} : \hat{M} = \hat{B}/\mathrm{H} \to \mathcal{C}_a = \Gamma_a/\mathrm{H}$ is a local diffeomorphism and the induced map $\imath : \tilde{M} := \mathrm{T}\backslash\hat{M} \to \mathrm{T}_a\backslash\mathcal{C}_a$ is connection preserving, where $\mathrm{T}_a\backslash\mathcal{C}_a$ is (locally) equipped with the special symplectic connection from Theorem 3.11. Thus, we have shown Theorem B from the introduction.

Remark 4.2. Theorem B generalizes immediately to orbifolds. Namely, if M is an *orbifold*, then a special symplectic orbifold connection consists of an *almost principal* H-*bundle* $B \to M$, i.e. H acts locally freely and properly on B such that $M = B/\mathrm{H}$, and a coframing $\theta + \eta \in \Omega^1(B) \otimes (V \oplus \mathfrak{h})$ on B such that $\eta(\xi_h) \equiv h \in \mathfrak{h}$ and $\theta(\xi_h) \equiv 0$ for all infinitesimal generators ξ_h of the H-action, and such that the structure equations (10) hold for some function $\rho : B \to \mathfrak{h}$.

Now the proofs of Propositions 2.13 and 4.1 as well as the proof of Theorem B go through verbatim as we never used the freeness of the H-action on B. In particular, the holonomy reduction $\hat{M}$ is a *manifold* on which $\hat{\mathrm{T}}$ acts locally freely, and $M = \hat{\mathrm{T}}\backslash\hat{M}$ as an orbifold.

5. Applications and global properties

Definition 5.1. Let (M, ∇) be a manifold with a connection. A *(local) symmetry* of the connection is a (local) diffeomorphism $\underline{\varphi} : M \to M$ which preserves ∇, i.e. such that $\nabla_{d\underline{\varphi}(X)}d\underline{\varphi}(Y) = d\underline{\varphi}(\nabla_X Y)$ for all vector fields X, Y on M. An *infinitesimal symmetry* of the connection is a vector field $\underline{\zeta}$ on M such that for all vector fields X, Y on M we have the relation

$$[\underline{\zeta}, \nabla_X Y] = \nabla_{[\underline{\zeta},X]}Y + \nabla_X[\underline{\zeta}, Y]\,.$$

Furthermore, let $\pi : B \to M$ be an H-structure compatible with ∇, and let θ, η denote the tautological and the connection form on B, respectively. A *(local) symmetry* on B is a (local) diffeomorphism $\varphi : B \to B$ such that $\varphi^*(\theta) = \theta$ and $\varphi^*(\eta) = \eta$. An *infinitesimal symmetry* on B is a vector field ζ on B such that $\mathfrak{L}_\zeta(\theta) = \mathfrak{L}_\zeta(\eta) = 0$.

The ambiguity of the terminology above is justified by the one-to-one correspondence between (local or infinitesimal) symmetries on M and B. Namely, if $\underline{\varphi} : M \to M$ is a (local) symmetry, then there is a unique (local) symmetry $\varphi : B \to B$ with $\pi \circ \varphi = \underline{\varphi} \circ \pi$, and vice versa. Likewise, for any infinitesimal symmetry $\underline{\zeta}$ on M, there is a unique infinitesimal symmetry ζ on B such that $\underline{\zeta} = d\pi(\zeta)$.

The infinitesimal symmetries form the Lie algebra of the (local) group of (local) symmetries. We also observe that an infinitesimal symmetry on B is uniquely determined by its value at any point. (The corresponding statement fails for infinitesimal symmetries on M in general.)

Proof of Corollary C. The first part follows immediately from Theorem B since $\mathcal{C}_a \subset \mathcal{C}$ is an open subset of the analytic manifold $\mathcal{C}$, and the action of T_a on $\mathcal{C}_a$ is analytic as well. Also, the C^4-germ of the connection at a point determines uniquely the G-orbit of $a \in \mathfrak{g}$ by (9) and hence the connection by Theorem B.

Note that the generic element $a \in \mathfrak{g}$ is G-conjugate to an element in the Cartan subalgebra which is uniquely determined up to the action of the (finite) Weyl group. Since multiplying $a \in \mathfrak{g}$ by a scalar does not change the connection, it follows that the generic special symplectic connection associated to $\mathfrak{g}$ depends on $(\mathrm{rk}(\mathfrak{g})-1)$ parameters.

For the second part, by virtue of Theorem B it suffices to show the statement for manifolds of the form $M = M_U$ where $U \subset \mathcal{C}_a$ is a regular open subset for some $a \in \mathfrak{g}$. Let $\Gamma_U \subset \Gamma_a \subset \mathrm{G}$ be the H-invariant subset such that we have the principal H-bundle $\Gamma_U \to U$, and let $B_U := \mathrm{T}_a \backslash \Gamma_U$ so that $B_U \to M_U$ is the associated H-structure.

Let $x \in \hat{\mathfrak{s}}$, and denote by $\hat{\zeta}_x$ the right invariant vector field on G corresponding to $-x$, so that the map $x \mapsto \hat{\zeta}_x$ is a Lie algebra homomorphism. Then $\mathfrak{L}_{\hat{\zeta}_x}(\mu) = 0$ where μ denotes the Maurer-Cartan form. By (27), it follows that the restriction of $\hat{\zeta}_x$ to Γ_a is tangent, and since $\Gamma_U \subset \Gamma_a$ is open, we may regard $\hat{\zeta}_x$ as a vector field on Γ_U. Since $\hat{\zeta}_x$ commutes with the action of T_a, it follows that there is a related vector field ζ_x on the quotient $B_U = \mathrm{T}_a^{\mathrm{loc}} \backslash \Gamma_U$, and since the tautological and curvature form of the induced connection on B_U pull back to components of μ, it follows that ζ_x is an infinitesimal symmetry on B_U.

Conversely, suppose that ζ is an infinitesimal symmetry on B_U. Since an infinitesimal symmetry must preserve the curvature and its covariant derivatives, we must have $\zeta(A) = 0$. But the tangent of the fiber of the map $A : \Gamma_U \to \mathfrak{g}$ is spanned by the vector fields ζ_x, $x \in \hat{\mathfrak{s}}$, and since infinitesimal symmetries are uniquely determined by their value at a point, it follows that $\zeta = \zeta_x$ for some $x \in \hat{\mathfrak{s}}$.

Finally, it is evident that $\zeta_x = 0$ iff $\hat{\zeta}_x$ is tangent to T_a iff $x \in \mathbb{R}a$, hence the claim follows. □

We also mention the following rigidity result from [CS].

Theorem 5.2. *Let $\mathfrak{g}$ be a 2-gradable simple Lie algebra, let* G *be the connected Lie group with Lie algebra $\mathfrak{g}$ and trivial center, and let $\hat{\mathrm{S}} \subset \mathrm{G}$ be a maximal compact subgroup. Then $\mathcal{C} = \hat{\mathrm{S}}/\mathrm{K}$ for some compact subgroup $\mathrm{K} \subset \hat{\mathrm{S}}$ where $\mathcal{C} \subset \mathbb{P}^o(\mathfrak{g})$ is the root cone. Moreover, let $\mathrm{T} \subset \hat{\mathrm{S}}$ be the identity component of the center of $\hat{\mathrm{S}}$. Then the following are equivalent:*

1. *There is a compact simply connected symplectic manifold M with a special symplectic connection associated to the simple Lie algebra $\mathfrak{g}$.*
2. $\dim \mathrm{T} = 1$, *i.e.* $\mathrm{T} \cong S^1$.
3. $\mathrm{T} \neq \{e\}$.

If these conditions hold then $\mathrm{T}\backslash\mathcal{C} \cong \hat{\mathrm{S}}/(\mathrm{T}\cdot\mathrm{K})$ is a compact Hermitean symmetric space, and the map $\imath : M \to \mathrm{T}\backslash\mathcal{C}$ from Theorem B is a connection preserving covering. Thus, M is a Hermitean symmetric space as well.

This theorem allows us to classify all compact simply connected manifolds with special symplectic connections, as the maximal compact subgroups of semisimple Lie groups are fully classified (e.g. [OV]). Thus, we obtain Theorem D from the introduction.

We shall only sketch the proof of Theorem 5.2. If M is simply connected, then there is a principal T_a-bundle $\pi : \hat{M} \to M$ by Theorem B with a connection κ whose curvature equals ω. If $\mathrm{T}_a \cong \mathbb{R}$, then π would be a homotopy equivalence, and since $\pi^*(\omega) = d\kappa$ is exact, this would imply that $\omega \in \Omega^2(M)$ was exact, which is impossible if M is compact.

We conclude that $\mathrm{T}_a \cong S^1$, so that $\hat{M}$ is compact as well. Thus, the local diffeomorphism $\hat{\imath} : \hat{M} \to \mathcal{C}_a \subset \mathcal{C}$ must be a covering, and, in particular, $\mathcal{C}_a = \mathcal{C}$.

Thus, we have to consider those $a \in \mathfrak{g}$ for which $\mathrm{T}_a \cong S^1$, $\mathcal{C}_a = \mathcal{C}$, and such that the action of T_a on the universal cover of $\mathcal{C}$ is free.

By a thorough investigation of the elements $a \in \mathfrak{g}$ with the above properties, one finds that the stabilizer $\hat{\mathrm{S}} \subset \mathrm{G}$ of a is a maximal compact subgroup such that $\mathrm{T}_a \subset \hat{\mathrm{S}}$ is the connected component of the center. Since $\hat{\mathrm{S}}$ also acts transitively on $\mathcal{C}$, i.e. $\mathcal{C} = \hat{\mathrm{S}}/\mathrm{K}$ for some compact subgroup $\mathrm{K} \subset \hat{\mathrm{S}}$, it follows that M is a cover of $\mathrm{T}_a\backslash\hat{\mathrm{S}}/\mathrm{K} = \hat{\mathrm{S}}/(\mathrm{T}_a\cdot\mathrm{K})$, and finally, one shows that $\hat{\mathrm{S}}/(\mathrm{T}_a \cdot \mathrm{K})$ is Hermitean symmetric.

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