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ON HOLOMORPHICALLY PROJECTIVE MAPPINGS FROM EQUIAFFINE SYMMETRIC AND RECURRENT SPACES ONTO KÄHLERIAN SPACES

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ABSTRACT. In this paper we consider holomorphically projective mappings from the symmetric and recurrent equiaffine spaces A_n onto (pseudo-) Kählerian spaces $\bar{K}_n$. We proved that in this case space A_n is holomorphically projective flat and $\bar{K}_n$ is space with constant holomorphic curvature.

These results are the generalization of results by T. Sakaguchi, J. Mikeš, V. V. Domashev, which were done for holomorphically projective mappings of symmetric, recurrent and semisymmetric Kählerian spaces.

1. INTRODUCTION

Diffeomorphisms and automorphisms of geometrically generalized spaces constitute one of the current main directions in differential geometry. A large number of papers are devoted to geodesic, quasigeodesic, almost geodesic, holomorphically projective and other mappings (see [1], [4], [11], [12], [13], [17], [19], [21], [22], [24], [25], [26], [28]). On the other hand, one line of thought is now the most important one, namely, the investigation of special affine-connected, Riemannian, Kählerian and Hermitian spaces.

As we know, Kählerian spaces are the special case of Hermitian spaces [5]. In many papers, holomorphically projective mappings and transformations of Hermitian spaces are studied (for example see [1], [3], [6], [8], [13], [14], [18], [15], [20], [23], [24], [25], [28]). These are special cases of F_1-planar mappings. In [11], [13], F_1-planar mappings from the space A_n with affine connection onto a Riemannian space $\bar{V}_n$ are defined and studied.

In this paper, we present some new results obtained for holomorphically projective mappings from equiaffine semisymmetric, symmetric and recurrent spaces A_n onto Kählerian spaces $\bar{K}_n$.

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Key words and phrases: holomorphically projective mapping, equiaffine space, affine-connected space, symmetric space, recurrent space, semisymmetric space, Riemannian space, Kählerian space.

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2. Holomorphically projective mappings

J. Mikeš and O. Pokorná [14] considered holomorphically projective mappings from equiaffine spaces A_n onto Kählerian spaces $\bar{K}_n$.

A space A_n with the affine connection Γ is *equiaffine* [19] if in A_n the Ricci tensor is symmetric. These spaces are characterized by a coordinate system x, such that $\Gamma^\alpha_{\alpha i}(x) = \partial f(x)/\partial x^i$, where $f(x)$ is a function on A_n, and Γ^h_{ij} are components of the connection Γ.

A (pseudo-) Riemannian space $\bar{K}_n$ is called a *Kählerian space* if it contains, along with the metric tensor $\bar{g}_{ij}(x)$, an affine structure $F^h_i(x)$ satisfying the following relations

$$(1) \qquad F^h_\alpha F^\alpha_i = -\delta^h_i\,, \qquad \bar{g}_{\alpha(i}F^\alpha_{j)} = 0\,, \qquad F^h_{i|j} = 0\,.$$

Here δ^h_i is the Kronecker symbol, $(i\,j)$ denotes the symmetrization without division, and "$|$" denotes the covariant derivative in $\bar{K}_n$.

As it is known, in Kählerian spaces $\bar{K}_n$ there hold the following properties

$$(2) \qquad \bar{g}^{\alpha\beta}F^i_\alpha F^j_\beta = \bar{g}^{ij}\,,\ \ \bar{R}^h_{\alpha jk}F^\alpha_i = \bar{R}^\alpha_{ijk}F^h_\alpha\,,\ \ \bar{R}^h_{i\alpha\beta}F^\alpha_j F^\beta_k = \bar{R}^h_{ijk}\,,\ \ \bar{R}_{\alpha\beta}F^\alpha_i F^\beta_j = \bar{R}_{ij}\,,$$

where $\|\bar{g}^{ij}\| = \|\bar{g}_{ij}\|^{-1}$, $\bar{R}^h_{ijk}$ and $\bar{R}_{ij}$ $(= \bar{R}^\alpha_{ij\alpha})$ are the components of the Riemannian and Ricci tensors in $\bar{K}_n$, respectively. We note that tensors $\bar{g}_{i\alpha}F^\alpha_j$, $\bar{g}^{i\alpha}F^j_\alpha$ and $\bar{R}_{i\alpha}F^\alpha_j$ are skew symmetric.

The following criteria from the paper [14] hold for holomorphically projective mappings from an equiaffine space A_n onto a Kählerian space $\bar{K}_n$.

Consider concrete mappings $f\colon A_n \to \bar{K}_n$, both spaces being referred to the general coordinate system x with respect to this mapping. This is a coordinate system where two corresponding points $M \in A_n$ and $f(M) \in \bar{K}_n$ have equal coordinates $x = (x^1, x^2, \ldots, x^n)$; the corresponding geometric objects in $\bar{K}_n$ will be marked with a bar. For example, Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are components of the affine connection on A_n and $\bar{K}_n$, respectively.

The equiaffine space A_n admits a *holomorphically projective mapping* f onto the Kählerian space $\bar{K}_n$ if and only if the following conditions in the common coordinate system x hold

$$(3) \qquad \bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_{(i}\psi_{j)} - F^h_{(i}F^\alpha_{j)}\psi_\alpha\,,$$

where $\psi_i(x)$ is a gradient, i.e. there is a function $\psi(x)$ for $\psi_i(x) = \partial\psi(x)/\partial x^i$.

If $\psi_i \not\equiv 0$ then a holomorphically projective mapping is called *nontrivial*; otherwise it is said to be *trivial* or *affine*.

In this space A_n there is a complex structure F covariantly constant, i.e. $F^h_{i,j} = 0$ (*comma* denotes the covariant derivative on A_n) and this condition implies the properties $R^h_{\alpha jk}F^\alpha_i = R^\alpha_{ijk}F^h_\alpha$ for the Riemannian tensor R^h_{ijk} of A_n.

The following theorem is the result of reduction of Theorem 2 from [14]:

Theorem 1. *Let in an equiaffine space A_n exist the solution of the following system of linear differential equations with respect to the unknown functions $a^{ij}(x)$ and $\lambda^i(x)$:*

$$(4) \qquad a^{ij}{}_{,k} = \lambda^{(i}\delta^{j)}_k + \lambda^\alpha F^{(i}_\alpha F^{j)}_k\,,$$

where the matrix $\|a^{ij}\|$ should further satisfy $\det\|a^{ij}\| \neq 0$ *and the algebraic conditions $a^{ij} = a^{ji}$ and $a^{ij} = a^{\alpha\beta}F^i_\alpha F^j_\beta$.*

Then A_n admits a holomorphically projective mapping onto a Kählerian spaces $\bar{K}_n$. The metric tensor $\bar{g}_{ij}$ of $\bar{K}_n$ and solutions of (4) *are connected by relations*

$$\text{(5)} \qquad \text{a) } a^{ij} = e^{-2\psi}\bar{g}^{ij}, \qquad \text{b) } \lambda^i = -a^{i\alpha}\psi_\alpha .$$

This Theorem is a generalization of results in [3], [13], [24], [25].

The question of existence of a solution of (4) leads to the study of integrability conditions and their differential prolongations. The general solution of (4) does not depend on more than $N_o = 1/4\,(n+1)^2$ parameters. The number of essential parameters $r \le N_o$ of a solution of (4) is called the *degree of mobility of an equiaffine space A_n relative to holomorphically projective mappings onto Kählerian spaces* [14].

Let in an equiaffine space A_n the condition

$$\text{(6)} \qquad R^h_{ijk} = \delta^h_i \overset{1}{v}_{jk} + \delta^h_j \overset{2}{v}_{ik} - \delta^h_k \overset{2}{v}_{ij} + F^h_i \overset{3}{v}_{jk} + F^h_j \overset{4}{v}_{ik} - F^h_k \overset{4}{v}_{ij}$$

hold, where $\overset{\sigma}{v}$ are tensors.

If an equiaffine space A_n with condition (6) (*holomorphically projective flat space*) admits a holomorphically projective mapping onto a Kählerian space $\bar{K}_n$, then $\bar{K}_n$ has the constant holomorphic curvature, and A_n has the degree $r = N_o$ [14].

From equations (3) for the Riemannian and Ricci tensors of A_n and $\bar{K}_n$ follow

$$\text{(7)} \qquad \bar{R}^h_{ijk} = R^h_{ijk} + \delta^h_k\psi_{ij} - \delta^h_j\psi_{ik} + (F^h_j\psi_{\alpha k} - F^h_k\psi_{\alpha j})F^\alpha_i + F^h_i(\psi_{\alpha k}F^\alpha_j - \psi_{\alpha j}F^\alpha_k),$$

$$\text{(8)} \qquad \bar{R}_{ij} = R_{ij} + n\psi_{ij} + 2\psi_{\alpha\beta}F^\alpha_i F^\beta_j,$$

where

$$\text{(9)} \qquad \psi_{ij} = \psi_{i,j} - \psi_i\psi_j + \psi_\alpha\psi_\beta F^\alpha_i F^\beta_j .$$

3. Holomorphically projective mappings with a preservation of the Ricci tensor in respect of the structure onto Kählerian spaces

Hereafter we shall assume that in the equiaffine space A_n *the Ricci tensor R_{ij} with respect to the structure F will be preserved*, i.e.

$$\text{(10)} \qquad R_{\alpha\beta}F^\alpha_i F^\beta_j = R_{ij} .$$

Because the conditions (2) and (10) hold for the Ricci tensors, from the formula (8) follows

$$\text{(11)} \qquad \psi_{\alpha\beta}F^\alpha_i F^\beta_j = \psi_{ij} .$$

Then the form (8) becomes simply

$$\text{(12)} \qquad \bar{R}_{ij} = R_{ij} + (n+2)\psi_{ij} .$$

In this case we see that in A_n holds the analogical formula of (2): $R^h_{i\alpha\beta}F^\alpha_j F^\beta_k = R^h_{ijk}$. We note that from (10) and (11) the tensors $R_{\alpha i}F^\alpha_j$ and $\psi_{\alpha i}F^\alpha_j$ are skew symmetric.

Using (12), we eliminate ψ_{ij} from (8). In this case $\bar{P}^h_{ijk} = P^h_{ijk}$ holds, where

$$\text{(13)} \qquad P^h_{ijk} = R^h_{ijk} - \frac{1}{n+2}\,(\delta^h_k R_{ij} - \delta^h_j R_{ik} + (F^h_j R_{\alpha k} - F^h_k R_{\alpha j})F^\alpha_i + 2\,F^h_i R_{\alpha k}F^\alpha_j)$$

is the *tensor of the holomorphically projective curvature* of A_n. This tensor is an invariant object of the holomorphically projective mappings, see [13], [24], [25], [28]. This tensor is F-traceless [7].

4. Holomorphically projective mappings from semisymmetric equiaffine spaces

As it is known, *semisymmetric* spaces A_n are characterized by the condition $R \circ R = 0$ [2]. These spaces were characterized by N. S. Sinyukov [24] by the following differential conditions on the Riemannian tensor: $R^h_{ijk,[lm]} = 0$. On base of the Ricci identity, this condition is written as follows

$$R^h_{\alpha jk} R^\alpha_{ilm} + R^h_{i\alpha k} R^\alpha_{jlm} + R^h_{ij\alpha} R^\alpha_{klm} - R^\alpha_{ijk} R^h_{\alpha lm} = 0 . \tag{14}$$

We assume that the semisymmetric equiaffine space A_n (where the Ricci tensor R_{ij} with respect to the structure F will be preserved) admits the holomorphically projective mapping onto Kählerian spaces $\bar{K}_n$.

We use (7) to eliminate the components R^h_{ijk} from (14). Through the metric tensor $\bar{g}_{ij}$ of $\bar{K}_n$ we lower the index h. Further we make the symmetrization of this term in the indices h and i. The term obtained we multiply with $\bar{g}^{kl}$ and contract with respect to k and l. After we make the symmetrization of the form received in the indices j and m. Finally we obtain

$$\begin{aligned}
&\bar{R}_{hj}\psi_{im} + \bar{R}_{ij}\psi_{hm} + \bar{R}_{hm}\psi_{ij} + \bar{R}_{im}\psi_{hj} \\
&\quad - \bar{R}^\alpha_j \psi_{h\alpha}\bar{g}_{im} - \bar{R}^\alpha_j \psi_{i\alpha}\bar{g}_{hm} - \bar{R}^\alpha_m \psi_{h\alpha}\bar{g}_{ij} - \bar{R}^\alpha_m \psi_{i\alpha}\bar{g}_{hj} \\
&\quad + (\bar{R}_{\delta j}\psi_{\gamma m} + \bar{R}_{\gamma j}\psi_{\delta m} + \bar{R}_{\delta m}\psi_{\gamma j} + \bar{R}_{\gamma m}\psi_{\delta j} - \bar{R}^\alpha_j \psi_{\delta\alpha}\bar{g}_{\gamma m} \\
&\quad - \bar{R}^\alpha_j \psi_{\gamma\alpha}\bar{g}_{\delta m} - \bar{R}^\alpha_m \psi_{\delta\alpha}\bar{g}_{\gamma j} - \bar{R}^\alpha_m \psi_{\gamma\alpha}\bar{g}_{\delta j}) F^\gamma_i F^\delta_h = 0 ,
\end{aligned} \tag{15}$$

where $\bar{R}^h_i = \bar{g}^{h\alpha}\bar{R}_{\alpha i}$.

We contract the formula (15) with $\bar{g}^{im}$. Because the tensors $\bar{R}_{hj}$, $\bar{g}_{hj}$ and ψ_{hj} are symmetric, it follows $\bar{R}^\alpha_j \psi_{\alpha h} = \bar{R}^\alpha_h \psi_{\alpha j}$. We make the symmetrization in all indices, we obtain

$$\begin{aligned}
&\bar{R}_{hj}\psi_{im} + \bar{R}_{ij}\psi_{hm} + \bar{R}_{hm}\psi_{ij} + \bar{R}_{im}\psi_{hj} + \bar{R}_{hi}\psi_{jm} + \bar{R}_{jm}\psi_{hi} \\
&\quad - \bar{R}^\alpha_j \psi_{h\alpha} g_{im} - \bar{R}^\alpha_j \psi_{i\alpha} g_{hm} - \bar{R}^\alpha_m \psi_{h\alpha} g_{ij} - \bar{R}^\alpha_m \psi_{i\alpha} g_{hj} \\
&\quad - \bar{R}^\alpha_i \psi_{h\alpha} g_{mj} - \bar{R}^\alpha_m \psi_{j\alpha} g_{hi} = 0 .
\end{aligned} \tag{16}$$

We subtract from (15) formula (16) and also formula (16), which was contracted with $F^h_{h'} F^i_{i'}$ after the elimination "$'$". We obtain

$$\bar{R}_{hi}\psi_{jm} + \bar{R}_{jm}\psi_{hi} - \bar{R}^\alpha_i \psi_{h\alpha}\bar{g}_{jm} - \bar{R}^\alpha_j \psi_{m\alpha}\bar{g}_{hi} = 0 . \tag{17}$$

We contract (17) with $\bar{g}^{hi}\bar{g}^{jm}$:

$$\bar{R}^\alpha_\beta \psi_{\alpha\gamma}\bar{g}^{\beta\gamma} = \frac{\Delta\,\bar{R}}{n}, \quad \text{where} \quad \Delta = \psi_{\alpha\beta}\bar{g}^{\alpha\beta} , \quad \bar{R} = \bar{R}_{\alpha\beta}\bar{g}^{\alpha\beta}$$

is the scalar curvature of $\bar{K}_n$. And after contracting (18) with $\bar{g}^{jm}$ we have:

$$\bar{R}^\alpha_i \psi_{h\alpha} = \frac{\Delta}{n}\,\bar{E}_{hi} + \frac{\bar{R}}{n}\,\psi_{hi} , \quad \text{where} \quad \bar{E}_{hi} = \bar{R}_{hi} - \frac{\bar{R}}{n}\,\bar{g}_{hi} .$$

We use the last formula to eliminate the components $\bar{R}_i^\alpha \psi_{h\alpha}$ from (17):

$$\bar{E}_{hi}\Sigma_{jm} + \bar{E}_{jm}\Sigma_{hi} = 0\,, \quad \text{where} \quad \Sigma_{hi} = \psi_{hi} - \frac{\Delta}{n}\,\bar{g}_{hi}\,.$$

This formula implies $\bar{E}_{hi} = 0$ or $\Sigma_{hi} = 0$, i.e.

$$\text{(18)} \qquad \text{a)}\ \bar{R}_{hi} = \frac{\bar{R}}{n}\,\bar{g}_{hi} \qquad \text{or} \qquad \text{b)}\ \psi_{hi} = \frac{\Delta}{n}\,\bar{g}_{hi}\,.$$

We have the following

Lemma 1. *If an equiaffine semisymmetric space A_n, where the Ricci tensor R_{ij} with respect to the structure F will be preserved, admits the holomorphically projective mapping onto a Kählerian space $\bar{K}_n$, then $\bar{K}_n$ is the Einstein space or $\bar{K}_n$ admits a K-concircular vector field ψ_i, which satisfies* (18b).

The proof consists in the fact that the condition (18a) characterizes $\bar{K}_n$ as the Einstein space and condition (18b) together with formulas (3) and (9) characterizes $\bar{K}_n$ as the K-concircular vector field. [18].

This Lemma is analogical to results for geodesic and holomorphically projective mappings of semisymmetric spaces which were studied by N.S. Sinyukov [24] and T. Sakaguchi [23], see [12], [13].

We shall prove more powerful

Theorem 2. *If an equiaffine semisymmetric space A_n, where the Ricci tensor R_{ij} with respect to the structure F will be preserved, admits a holomorphically projective mapping onto a Kählerian space $\bar{K}_n$, then $\bar{K}_n$ is the space with the constant holomorphically projective curvature or $\bar{K}_n$ admits a K-concircular vector field ψ_i, which satisfies* (18b), *and $\bar{K}_n$ is quasisymmetric.*

Remark. A space $\bar{K}_n$ is called *quasisymmetric* if (see [13])

$$\text{(19)} \qquad \bar{R}^h_{\alpha jk}\bar{Y}^\alpha_{ilm} + \bar{R}^h_{i\alpha k}\bar{Y}^\alpha_{jlm} + \bar{R}^h_{ij\alpha}\bar{Y}^\alpha_{klm} - \bar{R}^\alpha_{ijk}\bar{Y}^h_{\alpha lm} = 0\,,$$

where

$$\bar{Y}^h_{ijk} = \bar{R}^h_{ijk} - \frac{\Delta}{n}\big(\delta^h_k\bar{g}_{ij} - \delta^h_j\bar{g}_{ik} + (F^h_j\bar{g}_{\alpha k} - F^h_k\bar{g}_{\alpha j})\,F^\alpha_i + 2F^h_i\bar{g}_{\alpha k}F^\alpha_j\big)\,.$$

Proof. We will suppose (18b) not to hold. Hence $\bar{K}_n$is the Einstein space and (18a) holds. In this case the tensor $\bar{P}_{hijk} = \bar{g}_{h\alpha}\bar{P}^\alpha_{ijk} = \bar{g}_{h\alpha}P^\alpha_{ijk}$ is skew symmetric with respect to indices h and i.

The condition $P^h_{ijk,[lm]} = 0$ for the tensor of the holomorphically projective curvature follows from (14). This condition is written in the following form

$$\text{(20)} \qquad P^h_{\alpha jk}R^\alpha_{ilm} + P^h_{i\alpha k}R^\alpha_{jlm} + P^h_{ij\alpha}R^\alpha_{klm} - P^\alpha_{ijk}R^h_{\alpha lm} = 0\,.$$

We use (7) to eliminate the components R^h_{ijk} from (20). Through the metric tensor $\bar{g}_{ij}$ of $\bar{K}_n$ we lower the index h. Then we make the symmetrization of this term in the indices h and i. We obtain

$$\text{(21)} \qquad \begin{aligned} &\psi_{l(i}\bar{P}_{h)mjk} - \psi_{m(i}\bar{P}_{h)ljk} - \bar{g}_{l(i}\bar{P}^\gamma_{h)jk}\psi_{\gamma m} + \bar{g}_{m(i}\bar{P}^\gamma_{h)jk}\psi_{\gamma l} \\ &\qquad + (\psi_{l\alpha}\bar{P}_{\beta mjk} - \psi_{m\alpha}\bar{P}_{\beta ljk} - \bar{g}_{l\alpha}\bar{P}^\gamma_{\beta jk}\psi_{\gamma m} + \bar{g}_{m\alpha}\bar{P}^\gamma_{\beta jk}\psi_{\gamma l})F^\alpha_{(h}F^\beta_{i)} = 0\,. \end{aligned}$$

We contract (21) with $F^{i}_{i'}F^{l}_{l'}$, after the elimination "$'$" we add up the obtained formula and (21). Further, we make the symmetrization of this formula in the indices i and l. Finally we obtain

$$(22)\qquad \begin{aligned}\psi_{li}\bar{P}_{hmjk} - \psi_{m\alpha}F^{\alpha}_{h}F^{\beta}_{i}\bar{P}_{\beta ljk} - \bar{g}_{li}P^{\alpha}_{hjk}\psi_{\alpha m}\\ + \frac{1}{2}\bar{g}_{mh}\psi_{\alpha(l}P^{\alpha}_{i)jk} + \frac{1}{2}\bar{g}_{m\beta}F^{\beta}_{h}\psi_{\alpha(l}F^{\gamma}_{i)}P^{\alpha}_{\gamma jk} = 0\,.\end{aligned}$$

We contract (22) with $\bar{g}^{il}$:

$$(23)\qquad P^{\alpha}_{hjk}\psi_{\alpha m} = \frac{\Delta}{n}\bar{P}_{hmjk} - \bar{g}_{m\beta}F^{\beta}_{h}Q_{jk}\,,$$

where Q_{jk} is a tensor.

We make the cyclization of indices h, j and k. From the property of the tensor of the holomorphically projective curvature in the Einsten space $\bar{K}_n$ follows

$$\bar{g}_{m\beta}F^{\beta}_{h}Q_{jk} + \bar{g}_{m\beta}F^{\beta}_{j}Q_{kh} + \bar{g}_{m\beta}F^{\beta}_{k}Q_{hj} = 0\,.$$

Hence it is necessary that $Q_{hj} = 0$. Therefore (23) is more simply: $P^{\alpha}_{hjk}\psi_{\alpha m} = \frac{\Delta}{n}\bar{P}_{hmjk}$. After the substitution into (22), we obtain

$$\Sigma_{li}\bar{P}_{hmjk} - \Sigma_{m\alpha}F^{\alpha}_{h}F^{\beta}_{i}\bar{P}_{\beta ljk} = 0\,.$$

By the analysis of this term we see that

$$\Sigma_{li} = 0 \qquad \text{or} \qquad \bar{P}_{hmjk} = 0\,.$$

On the assumption $\Sigma_{li} \neq 0$, it is easy to see that $\bar{P}_{hmjk} = 0$. It means, that $\bar{K}_n$ is the space with the constant holomorphic curvature.

In case the space $\bar{K}_n$ has not constant holomorphic curvature, (18b) holds. After the substitution (18b) into (7) and the elimination R^{h}_{ijk} from (20) we obtain the condition(19), i.e. $\bar{K}_n$ is quasisymmetric [13], [15]. Theorem 2 is completely proved.

This result is an analogue of the similar theorem from the articles by J. Mikeš (see [24], [9], [10], [12], [13]) which holds for geodesic mappings of semisymmetric spaces.

Finally we prove

Theorem 3. *Let an equiaffine semisymmetric space A_n, where a Ricci tensor R_{ij} with respect to the structure F will be preserved, admit holomorphically projective mapping onto a Kählerian space $\bar{K}_n$. If A_n is not a holomorphically projective flat space then A_n admits a convergent vector field λ^h ($\lambda^h_{,i} = \text{const}\,\delta^h_i$), which satisfies* (4).

Proof. We derivate covariantly (5b). Based on the previous theorem, the formula (18b) holds. We accept (3), (5) and (9) and after their analysis we see that the vector field λ^h is concircular ([27], [12], [16]), i.e.

$$(24)\qquad \lambda^h_{,i} = \varrho\,\delta^h_i\,,$$

where ϱ is a function.

The conditions of integrability (24) have the form: $\lambda^{\alpha}R^{h}_{\alpha jk} = \varrho_{,j}\delta^h_k - \varrho_{,k}\delta^h_j$. Because in A_n $R^{h}_{i\alpha\beta}F^{\alpha}_{j}F^{\beta}_{k} = R^{h}_{ijk}$ holds, then $\varrho = \text{const}$. □

Analogical results were proved by J. Mikeš for the geodesic mappings of the semisymmetric Riemannian spaces and space with the affine connection, see [10], [12], [24], and for the holomorphically projective mappings of the semisymmetric Kählerian spaces, see [13].

5. Holomorphically projective mappings from symmetric and recurrent equiaffine spaces onto Kählerian spaces

As it is known, *symmetric* and *recurrent* spaces A_n are characterized by differential conditions on the Riemannian tensor

$$\text{(25)} \qquad \text{a)} \quad R^h_{ijk,l} = 0 \qquad \text{and} \qquad \text{b)} \quad R^h_{ijk,l} = \varphi_l R^h_{ijk}\,,$$

respectively, where $\varphi_l \neq 0$ is a covector.

We shall use the formula (25) for symmetric ($\varphi_l = 0$) and recurrent ($\varphi_l \neq 0$) spaces. It is easy to see that the symmetric spaces are semisymmetric. For the recurrent spaces it holds only in case when covector φ_l is a gradient, i.e. $\varphi_l = \varphi_{,l}$. It is known that all recurrent Riemannian spaces are semisymmetric.

Holomorphically projective mappings of symmetric and recurrent Kählerian spaces were studied by T. Sakaguchi [23], J. Mikeš and V. V. Domashev [3], [13], [24], [25]. These results are generalized in the following theorem:

Theorem 4. *Let an equiaffine symmetric (or semisymmetric recurrent) space A_n, where the Ricci tensor R_{ij} with respect to the structure F will be preserved, admit a nontrivial holomorphically projective mapping onto a Kählerian space $\bar{K}_n$. Then A_n is holomorphically projective flat and the space $\bar{K}_n$ has the constant holomorphically sectional curvature.*

Proof. Let an equiaffine symmetric (or semisymmetric recurrent) space A_n admit a nontrivial holomorphically projective mapping onto a Kählerian space $\bar{K}_n$. We suppose that A_n is not holomorphically projective flat. Based on the Theorem 3, the vector λ^h of equations (4) is convergent and (24) holds for $\varrho = \text{const}$. Hence for λ^h we have $\lambda^\alpha R^i_{\alpha jk} = 0$. The equation (4) has the solution with $\lambda^i \neq 0$. The integrability conditions of these equations based on formula (24) have the simple form

$$a^{i\alpha} R^j_{\alpha kl} + a^{j\alpha} R^i_{\alpha kl} = 0\,.$$

We derivate covariantly the last formula by x^l. Based on (4) and (25), we obtain

$$\lambda^{(i} R^{j)}_{mkl} + \lambda^\alpha F^{(i}_\alpha F^{j)}_\beta R^\beta_{mkl} = 0\,.$$

By the analysis of these formulas we see that $R^h_{ijk} = 0$, i.e. A_n is a flat space. Herewith the proof is complete. □

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