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ISOMETRY GROUPS OF k-CURVATURE HOMOGENEOUS PSEUDO-RIEMANNIAN MANIFOLDS

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ABSTRACT. We study the isometry groups of a family of complete $p+2$-curvature homogeneous pseudo-Riemannian metrics on $\mathbb{R}^{6+4p}$ which have neutral signature $(3+2p, 3+2p)$, and which are 0-curvature modeled on an indecomposible symmetric space.

1. INTRODUCTION

Let $\mathcal{M} := (M, g)$ be a pseudo-Riemannian manifold of signature (p, q). Let $g_P := g|_{T_PM}$ (resp. $\nabla^i R_P := \nabla^i R|_{T_PM}$) be the restriction of the metric (resp. the i^{th} covariant derivative of the curvature tensor) to the tangent space at $P \in M$. We define the *k-model of $\mathcal{M}$ at P* by setting:

$$\mathfrak{M}_k(\mathcal{M}, P) := (T_PM, g_P, R_P, \dots, \nabla^k R_P).$$

One says that $\phi : \mathfrak{M}_k(\mathcal{M}_1, P_1) \to \mathfrak{M}_k(\mathcal{M}_2, P_2)$ is an *isomorphism* from the k-model of $\mathcal{M}_1$ at P_1 to the k-model of $\mathcal{M}_2$ at P_2 if ϕ is a linear isomorphism from $T_{P_1}M_1$ to $T_{P_2}M_2$ with

$$\phi^* g_{2,P_2} = g_{1,P_1} \quad \text{and} \quad \phi^* \nabla_2^i R_{\mathcal{M}_2,P_2} = \nabla_1^i R_{\mathcal{M}_1,P_1} \quad \text{for} \quad 0 \le i \le k.$$

One says that $\mathcal{M}$ is *k-curvature homogeneous* if the k-models $\mathfrak{M}_k(\mathcal{M}, P)$ and $\mathfrak{M}_k(\mathcal{M}, Q)$ are isomorphic for any $P, Q \in M$.

In the Riemannian setting ($p = 0$), Sekigawa and Takagi constructed first examples of complete 0-curvature homogeneous Riemannian manifolds which are not locally homogeneous, see e.g. [14]. These examples are all noncompact. Compact examples (only in large dimensions) can be found in the paper by Ferus, Karcher, and Münzner [5]. Although many other examples have been constructed, there are no known Riemannian manifolds which are 1-curvature homogeneous but not locally homogeneous and it is natural to conjecture that any 1-curvature homogeneous Riemannian manifold is locally homogeneous.

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In the Lorentzian setting ($p = 1$), curvature homogeneous manifolds which are not locally homogeneous were constructed by Cahen et. al. [4]; 1-curvature homogeneous Lorentzian manifolds which are not locally homogeneous have been exhibited by Bueken and Djorić [2] and by Bueken and Vanhecke [3]. One could conjecture that a 2-curvature homogeneous Lorentzian manifold must be locally homogeneous.

It is clear that local homogeneity implies k-curvature homogeneity for any k. The following result, due to Singer [11] in the Riemannian setting and to F. Podesta and A. Spiro [10] in the general context, provides a partial converse:

Theorem 1.1 (Singer, Podesta-Spiro). *There exists an integer $k_{p,q}$ so that if $\mathcal{M}$ is a complete simply connected pseudo-Riemannian manifold of signature (p,q) which is $k_{p,q}$-curvature homogeneous, then (M,g) is homogeneous.*

Sekigawa, Suga, and Vanhecke [12, 13] showed any 1-curvature homogeneous complete simply connected Riemannian manifold of dimension $m < 5$ is homogeneous; thus $k_{0,2} = k_{0,3} = k_{0,4} = 1$. The estimate $k_{0,m} < \frac{3}{2}m - 1$ was claimed by Gromov [9]. Results of [6] can be used to show $k_{p,q} \geq \min(p,q)$; we conjecture $k_{p,q} = \min(p,q) + 1$.

If $\mathcal{H}$ is a homogeneous space, let $\mathfrak{M}_k(\mathcal{H}) := \mathfrak{M}_k(\mathcal{H}, Q)$ for any point $Q \in H$; the isomorphism class of $\mathfrak{M}_k(\mathcal{H})$ is independent of the point $Q \in H$. We say that $\mathcal{M}$ is *k-modeled on $\mathcal{H}$* and that $\mathfrak{M}_k(\mathcal{H})$ is *a k-model for $\mathcal{M}$* if $\mathfrak{M}_k(\mathcal{H})$ and $\mathfrak{M}_k(\mathcal{M}, P)$ are isomorphic for any $P \in M$.

Throughout this paper, we shall adopt the notational convention that

$$p \geq 1 .$$

In [7], we exhibited complete metrics on $\mathbb{R}^{6+4p}$ of neutral signature $(3+2p, 3+2p)$ which are $(p+2)$-curvature homogeneous, which are 0-modeled on an indecomposible symmetric space, but which are not $(p+3)$-curvature homogeneous; these examples show that the constants $k_{p,q} \to \infty$ as $(p,q) \to \infty$. The proof of Theorem 1.1 rested on a careful analysis of the isometry groups of the model spaces. In this paper, we continue our study of the manifolds introduced in [7] by examining their isometry groups and the isometry groups of their k-models.

We recall the definition of the metrics on $\mathbb{R}^{6+4p}$ which were introduced in [7]. We will be defining a number of tensors in this paper and, in the interests of brevity, we shall only give the non-zero components up to the usual symmetries. Let $x = (x_1, \ldots, x_m)$ be the usual coordinates on $\mathbb{R}^m$. Let

$$\{x, y, z_1, \ldots, z_p, \tilde{y}, \tilde{z}_1, \ldots, \tilde{z}_p, x^*, y^*, z_1^*, \ldots, z_p^*, \tilde{y}^*, \tilde{z}_1^*, \ldots, \tilde{z}_p^*\}$$

be coordinates on $\mathbb{R}^{6+4p}$. Let $F = F(y, z_1, \ldots, z_p) \in C^\infty(\mathbb{R}^{p+1})$. Let

$$\mathcal{M}_{6+4p,F} := (\mathbb{R}^{6+4p}, g_{6+4p,F})$$

where $g_{6+4p,F}$ is the metric of neutral signature $(3+2p, 3+2p)$ on $\mathbb{R}^{6+4p}$ with:

$$\begin{aligned}
g_{6+4p,F}(\partial_x, \partial_x) &= -2\{F(y, z_1, \ldots, z_p) + y\tilde{y} + z_1\tilde{z}_1 \cdots + z_p\tilde{z}_p\}, \\
g_{6+4p,F}(\partial_x, \partial_{x^*}) &= g_{6+4p,F}(\partial_y, \partial_{y^*}) = g_{6+4p,F}(\partial_{\tilde{y}}, \partial_{\tilde{y}^*}) = 1, \\
g_{6+4p,F}(\partial_{z_i}, \partial_{z_i^*}) &= g_{6+4p,F}(\partial_{\tilde{z}_i}, \partial_{\tilde{z}_i^*}) = 1 .
\end{aligned}$$

Theorem 1.2 (Gilkey-Nikčević [7]). *Let $\mathcal{M} = \mathcal{M}_{6+4p,F}$. Then:*

(1) *All geodesics in $\mathcal{M}$ extend for infinite time.*

(2) $\exp_P : T_P\mathbb{R}^{6+4p} \to \mathbb{R}^{6+4p}$ *is a diffeomorphism for all* $P \in \mathbb{R}^{6+4p}$.

(3) $\nabla^k R(\partial_x, \partial_{\xi_1}, \partial_{\xi_2}, \partial_x; \partial_{\xi_3}, \dots, \partial_{\xi_{k+2}}) = -\frac{1}{2}(\partial_{\xi_1} \cdots \partial_{\xi_{k+2}}) g_{6+4p,F}(\partial_x, \partial_x)$ *are the non-zero components of* $\nabla^k R$ *where* $\xi_i \in \{y, z_1, \dots, z_p, \tilde{y}, \tilde{z}_1, \dots, \tilde{z}_p\}$.

(4) *All scalar Weyl invariants of* $\mathcal{M}$ *vanish.*

(5) $\mathcal{M}$ *is a symmetric space if and only if* F *is at most quadratic.*

1.1 The manifolds $\mathcal{M}_{6+4p,k} = (\mathbb{R}^{6+4p}, g_{6+4p,k})$. We can specialize this construction as follows. Let $g_{6+4p,k}$ be defined by setting $F = f_{p,k}$ where we let:

$$f_{p,0}(y, z_1, \dots, z_p) := 0\,,$$
$$f_{p,k}(y, z_1, \dots, z_p) := z_1 y^2 + \cdots + z_k y^{k+1} \quad \text{if} \quad 1 \le k \le p\,.$$

As exceptional cases, we set:

$$f_{p,p+1}(y, z_1, \dots, z_p) := z_1 y^2 + \cdots + z_p y^{p+1} + y^{p+3}\,,$$
$$f_{p,p+2}(y, z_1, \dots, z_p) := z_1 y^2 + \cdots + z_p y^{p+1} + e^y\,.$$

Theorem 1.3 (Gilkey-Nikčević [7]). *Let* $1 \le k \le p+2$.

(1) $\mathcal{M}_{6+4p,0}$ *is an indecomposible symmetric space.*

(2) $\mathcal{M}_{6+4p,k}$ *is an indecomposible homogeneous space which is not symmetric.*

1.2 The manifolds $\mathcal{N}_{6+4p,\psi} = (\mathbb{R}^{6+4p}, g_{6+4p,\psi})$. Let $\psi = \psi(y)$ be a real analytic function of one variable such that

$$\psi^{(p+3)} > 0\,, \quad \psi^{(p+4)} > 0\,, \quad \text{and} \quad \psi^{(p+3)} \ne ae^{by}\,.$$

Define a metric $g_{6+4p,\psi}$ on $\mathbb{R}^{6+4p}$ by taking $F = f_\psi$ where

$$f_\psi(y, z_1, \dots, z_p) := \psi(y) + z_1 y^2 + \cdots + z_p y^{p+1}\,.$$

The following result shows that the geometry of a homogeneous pseudo-Riemannian manifold need not determined by the k-model:

Theorem 1.4 (Gilkey-Nikčević [7]). *Let* $0 \le j < k \le p+2$.

(1) $\mathcal{M}_{6+4p,k}$ *is* j*-modeled on* $\mathcal{M}_{6+4p,j}$*;* $\mathcal{M}_{6+4p,j}$ *is not* k*-modeled on* $\mathcal{M}_{6+4p,k}$.

(2) $\mathcal{N}_{6+4p,\psi}$ *is* $p+2$*-curvature homogeneous and* $p+2$*-modeled on* $\mathcal{M}_{6+4p,p+2}$.

(3) $\mathcal{N}_{6+4p,\psi}$ *is not* $p+3$*-curvature homogeneous and not locally homogeneous.*

1.3 Isometry groups. Let $G(\mathcal{M})$ (resp. $G(\mathfrak{M}_k)$) be the isometry group of a pseudo-Riemannian manifold $\mathcal{M}$ (resp. of a k-model $\mathfrak{M}_k$). In this paper, we study the groups $G(\mathcal{M}_{6+4p,k})$, $G(\mathcal{N}_{6+4p,\psi})$, and $G(\mathfrak{M}_k(\mathcal{M}_{6+4p,k}, P))$ for any point P of $\mathbb{R}^{6+4p}$. A byproduct of our study is the following result that shows, not surprisingly, that the symmetric space $\mathcal{M}_{6+4p,0}$ has the largest isometry group.

Theorem 1.5. *Let* $1 \le k \le p$. *Let* $n_p := (6+4p) + (p+1)(3+2p) + (2p+3)$.

(1) $\dim\{G(\mathcal{M}_{6+4p,0})\} = n_p + (p+1)(2p+1)$.

(2) $\dim\{G(\mathcal{M}_{6+4p,k})\} = n_p + (2p+2) + \frac{1}{2}(2p-k)(2p-k-1)$.

(3) $\dim\{G(\mathcal{M}_{6+4p,p+1})\} = \dim\{G(\mathcal{M}_{6+4p,p})\} - 1$.

(4) $\dim\{G(\mathcal{M}_{6+4p,p+2})\} = \dim\{G(\mathcal{M}_{6+4p,p+1})\} - 1$.

(5) $\dim\{G(\mathcal{N}_{6+4p,\psi})\} = \dim\{G(\mathcal{M}_{6+4p,p+2})\} - 1$.

Here is a brief outline to the remainder of this paper. In Section 2, we review some results from [7]. In Section 3, we reduce the proof of Theorem 1.5 to a purely algebraic problem by showing for any $P \in \mathbb{R}^{6+4p}$ that for $0 \le k \le p+2$, we have:

$$\dim\{G(\mathcal{M}_{6+4p,k})\} = 6 + 4p + \dim\{G(\mathfrak{M}_k(\mathcal{M}_{6+4p,k}, P))\},$$
$$\dim\{G(\mathcal{N}_{6+4p,\psi})\} = 5 + 4p + \dim\{G(\mathfrak{M}_{p+2}(\mathcal{M}_{6+4p,p+2}, P))\}.$$

In Section 4, we complete the proof by determining $\dim\{G(\mathfrak{M}_k(\mathcal{M}_{6+4p,k}, P))\}$ for $0 \le k \le p+2$.

2. Models

It is convenient to work in the purely algebraic setting. Let

$$\mathfrak{M}_\nu := (V, \langle\cdot,\cdot\rangle, A^0, \dots, A^\nu)$$

where $\langle\cdot,\cdot\rangle$ is a non-degenerate inner product of signature (p,q) on a finite dimensional vector space V of dimension $m = p+q$ and where $A^\mu \in \otimes^{4+\mu}V^*$ satisfies the appropriate symmetries of the covariant derivatives of the curvature tensor for $0 \le \mu \le \nu$; if $\nu = \infty$, then the sequence is infinite. We say that $\mathfrak{M}_\nu$ is a *ν-model* for a pseudo-Riemannian manifold $\mathcal{M} = (M,g)$ if for each point $P \in M$, there is an isomorphism $\phi_P : T_PM \to V$ so that

$$\phi_P^*\langle\cdot,\cdot\rangle = g_P \quad\text{and}\quad \phi_P^*A^\mu = \nabla^\mu R_P \quad\text{for}\quad 0 \le \mu \le \nu.$$

Clearly $\mathcal{M}$ is ν-curvature homogeneous if and only if it admits a ν-model.

2.1 Models for the manifolds $\mathcal{M}_{6+4p,k}$ and $\mathcal{N}_{6+4p,\psi}$. Let

$$\mathcal{B} = \{X, Y, Z_1 \dots, Z_p, \tilde{Y}, \tilde{Z}_1, \dots, \tilde{Z}_p, X^*, Y^*, Z_1^*, \dots, Z_p^*, \tilde{Y}^*, \tilde{Z}_1^*, \dots, \tilde{Z}_p^*\}$$

be a basis for $\mathbb{R}^{6+4p}$. Define a hyperbolic inner-product on $\mathbb{R}^{6+4p}$ by pairing ordinary variables with the corresponding dual $\star$-variables:

$$\langle X, X^*\rangle = \langle Y, Y^*\rangle = \langle \tilde{Y}, \tilde{Y}^*\rangle = \langle Z_i, Z_i^*\rangle = \langle \tilde{Z}_i, \tilde{Z}_i^*\rangle = 1. \tag{2.a}$$

Define $A^0 \in \otimes^4(\mathbb{R}^{6+4p})^*$ with non-zero components:

$$A^0(X, Y, \tilde{Y}, X) = A^0(X, Z_i, \tilde{Z}_i, X) = 1.$$

Define tensors $A^i \in \otimes^{4+i}(\mathbb{R}^{6+4p})^*$ for $1 \le i \le p$ with non-zero components:

$$A^i(X, Y, Z_i, X; Y, \dots, Y) = 1,$$
$$A^i(X, Y, Y, X; Z_i, Y, \dots, Y) = 1, \dots,$$
$$A^i(X, Y, Y, X; Y, \dots, Y, Z_i) = 1.$$

Finally define $A^{p+1} \in \otimes^{5+p}(\mathbb{R}^{6+4p})^*$ and $A^{p+2} \in \otimes^{6+p}(\mathbb{R}^{6+4p})^*$ by setting

$$A^{p+1}(X, Y, Y, X; Y, \dots, Y) = 1,$$
$$A^{p+2}(X, Y, Y, X; Y, \dots, Y) = 1.$$

Define models:

$$\mathfrak{M}_{6+4p,k} := (\mathbb{R}^{6+4p}, \langle\cdot,\cdot\rangle, A^0, \ldots, A^k) \quad \text{for} \quad 0 \le k \le p+2\,.$$

Lemma 2.1 (Gilkey-Nikčević [7]). *Let $0 \le k \le p+2$.*

(1) *$\mathfrak{M}_{6+4p,k}$ is a k-model for $\mathcal{M}_{6+4p,k}$.*

(2) *$\mathfrak{M}_{6+4p,p+2}$ is a $p+2$-model for $\mathcal{N}_{6+4p,\psi}$.*

3. Isometry groups in the geometric setting

In this section we will reduce the proof of Theorem 1.5 to a purely algebraic problem by showing:

Theorem 3.1. *Let $0 \le k \le p+2$.*

(1) $\dim\{G(\mathcal{M}_{6+4p,k})\} = 6 + 4p + \dim\{G(\mathfrak{M}_{6+4p,k})\}$.

(2) $\dim\{G(\mathcal{N}_{6+4p,\psi})\} = 5 + 4p + \dim\{G(\mathfrak{M}_{6+4p,p+2})\}$.

The proof of Theorem 3.1 will be based on several Lemmas. In Lemma 3.2, we review a basic result about group actions. In Lemma 3.3, we relate the full isometry group $G(\cdot)$ to the isotropy subgroup. In Lemma 3.4, we relate the isotropy subgroup to the isometry group of the ∞-model. In Lemma 3.5, we relate isometry group of the ∞-model to the isometry group of an appropriate finite model.

The following result is well known.

Lemma 3.2. *Let G be a Lie group which acts continuously on a metric space X. If $x \in X$, let $G \cdot x$ be the orbit and let $G_x = \{g \in G : gx = x\}$ be the isotropy subgroup.*

(1) *We have a smooth principle bundle $G_x \to G \to G \cdot x$.*

(2) $\dim\{G\} = \dim\{G_x\} + \dim\{G \cdot x\}$.

We can relate $\dim\{G(\mathcal{M})\}$ to $\dim\{G_P(\mathcal{M})\}$ for $\mathcal{M} = \mathcal{M}_{6+4p,k}$ or $\mathcal{M} = \mathcal{N}_{6+4p,\psi}$.

Lemma 3.3. *Let $P \in \mathbb{R}^{6+4p}$. Let $0 \le k \le p+2$.*

(1) $\dim\{G(\mathcal{M}_{6+4p,k})\} = 6 + 4p + \dim\{G_P(\mathcal{M}_{6+4p,k})\}$.

(2) $\dim\{G(\mathcal{N}_{6+4p,\psi})\} = 6 + 4p - 1 + \dim\{G_P(\mathcal{N}_{6+4p,\psi})\}$.

Proof. We apply Lemma 3.2 to the canonical action of $G(\mathcal{M})$ on $\mathbb{R}^{6+4p}$. Assertion (1) follows as $\mathcal{M}_{6+4p,k}$ is a homogeneous space. Let $\nu \ge 2$. Set

$$\alpha_{6+4p,\nu}(\psi) := \psi^{(\nu+p+3)}\{\psi^{(p+3)}\}^{\nu-1}\{\psi^{(p+4)}\}^{-\nu}\,.$$

We showed [7] that if $\mathcal{B}$ is a basis satisfying the normalizations of Section 2, then the only non-zero components of $\nabla^{\nu+p+1}R$ are given by:

$$\nabla^{\nu+p+1}R(X, Y, Y, X; Y, \ldots, Y) = \alpha_{6+4p,\nu}(\psi)\,. \tag{3.a}$$

We also showed that the following assertions are equivalent:

(1) $\alpha_{6+4p,\nu}(\psi_1)(P_1) = \alpha_{6+4p,\nu}(\psi_2)(P_2)$ for all $\nu \ge 2$.

(2) There exists an isometry $\phi : \mathcal{N}_{6+4p,\psi_1} \to \mathcal{N}_{6+4p,\psi_2}$ with $\phi(P_1) = P_2$.

The functions $\alpha_{6+4p,\nu}(\psi)$ are constant on the hyperplanes $y = c$; thus the group of isometries acts transitively on such a hyperplane. Consequently

$$\dim\{G(\mathcal{N}_{6+4p,\psi})\} \geq \dim\{G_P(\mathcal{N}_{6+4p,\psi})\} + 6 + 4p - 1\,.$$

Since $\mathcal{N}_{6+4p,\psi}$ is not a homogeneous space, equality holds. □

Let $P \in M$. We can show that $G_P(\mathcal{M})$ is isomorphic to $G(\mathfrak{M}_\infty(\mathcal{M}, P))$ under certain circumstances.

Lemma 3.4.

(1) *Let $\mathcal{M}_1 := (M_1, g_1)$ and $\mathcal{M}_2 := (M_2, g_2)$ be real analytic. Assume for $\varrho = 1, 2$ that there are points $P_\varrho \in M_\varrho$ so $\exp_{P_\varrho} : T_{P_\varrho}M_\varrho \to M_\varrho$ is a diffeomorphism. If $\phi : T_{P_1}M_1 \to T_{P_2}M_2$ induces an isomorphism from $\mathfrak{M}_\infty(\mathcal{M}_1, P_1)$ to $\mathfrak{M}_\infty(\mathcal{M}_2, P_2)$, then $\Phi := \exp_{P_2} \circ \phi \circ \exp_{P_1}^{-1}$ is an isometry from $\mathcal{M}_1$ to $\mathcal{M}_2$.*

(2) *If $\mathcal{M} = \mathcal{M}_{6+4p,k}$ or if $\mathcal{M} = \mathcal{N}_{6+4p,\psi}$, then $G_P(\mathcal{M}) = G\big(\mathfrak{M}_\infty(\mathcal{M}, P)\big)$ for any point $P \in \mathbb{R}^{6+4p}$.*

Proof. An analytic pseudo-Riemannian metric g is uniquely determined, up to local isometry, by the tensors R, ∇R, …, $\nabla^k R$, … at one point, see Belger and Kowalski [1] and Gray [8] for related work. The first assertion now follows; the second follows immediately from the first and from Theorem 1.2. □

We now replace the infinite model by a finite model:

Lemma 3.5. *Let $P \in \mathbb{R}^{6+4p}$. Let $0 \leq k \leq p + 2$. Then:*

(1) $G\big(\mathfrak{M}_\infty(\mathcal{M}_{6+4p,k}, P)\big) = G(\mathfrak{M}_{6+4p,k})$.

(2) $G\big(\mathfrak{M}_\infty(\mathcal{N}_{6+4p,\psi}, P)\big) = G(\mathfrak{M}_{6+4p,p+2})$.

Proof. If $\mathcal{M}$ is a pseudo-Riemannian manifold, restriction induces an injective map

$$r : G\big(\mathfrak{M}_\infty(\mathcal{M}, P)\big) \to G\big(\mathfrak{M}_k(\mathcal{M}, P)\big)\,.$$

Suppose that $\mathcal{M} = \mathcal{M}_{4p+6,k}$ for $k < p+2$. Then $\nabla^j R = 0$ for $j > k$; consequently any isomorphism of the k-model is an isomorphism of the ∞-model; this proves Assertion (1) for $0 \leq k \leq p+1$.

To deal with the remaining cases, we suppose that $\psi^{(p+3)}$ and $\psi^{(p+4)}$ are always positive, but drop the restriction that $\psi^{(p+3)} \neq ae^{by}$. Choose a basis $\mathcal{B}$ for $T_P M$ satisfying the normalizations of Section 2. If $g \in G\big(\mathfrak{M}_{p+2}(\mathcal{M}_{6+4p,p+2}, P)\big)$, then $g\mathcal{B}$ also satisfies the normalizations of Section 2. We may then apply Equation (3.a) to see that g is in fact an isomorphism of the ∞-model since g preserves $\nabla^k R$ for any $k > p + 2$. The first assertion with $k = p + 2$ and the second assertion of the Lemma now follow; this also completes the proof of Theorem 3.1. □

4. Isometry groups of the models

Let $\mathbb{R}^{3+2p} := \operatorname{Span}\{X, Y, Z_1, \ldots, Z_p, \tilde{Y}\tilde{Z}_1, \ldots, \tilde{Z}_p\}$ and let $B^i \in \otimes^{4+i}(\mathbb{R}^{3+2p})^*$ be the restriction of A^i to $\mathbb{R}^{3+2p}$. We introduce the affine models by restricting the domain and suppressing the metric:

$$\mathfrak{A}_{3+2p,k} := \big(\mathbb{R}^{3+2p}, B^0, \ldots, B^k\big)\,.$$

Lemma 4.1. $\dim\{G(\mathfrak{M}_{6+4p,k})\} = \dim\{G(\mathfrak{A}_{3+2p,k})\} + (p+1)(3+2p)$.

Proof. Let $\mathfrak{o}(s)$ be Lie algebra of skew-symmetric $s \times s$ real matrices. Set

$$\begin{aligned} \mathcal{S} &:= (S_1, \dots, S_{3+2p}) = (X, Y, Z_1 \dots, Z_p, \tilde{Y}, \tilde{Z}_1 \dots, \tilde{Z}_p), \\ \mathcal{S}^* &:= (S_1^*, \dots, S_{3+2p}^*) = (X^*, Y^*, Z_1^*, \dots, Z_p^*, \tilde{Y}^*, \tilde{Z}_1^*, \dots, \tilde{Z}_p^*), \\ \mathcal{K} &:= \{\xi \in \mathbb{R}^{6+4p} : A^0(\xi, \eta_1, \eta_2, \eta_3) = 0 \ \forall\ \eta_i \in \mathbb{R}^{6+4p}\} \\ &= \operatorname{Span}\{S_1^*, \dots, S_{3+2p}^*\}. \end{aligned}$$

Let $g \in G(\mathfrak{M}_{6+4p,k})$. The space $\mathcal{K}$ is preserved by g. Thus

$$gS_i = \textstyle\sum_{i,j}\{g_{0,ij}S_j + g_{1,ij}S_j^*\} \quad \text{and} \quad gS_i^* = \textstyle\sum_{i,j}\{g_{2,ij}S_j^*\}.$$

By Equation (2.a), $\langle gS_i, gS_j\rangle = 0$ and $\langle gS_i, gS_j^*\rangle = \delta_{ij}$. Thus

$$\textstyle\sum_k\{g_{0,ik}g_{1,jk} + g_{1,ik}g_{0,jk}\} = 0 \quad \text{and} \quad \textstyle\sum_k\{g_{0,ik}g_{2,jk}\} = \delta_{ij}.$$

for all i, j. Set $\gamma := g_0 g_1^t$. One then has

$$g_0 \in G(\mathfrak{A}_{3+2p,k}), \quad \gamma + \gamma^t = 0, \quad \text{and} \quad g_0 g_2^t = \mathrm{id}. \tag{4.a}$$

Conversely, if Equation (4.a) is satisfied then $g \in G(\mathfrak{M}_{6+4p,k})$. The map $g \to (g_0, \gamma)$ yields an identification of

$$G(\mathfrak{M}_{6+4p,k}) = G(\mathfrak{A}_{3+2p,k}) \times \mathfrak{o}(3+2p)$$

as a twisted product. The Lemma follows as $\dim\{\mathfrak{o}(3+2p)\} = \frac{1}{2}(3+2p)(2+2p)$. $\square$

There is a natural action of $G(\mathfrak{A}_{3+2p,k})$ on $\mathbb{R}^{3+2p}$. We continue our study by relating $G(\mathfrak{A}_{3+2p,k})$ and the isotropy subgroup $G_X(\mathfrak{A}_{3+2p,k})$.

Lemma 4.2.

(1) $\dim\{G(\mathfrak{A}_{3+2p,k})\} = \dim\{G_X(\mathfrak{A}_{3+2p,k})\} + 2p + 3$ *for* $k \le p+1$.

(2) $\dim\{G(\mathfrak{A}_{3+2p,p+2})\} = \dim\{G_X(\mathfrak{A}_{3+2p,p+2})\} + 2p + 2$.

Proof. Lemma 4.2 will follow from Lemma 3.2 and the following relations:

$$\begin{aligned} G(\mathfrak{A}_{3+2p,k})X &= \{\xi \in \mathbb{R}^{3+2p} : \langle \xi, X^*\rangle \neq 0\} \quad \text{if} \quad k \le p+1, \\ G(\mathfrak{A}_{3+2p,p+2})X &= \{\xi \in \mathbb{R}^{3+2p} : \langle \xi, X^*\rangle = \pm 1\}. \end{aligned} \tag{4.b}$$

We first show $\supset$ holds in Equation (4.b). Let $\xi \in \mathbb{R}^{3+2p}$. Assume that

$$a := \langle \xi, X^*\rangle \neq 0.$$

Set $gX = \xi$ and set

$$\begin{aligned} &\varepsilon_0 := (a^2)^{-1/(p+3)}, && gY := \varepsilon_0 Y, && g\tilde{Y} := a^{-2}\varepsilon_0^{-1}\tilde{Y}, \\ &\varepsilon_i := \{a^2\varepsilon_0^{i+1}\}^{-1}, && gZ_i := \varepsilon_i Z_i, && gZ_i^* := \varepsilon_i^{-1}a^{-2}\tilde{Z}_i. \end{aligned}$$

The non-zero components of $\nabla^i R$ for $1 \leq i \leq p+2$ are then given by

$$\begin{aligned}
&R(gX, gY, g\tilde{Y}, gX) = a^2\varepsilon_0 a^{-2}\varepsilon_0^{-1} = 1,\\
&R(gX, gZ_i, g\tilde{Z}_i, gX) = a^2\varepsilon_i\varepsilon_i^{-1}a^{-2} = 1,\\
&\nabla R(gX, gY, gZ_1, gX; gY) = \nabla R(gX, gY, gY, gX; gZ_1) = a^2\varepsilon_0^2\varepsilon_1 = 1, \ldots\\
&\nabla^p R(gX, gY, gZ_p, gX; gY, \ldots, gY) = \nabla^p R(gX, gY, gY, gX; gZ_p, gY, \ldots, gY) = \ldots\\
&\qquad = \nabla^p R(gX, gY, gY, gX; gY, \ldots, gY, gZ_p) = a^2\varepsilon_0^{p+1}\varepsilon_p = 1,\\
&\nabla^{p+1} R(gX, gY, gY, gX; gY, \ldots, gY) = a^2\varepsilon_0^{p+3} = 1,\\
&\nabla^{p+2} R(gX, gY, gY, gX; gY, \ldots, gY) = a^2\varepsilon_0^{p+4} = \varepsilon_0\,.
\end{aligned}$$

Thus $g \in G(\mathfrak{A}_{3+2p,p+1})$. Furthermore, $g \in G(\mathfrak{A}_{3+2p,p+2})$ if $a^2 = 1$. Consequently:

$$\begin{aligned}
&\{\xi \in \mathbb{R}^{3+2p} : \langle \xi, X^*\rangle \neq 0\} \subset G(\mathfrak{A}_{3+2p,k}) \cdot X \quad \text{for} \quad k \leq p+1\,,\\
&\{\xi \in \mathbb{R}^{3+2p} : \langle \xi, X^*\rangle = \pm 1\} \subset G(\mathfrak{A}_{3+2p,p+2}) \cdot X\,.
\end{aligned} \tag{4.c}$$

We must establish the reverse inclusions to complete the proof. Let $\xi \in \mathbb{R}^{3+2p}$. Let $J_\xi(\eta_1, \eta_2) := R(\xi, \eta_1, \eta_2, \xi)$ be the *Jacobi form*. Adopt the Einstein convention and sum over repeated indices to expand

$$\xi = aX + b^i Z_i + \tilde{b}^i \tilde{Z}_i$$

where $a = \langle \xi, X^*\rangle$. We have the following cases

(1) If $a = 0$, then $J_\xi = 0$ on $\operatorname{Span}\{Y, \tilde{Y}, Z_i, \tilde{Z}_i\}$ so $\operatorname{Rank}(J_\xi) \leq 1$.

(2) If $a \neq 0$, then $J_\xi(Y, \tilde{Y}) \neq 0$ so $\operatorname{Rank}(J_\xi) \geq 2$.

If $g \in G(\mathfrak{A}_{3+2p,k})$, then $\operatorname{Rank}\{J_\xi\} = \operatorname{Rank}\{J_{g\xi}\}$. Consequently

$$\langle \xi, X^*\rangle = 0 \Leftrightarrow \operatorname{Rank}(J_\xi) \leq 1 \Leftrightarrow \operatorname{Rank}(J_{g\xi}) \leq 1 \Leftrightarrow \langle g\xi, X^*\rangle = 0$$

Consequently we have

$$\begin{aligned}
&G(\mathfrak{A}_{3+2p,k}) \cdot X \subset \{\xi \in \mathbb{R}^{3+2p} : \langle \xi, X^*\rangle \neq 0\}\,,\\
&G(\mathfrak{A}_{3+2p,k}) \cdot \operatorname{Span}\{Y, Z_i, \tilde{Z}_i\} = \operatorname{Span}\{Y, Z_i, \tilde{Z}_i\}\,.
\end{aligned} \tag{4.d}$$

Suppose $k = p+2$. Since $\operatorname{Rank}(J_Y) = 0$, $\operatorname{Rank}(J_{gY}) = 0$ so $\langle gY, X^*\rangle = 0$. Expand

$$\begin{aligned}
gX &= aX + a_0 Y + \tilde{a}_0 \tilde{Y} + a^i Z_i + \tilde{a}^i \tilde{Z}_i\,,\\
gY &= \qquad\; b^0 Y + \tilde{b}^0 \tilde{Y} + b^i Z_i + \tilde{b}^i \tilde{Z}_i\,.
\end{aligned}$$

Then

$$\begin{aligned}
1 &= \nabla^{p+1} R(gX, gY, gY, gX; gY, \ldots, gY) = a^2(b^0)^{p+3}\,,\\
1 &= \nabla^{p+2} R(gX, gY, gY, gX; gY, \ldots, gY) = a^2(b^0)^{p+4}\,.
\end{aligned}$$

This shows that $a^2 = 1$ and $b^0 = 1$ so

$$\begin{aligned}
&G(\mathfrak{A}_{3+2p,p+2})X \subset \{\xi \in \mathbb{R}^{3+2p} : \langle \xi, X^*\rangle = \pm 1\}\,,\\
&G(\mathfrak{A}_{3+2p,p+2})Y \subset \{\xi \in \mathbb{R}^{3+2p} : \langle \xi, X^*\rangle = 0\,,\ \text{and}\ \langle \xi, Y^*\rangle = 1\}\,.
\end{aligned} \tag{4.e}$$

Equations (4.c), (4.d), and (4.e) now imply Equation (4.b); the Lemma follows. □

We now consider the double isotropy group

$$G_{X,Y}(\mathfrak{A}_{3+2p,k}) = \{g \in G(\mathfrak{A}_{3+2p,k}) : gX = X \text{ and } gY = Y\}\,.$$

Lemma 4.3.

(1) $\dim\{G_X(\mathfrak{A}_{3+2p,0})\} = (p+1)(2p+1)$.

(2) $\dim\{G_X(\mathfrak{A}_{3+2p,k})\} = \dim\{G_{X,Y}(\mathfrak{A}_{3+2p,k})\} + 2p + 2$ *for* $1 \leq k \leq p$.

(3) $\dim\{G_X(\mathfrak{A}_{3+2p,k})\} = \dim\{G_{X,Y}(\mathfrak{A}_{3+2p,k})\} + 2p + 1$ *for* $k = p+1, p+2$.

(4) $G_{X,Y}(\mathfrak{A}_{3+2p,p}) = G_{X,Y}(\mathfrak{A}_{3+2p,p+1}) = G_{X,Y}(\mathfrak{A}_{3+2p,p+2})$.

Proof. As noted above, the Jacobi form $J_X(\cdot,\cdot) = R(X,\cdot,\cdot,X)$ defines a non-singular bilinear form of signature $(p+1,p+1)$ on

$$W := \operatorname{Span}\{Y, Z_1, \ldots, Z_p, \tilde{Y}, \tilde{Z}_1, \ldots, \tilde{Z}_p\} = \{\xi : \operatorname{Rank}(J_\xi) \leq 1\}.$$

Let $O(W, J_X)$ be the associated orthogonal group. If $g \in G_X(\mathfrak{A}_{3+2p,k})$, then we have $gW = W$ by Equation (4.d). Since $gX = X$, we may safely identify g with $g|_W$. Furthermore,

$$J_X(\xi,\eta) = J_{gX}(g\xi, g\eta) = J_X(g\xi, g\eta) \quad \text{so} \quad G_X(\mathfrak{A}_{3+2p,k}) \subset O(W, J_X).$$

Conversely, if g is a linear map of W which preserves J_X, we may extend g to $\mathbb{R}^{3+2p}$ by defining $gX = X$ and thereby obtain an element of $G_X(\mathfrak{A}_{3+2p,0})$. Thus $G_X(\mathfrak{A}_{3+2p,0}) = O(W, J_X)$. Assertion (1) now follows since

$$\dim\{O(W, J_X)\} = \tfrac{1}{2}\dim W(\dim W - 1) = \tfrac{1}{2}(1+2p)(2+2p).$$

Assertions (2) and (3) will follow from Lemma 3.2 and from the relations:

$$\begin{aligned} &G_X(\mathfrak{A}_{3+2p,k}) \cdot Y = \{\xi \in W : \langle \xi, Y^* \rangle \neq 0\} \quad \text{for} \quad 1 \leq k \leq p, \\ &G_X(\mathfrak{A}_{3+2p,p+1}) \cdot Y = \{\xi \in W : \langle \xi, Y^* \rangle^{p+3} = 1\}, \\ &G_X(\mathfrak{A}_{3+2p,p+2}) \cdot Y = \{\xi \in W : \langle \xi, Y^* \rangle = 1\}. \end{aligned} \tag{4.f}$$

If $\xi \in W$, let $S_\xi(\eta) := \nabla R(X, \xi, \xi, X; \eta)$. Expand

$$\xi = b^0 Y + \tilde{b}^0 \tilde{Y} + b^i Z_i + \tilde{b}^i \tilde{Z}_i. \tag{4.g}$$

We then have that

$$\begin{aligned} &S_\xi(X) = 0, \quad S_\xi(\tilde{Z}_i) = 0, \quad S_\xi(Y) = 2b^0 b^1, \\ &S_\xi(Z_1) = (b^0)^2, \quad \text{and} \quad S_\xi(Z_i) = 0 \quad \text{for} \quad i \geq 2. \end{aligned}$$

Thus $S_\xi = 0$ if and only if $b^0 = \langle \xi, Y^* \rangle = 0$. It now follows that for $k \geq 1$ we have

$$\begin{aligned} &G_X(\mathfrak{A}_{3+2p,k})Y \subset \{\xi \in W : \langle \xi, Y^* \rangle \neq 0\}, \\ &G_X(\mathfrak{A}_{3+2p,k}) \operatorname{Span}\{Z_i, \tilde{Y}, \tilde{Z}_i\} \subset \operatorname{Span}\{Z_i, \tilde{Y}, \tilde{Z}_i\}. \end{aligned} \tag{4.h}$$

Since $a = 1$, the analysis used to prove Lemma 4.2 shows $(b^0)^{p+3} = 1$ if $k = p+1$ and $b^0 = 1$ if $k = p+2$. This establishes the inclusions $\subset$ in Equation (4.f).

We complete the proof by establishing the reverse inclusions in Equation (4.f). Expand ξ in the form given in Equation (4.g). Assume $b^0 \neq 0$. Let $gX = X$, $gY = \xi$, $g\tilde{Y} = (b^0)^{-1}\tilde{Y}$,

$$gZ_i := \varepsilon_i\{Z_i - (b^0)^{-1}\tilde{b}^i\tilde{Y}\} \quad \text{and} \quad g\tilde{Z}_i := \varepsilon_i^{-1}\{\tilde{Z}_i - (b^0)^{-1}b^i\tilde{Y}\}.$$

The possibly non-zero components of R are then given by

$$\begin{aligned}
&R(gX, gY, g\tilde{Y}, gX) = 1\,,\\
&R(gX, gY, gZ_i, gX) = \varepsilon_i\{\tilde{b}^i - (b^0)(b^0)^{-1}\tilde{b}^i\} = 0\,,\\
&R(gX, gY, g\tilde{Z}_i, gX) = \varepsilon_i^{-1}\{b^i - (b^0)(b^0)^{-1}b^i\} = 0\,,\\
&R(gX, gZ_i, g\tilde{Z}_i, gX) = \varepsilon_i^{-1}\varepsilon_i = 1\,.
\end{aligned}$$

The non-zero components of $\nabla^i R$ for $1 \leq i \leq p$ are given by

$$\begin{aligned}
&\nabla^i R(gX, gY, gZ_i, gX; gY, \dots, gY) = \dots\\
&\quad = \nabla^i R(gX, gY, gY, gX; gY, \dots, gZ_i) = (b^0)^{i+1}\varepsilon_i\,.
\end{aligned}$$

We therefore set $\varepsilon_i = (b^0)^{-i-1}$ for $1 \leq i \leq p$ to ensure $g \in G(\mathfrak{A}_{3+2p,p})$.

The non-zero components of $\nabla^i R$ for $i = p+1, p+2$ are

$$\nabla^i R(gX, gY, gY, gX; gY, \dots, gY) = (b^0)^{i+2}\,.$$

If $(b^0)^{p+3} = 1$, then $g \in G(\mathfrak{A}_{3+2p,p+1})$; if $b^0 = 1$, then $g \in G(\mathfrak{A}_{3+2p,p+2})$. This establishes the reverse inclusions in Equation (4.f) and completes the proof of Assertions (2) and (3); Assertion (4) is immediate. □

Let $W(p) := \text{Span}\{Z_1, \dots, Z_p, \tilde{Z}_1, \dots, \tilde{Z}_p\}$. Let $\{\beta_1, \dots, \beta_p, \tilde{\beta}_1, \dots, \tilde{\beta}_p\}$ be the corresponding dual basis for the dual space $\mathcal{W}(p) := W(p)^*$. The curvature tensor $R(X, \cdot, \cdot, X)$ defines a non-degenerate form $\langle\cdot,\cdot\rangle$ on $W(p)$; dually on $\mathcal{W}(p)$ we have:

$$\langle\beta_i, \beta_j\rangle = \langle\tilde{\beta}_i, \tilde{\beta}_j\rangle = 0, \quad \langle\beta_i, \tilde{\beta}_j\rangle = \delta_{ij}\,.$$

Let $\mathcal{O}(p)$ be the associated orthogonal group on $\mathcal{W}(p)$. Let

$$\mathcal{O}(p,k) := \{h \in \mathcal{O}(p) : h\beta_i = \beta_i \quad \text{for} \quad 1 \leq i \leq k\}$$

be the simultaneous isotropy group. We set $\mathcal{O}(p,0) = \mathcal{O}(p)$. Theorem 1.5 will now follow from the following result:

Lemma 4.4. *Let* $1 \leq k \leq p$.

(1) $G_{X,Y}(\mathfrak{A}_{3+2p,k}) = \mathcal{O}(p,k)$.
(2) $\mathcal{O}_{\tilde{\beta}_1}(p,k) = \mathcal{O}(p-1,k-1)$.
(3) $\dim\{\mathcal{O}(p,k)\} = \dim\{\mathcal{O}(p-1,k-1)\} + 2p - k - 1$.
(4) $\dim\{\mathcal{O}(p,k)\} = \frac{1}{2}(2p-k)(2p-k-1)$.

Proof. Let $g \in G_{X,Y}(\mathfrak{A}_{3+2p,k})$. Let $\xi \in \text{Span}\{Z_1, \dots, Z_p, \tilde{Y}, \tilde{Z}_1, \dots, \tilde{Z}_p\}$. We may use Equation (4.h) and the relation $R(X, Y, g\xi, X) = R(X, Y, \xi, X)$, to see

$$g\tilde{Y} = \tilde{Y} + a^i Z_i + a^{\bar{i}}\tilde{Z}_i\,, \quad gZ_i = a_i^j Z_j + a_i^{\bar{j}}\tilde{Z}_{\bar{j}}\,, \quad g\tilde{Z}_{\bar{i}} = a_{\bar{i}}^j Z_j + a_{\bar{i}}^{\bar{j}}\tilde{Z}_{\bar{j}}\,.$$

Consequently $\text{Span}_{1\leq i\leq p}\{gZ_i, g\tilde{Z}_{\bar{i}}\} = \text{Span}_{1\leq i\leq p}\{Z_i, \tilde{Z}_{\bar{i}}\}$ and the relation

$$R(X, gZ_i, g\tilde{Y}, X) = R(X, g\tilde{Z}_{\bar{i}}, g\tilde{Y}, X) = 0$$

implies $a^i = a^{\bar{i}} = 0$. Thus $g\tilde{Y} = \tilde{Y}$ and $g : W(p) \to W(p)$; this shows that g is determined by its restriction to $W(p)$. Let $h := {}^*g$ denote the dual action of g on

$\mathcal{W}(p)$. The isomorphism of Assertion (1) now follows as:

$$R(X, g\xi_1, g\xi_2, R) = R(X, \xi_1, \xi_2, X)\ \forall \xi_1, \xi_2 \Leftrightarrow h \in \mathcal{O}(p),$$
$$\nabla^i R(X, Y, g\xi, X; Y, \dots, Y) = \nabla^i R(X, Y, \xi, X; Y, \dots, Y)\ \forall \xi \Leftrightarrow h\beta_i = \beta_i .$$

If $h(\beta_1) = \beta_1$ and $h(\tilde\beta_1) = \tilde\beta_1$, then h preserves

$$\operatorname{Span}\{\beta_1, \tilde\beta_1\}^\perp = \operatorname{Span}\{\beta_2, \dots, \beta_p, \tilde\beta_2, \dots, \tilde\beta_p\}.$$

The isomorphism of Assertion (2) now follows by restricting h to this subspace and by renumbering the variables appropriately.

We set

$$\mathcal{W}(p,k) := \{\xi \in \mathcal{W}(p) : \langle \xi, \xi\rangle = 0\,,\ \langle \xi, \beta_1\rangle = 1\,,\ \langle \xi, \beta_i\rangle = 0 \quad \text{for} \quad 2 \le i \le k\}.$$

If $h \in \mathcal{O}(p,k)$, then h preserves $\langle \cdot, \cdot \rangle$ and h preserves $\{\beta_1, \dots, \beta_k\}$. Consequently $h\tilde\beta_1 \in \mathcal{W}(p,k)$ as $\tilde\beta_1$ satisfies these relations. Conversely, $\xi \in \mathcal{W}(p,k)$ if and only if

$$\xi = b^1\beta_1 + \sum_{1<i} b^i\beta_i + \tilde\beta_1 + \sum_{k<i} \tilde b^i \tilde\beta_i \quad \text{where} \quad b^1 + \sum_{k<i} b^i \tilde b^i = 0\,.$$

Since the variables $\{b^2, \dots, b^p, \tilde b^{k+1}, \dots, \tilde b^p\}$ can be chosen arbitrarily,

$$\mathcal{W}(p,k) = \mathbb{R}^{p-1+p-k} \quad \text{so} \quad \dim \mathcal{W}(p,k) = 2p - k - 1\,.$$

We show that $\xi \in \mathcal{O}(p,k)\tilde\beta_1$ by finding $h \in \mathcal{O}(p,k)$ so $h\tilde\beta_1 = \xi$. Set:

$$\begin{aligned} &h\beta_i = \beta_i \quad \text{for } 1 \le i \le k\,, && h\beta_i = \beta_i - \tilde b^i \beta_1 \quad \text{for } k < i\,,\\ &h\tilde\beta_1 = \xi\,, && h\tilde\beta_i = \tilde\beta_i - b^i\beta_1 \quad \text{for } 1 < i\,. \end{aligned}$$

This shows $\mathcal{O}(p,k) \cdot \tilde\beta_1 = \mathcal{W}(p,k)$. Assertion (3) now follows from Assertion (2) and from Lemma 3.2.

As $\dim\{\mathcal{O}(p-k)\} = \frac{1}{2}(2p-2k)(2p-2k-1)$, Assertion (4) follows by induction. □

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References

[1] Belger, M. and Kowalski, O., *Riemannian metrics with the prescribed curvature tensor and all its covariant derivatives at one point*, Math. Nachr. **168** (1994), 209–225.

[2] Bueken, P. and Djorić, M., *Three-dimensional Lorentz metrics and curvature homogeneity of order one*, Ann. Global Anal. Geom. **18** (2000), 85–103.

[3] Bueken, P. and Vanhecke, L., *Examples of curvature homogeneous Lorentz metrics*, Classical Quantum Gravity **14** (1997), L93–L96.

[4] Cahen, M., Leroy, J., Parker, M., Tricerri, F. and Vanhecke, L., *Lorentz manifolds modelled on a Lorentz symmetric space*, J. Geom. Phys. **7** (1990), 571-581.

[5] Ferus, D., Karcher, H. and Münzner, H., *Cliffordalgebren und neue isoparametrische Hyperflächen*, Math. Z. **177** (1981), 479–502.

[6] Gilkey, P. and Nikčević, S., *Complete k-curvature homogeneous pseudo-Riemannian manifolds*, Ann. Global Anal. Geom. **27** (2005), 87–100.

[7] Gilkey, P. and Nikčević, S., *Complete k-curvature homogeneous pseudo-Riemannian manifolds 0-modeled on an indecomposible symmetric space*, Topics in Almost Hermitian Geometry and the Related Fields, World Scientific (2005), 87–104.

[8] Gray, A., *The volume of a small geodesic ball of a Riemannian manifold*, Michigan Math. J. **20** (1973), 329–344.

[9] Gromov, M., *Partial differential relations*, Ergeb. Math. Grenzgeb 3. Folge, Band 9, Springer-Verlag (1986).

[10] Podesta, F. and Spiro, A., *Introduzione ai Gruppi di Trasformazioni*, Volume of the Preprint Series of the Mathematics Department "V. Volterra" of the University of Ancona, Via delle Brecce Bianche, Ancona, Italy (1996).

[11] Singer, I. M., *Infinitesimally homogeneous spaces*, Comm. Pure Appl. Math. **13** (1960), 685–697.

[12] Sekigawa, K., Suga, H. and Vanhecke, L., *Four-dimensional curvature homogeneous spaces*, Comment. Math. Univ. Carolin. **33** (1992), 261–268.

[13] Sekigawa, K., Suga, H. and Vanhecke, L., *Curvature homogeneity for four-dimensional manifolds*, J. Korean Math. Soc. **32** (1995), 93–101.

[14] Takagi, H., *On curvature homogeneity of Riemannian manifolds*, Tôhoku Math. J. **26** (1974), 581–585.

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