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Transferring A_∞ (strongly homotopy associative) structures

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TRANSFERRING A_∞ (STRONGLY HOMOTOPY ASSOCIATIVE) STRUCTURES

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ABSTRACT. The aim of this simple-minded "applied" note is to give explicit formulas for transfers of A_∞-structures and related maps and homotopies in the most easy situation in which these transfers exist. The existence of these transfers follows, in characteristic zero, from a general theory developed by the author in [5]. The easier half of our formulas was already known to Kontsevich-Soibelman and Merkulov [2, 9] who derived them, without explicit signs, under slightly stronger assumptions than those made in this note.

1. INTRODUCTION AND RESULTS

We will work in the category of (left) modules over an arbitrary commutative unital ring R. Therefore, by a chain complex we will understand a chain complex of R-modules, by a linear map an R-linear map, etc. In particular, results of this paper apply to the category of abelian groups and to the category of vector spaces over a field of arbitrary characteristic. Let us consider the following situation.

Situation 1. We are given chain complexes (V, ∂_V), (W, ∂_W) and chain maps $f : (V, \partial_V) \to (W, \partial_W)$, $g : (W, \partial_W) \to (V, \partial_V)$ such that the composition gf is chain homotopic to the identity $\mathbb{1}_V : V \to V$, via a chain-homotopy h.

A compact way to express Situation 1 is to say that $g : (W, \partial_W) \to (V, \partial_V)$ is a left chain-homotopy inverse of $f : (V, \partial_V) \to (W, \partial_W)$. Our assumptions are in particular satisfied when the complexes (V, ∂_V) and (W, ∂_W) are chain homotopy equivalent. In this note we address the following

Problem 2. Suppose we are given an A_∞-structure $\boldsymbol{\mu} = (\mu_2, \mu_3, \ldots)$ on (V, ∂_V). In Situation 1, give explicit formulas for the following objects:

(i) an A_∞-structure $\boldsymbol{\nu} = (\nu_2, \nu_3, \ldots)$ on (W, ∂_W),
(ii) an A_∞-map $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \ldots) : (V, \partial, \mu_2, \mu_3, \ldots) \to (W, \partial, \nu_2, \nu_3, \ldots)$,
(iii) an A_∞-map $\boldsymbol{\psi} = (\psi_1, \psi_2, \ldots) : (W, \partial, \nu_2, \nu_3, \ldots) \to (V, \partial, \mu_2, \mu_3, \ldots)$, and

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(iv) an A_∞-homotopy $\boldsymbol{H} = (H_1, H_2, \ldots)$ between $\boldsymbol{\psi\varphi}$ and $1\!\!1_V$

such that $\boldsymbol{\varphi}$ extends f, $\boldsymbol{\psi}$ extends g and $\boldsymbol{H}$ extends h or, expressed more formally, $\varphi_1 = f$, $\psi_1 = g$ and $H_1 = h$.

Our strategy will be to construct suitable degree $n-2$ maps $\{\boldsymbol{p}_n : V^{\otimes n} \to V\}_{n\geq 2}$ (the *p-kernels*) and suitable degree $n-1$ maps $\{\boldsymbol{q}_n : V^{\otimes n} \to V\}_{n\geq 1}$ (the *q-kernels*) such that ν_n, φ_n, ψ_n and H_n defined by the following Anzatz:

(1) $\nu_n := f \circ \boldsymbol{p}_n \circ g^{\otimes n}, \quad \varphi_n := f \circ \boldsymbol{q}_n, \quad \psi_n := h \circ \boldsymbol{p}_n \circ g^{\otimes n}$ and $H_n := h \circ \boldsymbol{q}_n$

answer Problem 2. We give both inductive (formulas (7) and (8) in Section 3) and non-inductive (Propositions 6 and 7 of Section 4) formulas for the kernels.

Remark 3. We already mentioned in the Abstract that the formulas for ν_n and ψ_n were given, without explicit signs, in [2] (non-inductive formulas) and also in [9] (inductive formulas). Kontsevich and Soibelman [2] assumed (in our notation) that (W, ∂_W) was a subcomplex of (V, ∂_V), $f : (V, \partial_V) \to (W, \partial_W)$ a projection, $g : (W, \partial_W) \hookrightarrow (V, \partial_V)$ the inclusion and, of course, that gf was chain homotopic to the identity $1\!\!1_V$. Merkulov [9] made similar assumptions and he moreover assumed that $(V, \partial_V, \boldsymbol{\mu})$ was an ordinary dg-associative algebra, that is, $\mu_n = 0$ for $n \geq 3$. Our formulas for φ_n and H_n are, to our best knowledge, new ones. A surprising interpretation of the p-kernel in terms of homotopy operads is suggested by [3].

Remark 4. In principle, the transfer demanded in Problem 2 could also be obtained by applying the Coalgebra Perturbation Lemma of Huebschmann and Kadeishvili [1, 2.1$_*$] to the induced maps $T^c(\downarrow f) : T^c(\downarrow V) \to T^c(\downarrow W)$ and $T^c(\downarrow g) : T^c(\downarrow W) \to T^c(\downarrow V)$. But to apply this lemma, one needs to assume that $fg = 1\!\!1_W$ and, moreover, also the *annihilation properties* (also called the *side conditions*, see [7] for an analysis of these conditions)

$$f \circ h = 0, \quad h \circ g = 0 \quad \text{and} \quad h \circ h = 0\,!$$

The formulas of [1] in fact also use the kernels, though the authors did not make this concept explicit. The rôle of the p-kernel is played by the summation $\sum_{n\geq 0}(\tilde{h}\circ\delta_\mu)^n$ and the q-kernel is represented by $\sum_{n\geq 0}(\delta_\mu \circ \tilde{h})^n$, where δ_μ is the square-zero coderivation of $T^c(\downarrow V)$ corresponding to the A_∞-structure $\boldsymbol{\mu}$ and $\tilde{h}$ is the extension of h as a coderivation homotopy, see [1, Perturbation Lemma 1.1]. It can be shown that, under the conditions formulated in the previous paragraph, these kernels coincide with the kernels used in the present paper. Without these conditions, the formulas of [1] are wrong.

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This work was stimulated by E. Getzler who indicated that there might be some need for explicit transfers. The A_∞-case discussed here in fact turned out to be more elementary than we expected, which we attribute to the existence of a canonical non-Σ polarization [6, Remark 25].

Acknowledgment. We are indebted to our wife Květoslava for sketching out the carp's head after Figure 4.

2. Conventions

In this unbelievably boring section we set up sign conventions used in this note. The signs in the axioms of A_∞-algebras and related objects are unique up to an action of the infinite product $\times_1^\infty C_2$ of the cyclic group $C_2 = \{-1, 1\}$. For example, $(\epsilon_2, \epsilon_3, \dots) \in C_2 \times C_2 \times \cdots$ acts on the signs in Axiom (2) below by

$$(\mu_2, \mu_3, \dots) \longmapsto (\epsilon_2\mu_2, \epsilon_3\mu_3, \dots).$$

The sign convention used here is compatible with the one of [4]. It differs from the original one of Jim Stasheff [10] by the action, in Axiom (2), of $(\epsilon_2, \epsilon_3, \dots)$ with $\epsilon_n = (-1)^{n(n-1)/2} = \uparrow^{\otimes n} \circ \downarrow^{\otimes n}$, where $\uparrow$ (resp. $\downarrow$) denotes the suspension (resp. desuspension) operator.

We are going to recall axioms for an A_∞-structure $\boldsymbol{\mu} = (\mu_2, \mu_3, \cdots)$ on (V, ∂_V) (Axiom (2)), an A_∞-structure $(\nu_2, \nu_3, \dots)$ on (W, ∂_W) (Axiom (3)), for an A_∞ map $\boldsymbol{\varphi} : (V, \partial_V, \boldsymbol{\mu}) \to (W, \partial_W, \boldsymbol{\nu})$ (Axiom (4)), for an A_∞ map $\boldsymbol{\psi} : (W, \partial_W, \nu) \to (V, \partial_V, \boldsymbol{\mu})$ (Axiom (5)) and for an A_∞-homotopy $\boldsymbol{H}$ between the composition $\boldsymbol{\psi\varphi}$ and $1\!\!1_V$ (Axiom (6)). In Axioms (2) and (3), $\mu_n : V^{\otimes n} \to V$ and $\nu_n : W^{\otimes n} \to W$ are n-multilinear degree $n-2$ maps, in Axioms (4) and (5), $\varphi_n : V^{\otimes n} \to W$ and $\psi_n : W^{\otimes n} \to V$ are n-multilinear maps of degree $n-1$, and finally in Axiom (6), $H_n : V^{\otimes n} \to V$ is an n-multilinear degree n map. Here are the axioms in their full glory:

$$\delta(\mu_n) := \sum_A (-1)^{i(l+1)+n} \mu_k(1\!\!1^{\otimes i-1} \otimes \mu_l \otimes 1\!\!1^{\otimes k-i}),\ n \geq 2, \tag{2}$$

$$\delta(\nu_n) := \sum_A (-1)^{i(l+1)+n} \nu_k(1\!\!1^{\otimes i-1} \otimes \nu_l \otimes 1\!\!1^{\otimes k-i}),\ n \geq 2, \tag{3}$$

$$\begin{aligned}\delta(\varphi_n) := &-\sum_B (-1)^{\vartheta(r_1,\dots,r_k)} \nu_k(\varphi_{r_1} \otimes \cdots \otimes \varphi_{r_k}) + \\ &-\sum_A (-1)^{i(l+1)+n} \varphi_k(1\!\!1^{\otimes i-1} \otimes \mu_l \otimes 1\!\!1^{\otimes k-i}),\ n \geq 1,\end{aligned} \tag{4}$$

$$\begin{aligned}\delta(\psi_n) := &-\sum_B (-1)^{\vartheta(r_1,\dots,r_k)} \mu_k(\psi_{r_1} \otimes \cdots \otimes \psi_{r_k}) + \\ &-\sum_A (-1)^{i(l+1)+n} \psi_k(1\!\!1^{\otimes i-1} \otimes \nu_l \otimes 1\!\!1^{\otimes k-i}),\ n \geq 1, \text{ and}\end{aligned} \tag{5}$$

$$\begin{aligned}\delta(H_n) := &-\sum_C (-1)^{n+r_i+\vartheta(r_1,\dots,r_i)} \mu_k((\boldsymbol{\psi\varphi})_{r_1} \otimes \cdots \otimes (\boldsymbol{\psi\varphi})_{r_{i-1}} \otimes H_{r_i} \otimes 1\!\!1^{\otimes k-i}) \\ &+\sum_A (-1)^{n+i(l+1)} H_k(1\!\!1^{\otimes i-1} \otimes \mu_l \otimes 1\!\!1^{\otimes k-i}) + (\boldsymbol{\psi\varphi})_n - (1\!\!1_V)_n,\ n \geq 1.\end{aligned} \tag{6}$$

In the above display,

$A := \{k, l \mid k + l = n + 1,\ k, l \geq 2,\ 1 \leq i \leq k\}$,
$B := \{k, r_1, \dots, r_k \mid 2 \leq k \leq n,\ r_1, \dots, r_k \geq 1,\ r_1 + \cdots + r_k = n\}$,
$C := \{k, i, r_1, \dots, r_i \mid 2 \leq k \leq n,\ 1 \leq i \leq k,\ r_1, \dots, r_i \geq 1,\ r_1 + \cdots + r_i + k - i = n\}$,

and, for integers $u_1, \ldots, u_s$, we denoted

$$\vartheta(u_1, \ldots, u_s) := \sum_{1 \le \alpha < \beta \le s} u_\alpha(u_\beta + 1).$$

The symbols δ in the left hand sides denote the induced differentials in the corresponding complex of multilinear maps. To interpret the above axioms in terms of elements, one must of course use the Koszul sign convention. For example, Axiom (2) evaluated at elements $v_1, \ldots, v_n \in V$, reads

$$\partial_V \mu_n(v_1, \ldots, v_n) - \sum_{1 \le i \le n} (-1)^{n+|v_1|+\cdots+|v_{i-1}|} \mu_n(v_1, \ldots, v_{i-1}, \partial_V(v_i), v_{i+1}, \ldots, v_n)$$
$$:= \sum_A (-1)^{i(l+1)+n+l(|v_1|+\cdots+|v_{i-1}|)} \mu_k(v_1, \ldots, v_{i-1}, \mu_l(v_i, \ldots, v_{i+l-1}), v_{i+l}, \ldots, v_n),$$

which is [4, Equation (1)]. If $T^c(-)$ denotes the tensor coalgebra functor, then

- $\boldsymbol{\mu}$ is the same as a degree -1 square-zero coderivation δ_μ of $T^c(\downarrow V)$ whose linear part is ∂_V,
- $\boldsymbol{\nu}$ is the same as a degree -1 square-zero coderivation δ_ν of $T^c(\downarrow W)$ with linear part ∂_W,
- $\boldsymbol{\varphi}$ is the same as a dg-algebra homomorphism $F : (T^c(\downarrow V), \delta_\mu) \to (T^c(\downarrow W), \delta_\nu)$,
- $\boldsymbol{\psi}$ is the same as a dg-algebra homomorphism $G : (T^c(\downarrow W), \delta_\nu) \to (T^c(\downarrow V), \delta_\mu)$, and
- $\boldsymbol{H}$ is the same as a coderivation homotopy between GF and the identity map of $T^c(\downarrow V)$.

3. Inductive formulas

In this section we give inductive formulas for the kernels. Let us start with the p-kernel. We set $\boldsymbol{p}_2 := \mu_2$ and

$$\boldsymbol{p}_n := \sum_B (-1)^{\vartheta(r_1, \ldots, r_k)} \mu_k(h \circ \boldsymbol{p}_{r_1} \otimes \cdots \otimes h \circ \boldsymbol{p}_{r_k}) \tag{7}$$

with the formal convention that $h\boldsymbol{p}_1 = 1\!\!1$. For our inductive definition of the q-kernel we need the following notation:

$$\boldsymbol{p}_n^i = \sum_D (-1)^{\vartheta(r_1, \ldots, r_{i-1})} \mu_k(h \circ \boldsymbol{p}_{r_1} \otimes \cdots \otimes h \circ \boldsymbol{p}_{r_{i-1}} \otimes 1\!\!1^{\otimes n-i+1})$$

where

$$D := \{k, r_1, \ldots, r_{i-1} \mid 2 \le k \le n, \ \ i \le k, \ r_1, \ldots, r_{i-1} \ge 1,$$
$$r_1 + \cdots + r_{i-1} + k - i + 1 = n\},$$

i is a fixed integer, $1 \le i \le n$, and where we again put $h\boldsymbol{p}_1 = 1\!\!1_V$. We then define $\boldsymbol{q}_1 := 1\!\!1_V$ and, inductively

$$\boldsymbol{q}_n = \sum_C (-1)^{n+r_i+\vartheta(r_1, \ldots, r_i)} \boldsymbol{p}_k^i(gf \circ \boldsymbol{q}_{r_1} \otimes \cdots \otimes gf \circ \boldsymbol{q}_{r_{i-1}} \otimes h \circ \boldsymbol{q}_{r_i} \otimes 1\!\!1^{\otimes k-i}). \tag{8}$$

The first result of this note is:

Theorem 5. *Let $\{\boldsymbol{p}_n\}_{n\ge 2}$ and $\{\boldsymbol{q}_n\}_{n\ge 1}$ be defined inductively by* (7) *and* (8). *Then ν_n, φ_n, ψ_n and H_n determined by these $\boldsymbol{p}_n$ and $\boldsymbol{q}_n$ as in formula* (1) *solve Problem 2.*

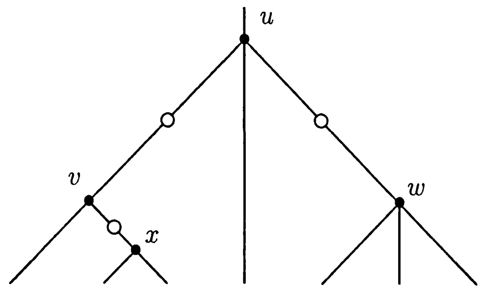


FIGURE 1. An element of P_7.

Proof. A straightforward but awfully technical induction shows that the kernels satisfy:

$$\delta(\boldsymbol{p}_n) = \sum_A (-1)^{i(l+1)+n} \boldsymbol{p}_k(\mathbb{1}^{\otimes i-1} \otimes gf \circ \boldsymbol{p}_l \otimes \mathbb{1}^{\otimes k-i}), \quad n \geq 2,$$

and

$$\delta(\boldsymbol{q}_n) = -\sum_B (-1)^{\vartheta(r_1,\ldots,r_k)} \boldsymbol{p}_k(gf \circ \boldsymbol{q}_{r_1} \otimes \cdots \otimes gf \circ \boldsymbol{q}_{r_k}) + \\ - \sum_A (-1)^{i(l+1)+n} \boldsymbol{q}_k(\mathbb{1}^{\otimes i-1} \otimes \mu_l \otimes \mathbb{1}^{\otimes k-i}), \quad n \geq 1.$$

It is then almost obvious that the above two equations imply Axioms (2)–(6) for ν_n, φ_n, ψ_n and H_n defined by (1). □

4. NON-INDUCTIVE FORMULAS

In this section we give non-inductive formulas for the kernels. Our formulas will be based on the language of trees which we use as names for maps and their compositions. Formally this means that we work in a certain free operad, but we are not going to use this fancy language here. The terminology of trees is recalled in Section II.1.5 of [8].

Let P_n denote the set of planar directed trees with at least binary vertices (that is, all vertices have at least two incoming edges), with interior edges decorated by the symbol ⦶, and n leaves. An example of such a tree is given in Figure 1. To each decorated tree $T \in \mathsf{P}_n$ we assign a map $F_T : V^{\otimes n} \to V$, by interpreting T as a "flow chart," with ⦶ denoting the homotopy $h : V \to V$ and a vertex of arity (= the number of incoming edges) k denoting the map $\mu_k : V^{\otimes k} \to V$. For example, the tree T in Figure 1 gives the degree 5 map

$$F_T = \mu_3(h \circ \mu_2(\mathbb{1}_V \otimes h \circ \mu_2) \otimes \mathbb{1}_V \otimes h \circ \mu_3) : V^{\otimes 7} \to V$$

which, evaluated at $(a,b,c,d,e,f,g) \in V^{\otimes 7}$, equals

$$F_T(a,b,c,d,e,f,g) = (-1)^{|a|}\mu_3(h \circ \mu_2(a, h \circ \mu_2(b,c)), d, h \circ \mu_3(e,f,g)).$$

Finally, we assign to each tree $T \in \mathsf{P}_n$ the sign $\vartheta(T)$ as follows. For a vertex $v \in \mathit{Vert}(T)$ of arity k and $1 \leq i \leq k$, let r_i be the number of legs (= leaves) e of T

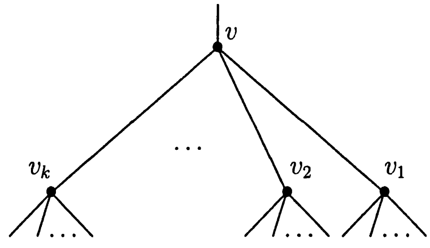


FIGURE 2. A subtree of S used in the definition of the total order $<$.

such that the unique path from e to the root of T contains the i-th input edge of v. We then define $\vartheta_T(v) := \vartheta(r_1, \ldots, r_k)$ and $\vartheta(T) := \sum_{v \in Vert(T)} \vartheta_T(v)$.

For example, for the tree T in Figure 1 we have, at the vertex u of arity 3, $r_1 = 3$, $r_2 = 1$, $r_3 = 3$, at the vertex v of arity 2, $r_1 = 1$, $r_2 = 2$, at the vertex w of arity 3, $r_1 = r_2 = r_3 = 1$ and, at the vertex x of arity 2, $r_1 = r_2 = 1$. Therefore, modulo 2, $\vartheta_T(u) = 3\cdot 2 + 4\cdot 4 = 0$, $\vartheta_T(v) = 1\cdot 3 = 1$, $\vartheta_T(w) = 1\cdot 2 + 2\cdot 2 = 0$ and $\vartheta_T(x) = 1\cdot 2 = 0$, which gives, again mod 2, $\vartheta(T) = 1$. We may finally formulate the following almost obvious:

Proposition 6. *The p-kernel $p_n : V^{\otimes n} \to V$, defined inductively by* (7), *can also be defined as*

$$p_n := \sum_{T \in \mathrm{P}_n} (-1)^{\vartheta(T)} \cdot F_T, \quad \textit{for each } n \geq 2.$$

Let us proceed to our non-inductive definition of the q-kernel based on a slightly more elaborate definition of a decoration of a planar tree. We need to observe first that each planar tree S admits a natural total order of its set of vertices $Vert(S)$ determined in the following way.

We say that a vertex u is *below* a vertex v if v lies on the (unique) directed path joining u with the root. This defines a *partial* order on the set of vertices of S. It is easy to see that there exists precisely one *total* order $<$ on the set $Vert(S)$ which satisfies the following two conditions:

(i) If u is below v, then $u < v$.
(ii) Suppose S contains a subtree of the form shown in Figure 2 and $1 \leq i \leq k-1$. Then v_i and all vertices below v_i are less, in the order $<$, than v_{i+1}.

See Figure 3 for an example of such an order.

The next step is to redraw the tree in such a way that the vertices are placed into different levels, according to their order, and then draw horizontal lines slightly below the vertices, as illustrated in Figure 4. Now we decorate some (not necessary all) of the intersections of the horizontal lines with the edges of the tree with symbols $\circ$ or $\bullet$, according to the following rules:

(i) Let $x_1, \ldots, x_k$ be the points at which a horizontal line intersects the edges of S, numbered from left to right. Then there is some $0 \leq s \leq k-1$ such that the points $x_1, \ldots, x_s$ are decorated by $\bullet$, x_{s+1} is decorated by $\circ$ and the points $x_{s+2}, \ldots, x_k$ are not decorated.
(ii) Each edge of S is decorated at most once.

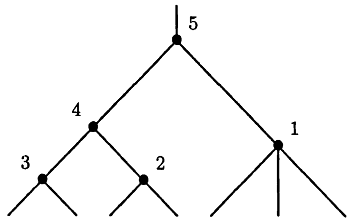


FIGURE 3. Ordering vertices of a planar tree. The vertices are numbered, from the biggest to the smallest one.

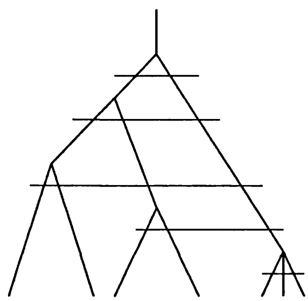


FIGURE 4. Drawing horizontal lines.

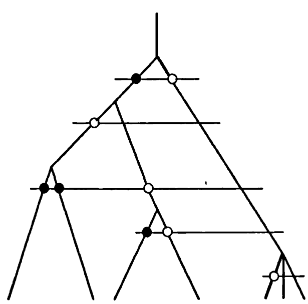
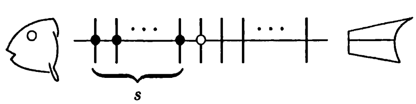


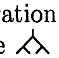
FIGURE 5. A decoration of the tree from the previous figure.

(iii) Each internal edge of S is decorated.

Condition (i) means that we may see the following pattern[1] on the horizontal lines:



with the case $s = 0$ (no black dot) allowed.

A decoration of the tree from Figure 4 is shown in Figure 5. All possible decorations of the tree  are shown in Figure 6.

[1]This should remind us about the time when this paper was finished – carp with potato salad is the most typical Czech Christmas dish.

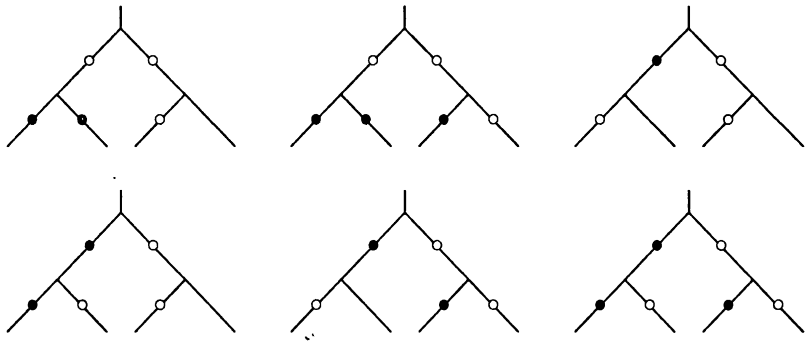


FIGURE 6. All possible decorations of a tree.

Let Q_n be the set of all decorated, in the above sense, planar directed trees with at least binary vertices and n leaves. To each $S \in \mathsf{Q}_n$ we assign a map $G_S : V^{\otimes n} \to V$, by interpreting S as a "flow chart," with $\circ$ denoting the homotopy $h : V \to V$, $\bullet$ denoting the composition gf, and a vertex of arity k the map $\mu_k : V^{\otimes k} \to V$. For example, the tree S in Figure 5 gives degree 6 map

$$\mu_2(gf \circ \mu_2(h \circ \mu_2(gf \otimes gf) \otimes h \circ \mu_2(gf \otimes h)) \otimes h \circ \mu_3(h \otimes \mathbb{1}^{\otimes 2})) : V^{\otimes 7} \to V\,.$$

Finally, we must define a sign $\varepsilon(S)$ of a tree $S \in \mathsf{Q}_n$. The definition is more difficult than the definition of the sign $\vartheta(T)$ of a tree $T \in \mathsf{P}_n$, because $\varepsilon(S)$ will depend also on the decoration, not only on the combinatorial type, of the tree S.

To calculate $\varepsilon(S)$, we must first decompose S into trees $T_1, \ldots, T_k$ representing summands of p-kernels, following the pattern of (8). The sign is then defined as

$$\varepsilon(S) := n + r_i + \vartheta(r_1, \ldots, r_i) + \sum_1^k \vartheta(T_j)\,,$$

where $r_1, \ldots, r_i$ have the same meaning as in (8). Let us calculate, as an example, the sign of the decorated tree from Figure 5. The decomposition of this tree into trees from P is shown in Figure 7. In this figure, T_1 is the decorated subtree with vertices u and v and T_2 is the subtree with vertices a, b and c. The sign of S is then the sum $\varepsilon(S) := 7 + 3 + 1 \cdot 4 + \vartheta(T_1) + \vartheta(T_2) = 0 \pmod 2$. The following proposition then follows from boring combinatorics argument.

Proposition 7. *The q-kernel* $\boldsymbol{q}_n : V^{\otimes n} \to V$, *defined inductively by* (8), *can also be defined as*

$$\boldsymbol{q}_n := \sum_{S \in \mathsf{Q}_n} (-1)^{\varepsilon(S)} \cdot G_S\,, \quad \text{for } n \geq 2\,,$$

and $q_1 := \mathbb{1}_V$ *for* $n = 1$.

5. WHY DO THE TRANSFERS EXIST

As we already observed, if the basic ring R is a field of characteristic 0, the existence of transfers follows from a general theory developed in [5] – see "move **(S)**" on page 141 of [5]. We want to make this statement more precise now. In this section we

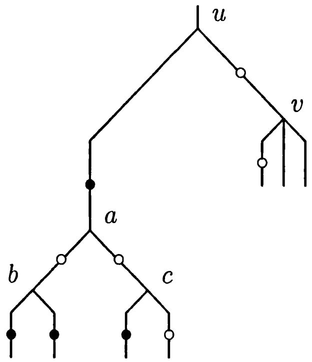


FIGURE 7. Decomposing a tree into p-kernels.

assume that the reader is familiar with colored operads which describe diagrams of algebras, see [5] again. The rest of the paper is independent of the material in this section.

Let $\mathcal{P}_{in}$ be the 2-colored operad describing structures consisting of an associative multiplication μ on a vector space V and linear maps of vector spaces $f : V \to W$, $g : W \to V$ such that $gf = 1\!\!1_V$. Let also $\mathcal{P}_{out}$ be the 2-colored operad describing diagrams consisting of an associative multiplication μ on V, an associative multiplication ν on W, and homomorphisms $f : V \to W$, $g : W \to V$ of these associative algebras such that $gf = 1\!\!1_V$. An explicit description of these operads can be found in Example 12 of [5], where $\mathcal{P}_{in}$ was denoted $\mathcal{P}_{(S,\underline{D})}$ and $\mathcal{P}_{out}$ was denoted $\mathcal{P}_{\underline{S}}$. Finally, let $S : \mathcal{P}_{out} \to \mathcal{P}_{in}$ be the map defined by

$$S(\mu) := \mu,\ S(f) := f\,,\quad S(g) := g \quad \text{and} \quad S(\nu) := f\mu(g \otimes g)\,.$$

This well-defined map of colored operads represents a solution of the following "classical limit" of Problem 2.

Problem 8. We are given two vector spaces V, W and linear maps $f : V \to W$, $g : W \to V$ such that $gf = 1\!\!1_V$ (in other words, $f : V \hookrightarrow W$ is an inclusion and g its retraction). Given an associative algebra structure $\mu : V \otimes V \to V$ on the vector space V, find an associative algebra structure $\nu : W \otimes W \to W$ on W such that f and g became homomorphisms of associative algebras.

Let $\mathcal{R}_{in}$ be the dg-operad representing the "input data" of our transfer problem for A_∞-algebras, that is, diagrams consisting of an A_∞-structure $\boldsymbol{\mu} = (\mu_2, \mu_3, \ldots)$ on (V, ∂_V), dg-maps $f : (V, \partial_V) \to (W, \partial_W)$, $g : (W, \partial_W) \to (V, \partial_V)$ and a chain homotopy h between gf and $1\!\!1_V$. Let $\rho_{in} : \mathcal{R}_{in} \to \mathcal{P}_{in}$ be the map of colored operad given by

$$\rho_{in}(\mu_2) := \mu\,,\ \rho_{in}(\mu_n) := 0 \quad \text{for } n \geq 3\,,\quad \rho_{in}(f) := f\,,\quad \rho_{in}(g) := g \quad \text{and} \quad \rho_{in}(h) := 0\,.$$

In the same vein, let $\mathcal{R}_{out}$ be the dg-operad representing a solution of our transfer problem, that is, diagrams consisting of an A_∞-structure $\boldsymbol{\mu} = (\mu_2, \mu_3, \ldots)$ on (V, ∂_V), an A_∞-structure $\boldsymbol{\nu} = (\nu_2, \nu_3, \ldots)$ on (W, ∂_W), A_∞-maps $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \ldots) : (V, \partial, \boldsymbol{\mu}) \to (W, \partial, \boldsymbol{\nu})$, $\boldsymbol{\psi} = (\psi_1, \psi_2, \ldots) : (W, \partial, \boldsymbol{\nu}) \to (V, \partial, \boldsymbol{\mu})$ and an A_∞-homotopy

$H = (H_1, H_2, \ldots)$ between $\psi\varphi$ and $1\!\!1_V$. Let $\rho_{out} : \mathcal{R}_{out} \to \mathcal{P}_{out}$ be the map defined by

$$\rho_{out}(\mu_2) := \mu\,,\ \rho_{out}(\mu_n) := 0 \ \text{ for } n \geq 3\,, \qquad \rho_{out}(\nu_2) := \nu\,,\ \rho_{out}(\nu_n) := 0 \ \text{ for } n \geq 3\,,$$

$$\rho_{out}(\varphi_1) := f\,,\ \rho_{out}(\varphi_n) := 0 \ \text{ for } n \geq 2\,, \qquad \rho_{out}(\psi_1) := g\,,\ \rho_{out}(\psi_n) := 0 \ \text{ for } n \geq 2\,,$$

$$\text{and} \quad \rho_{out}(H_n) := 0 \ \text{ for } n \geq 1\,.$$

The following proposition follows from the methods of [5] and [6].

Proposition 9. *The map $\rho_{in} : \mathcal{R}_{in} \to \mathcal{P}_{in}$ is a cofibrant resolution of the colored operad $\mathcal{P}_{in}$ and $\rho_{out} : \mathcal{R}_{out} \to \mathcal{P}_{out}$ is a cofibrant resolution of the colored operad $\mathcal{P}_{out}$.*

It follows from [5, Lemma 20] that there exists a lift $\widetilde{S} : \mathcal{R}_{out} \to \mathcal{R}_{in}$ making the following diagram commutative:

$$\begin{array}{ccc} \mathcal{R}_{out} & \xrightarrow{\ \widetilde{S}\ } & \mathcal{R}_{in} \\ {\scriptstyle \rho_{in}}\downarrow & & \downarrow{\scriptstyle \rho_{out}} \\ \mathcal{P}_{out} & \xrightarrow{\ S\ } & \mathcal{P}_{in} \end{array}$$

Each such lift $\widetilde{S}$ clearly provides a solution of Problem 2 while formulas (1) determine a specific lift of S.

Observe that, very crucially, we work with algebras *without units*. It is straightforward to realize that the unital version of Problem 8 does not have an affirmative answer. Indeed, given a unit $1_V \in V$ for μ, we would be forced to define $1_W := f(1_V)$. It is then easy to see that such 1_W is a unit for the transferred structure if and only if $fg = 1\!\!1_W$, that is, f and g are isomorphisms inverse to each other which we did not assume. This observation explains why our solution of Problem 2 which is, as we explained above, a lift of the classical Problem 8, works only for non-unital A_∞-algebras. Transfers of unital A_∞-structures present much harder problem, see the analysis in [5].

6. Some other properties of the transfer

In this section we analyze what happens if g is not just a left homotopy inverse of f, but if f and g are chain homotopy equivalences inverse to each other.

Let $\mathcal{A}_\infty(V, \partial)$ denote the set of *isomorphism classes* (with respect to A_∞-maps) of A_∞-structures on a given chain complex (V, ∂). Suppose we are given chain maps $f : (V, \partial_V) \to (W, \partial_W)$, $g : (W, \partial_W) \to (V, \partial_V)$ and a chain homotopy h between gf and $1\!\!1_V$. It can be easily shown that the first formula of (1) defines a set map

$$\mathsf{Tr}_{f,g,h} : \mathcal{A}_\infty(V, \partial_V) \to \mathcal{A}_\infty(W, \partial_W)\,.$$

Suppose we are given also a chain homotopy l between fg and $1\!\!1_W$, that is, f and g are now fully fledged chain homotopy equivalences inverse to each other. Then one may as well consider the map

$$\mathsf{Tr}_{g,f,l} : \mathcal{A}_\infty(W, \partial_W) \to \mathcal{A}_\infty(V, \partial_V)\,.$$

We found the following proposition surprising, because there is no relation between the homotopies h and l.

Proposition 10. *Let f and g be chain homotopy equivalences, with chain homotopies $h : gf \cong 1\!\!1_V$ and $l : fg \cong 1\!\!1_W$. Then both $\mathsf{Tr}_{f,g,h}$ and $\mathsf{Tr}_{g,f,l}$ are isomorphisms and*

$$\mathsf{Tr}_{g,f,l} = \mathsf{Tr}^{-1}_{f,g,h}\,.$$

Proof. Formulas (1) give an A_∞-structure $\boldsymbol{\nu}$ on (W, ∂_W) together with an A_∞-map $\boldsymbol{\varphi} : (V, \partial_V, \boldsymbol{\nu}) \to (W, \partial_W, \boldsymbol{\mu})$. Let us apply (1) once again, this time to construct an A_∞-structure $\bar{\boldsymbol{\mu}}$ on (V, ∂_V) together with an A_∞-map $\bar{\boldsymbol{\varphi}} : (W, \partial_W, \boldsymbol{\nu}) \to (V, \partial_V, \bar{\boldsymbol{\mu}})$, using g instead of f, f instead of g and l instead of h. We must prove that $\boldsymbol{\mu}$ is isomorphic to $\bar{\boldsymbol{\mu}}$. To this end, recall the following A_∞-case of "move **(M2)**" of [5].

Proposition 11. *Let $(A, \partial_A, \boldsymbol{\xi})$, $(B, \partial_B, \boldsymbol{\eta})$ be A_∞-algebras and $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots) : (A, \partial_A, \boldsymbol{\xi}) \to (B, \partial_B, \boldsymbol{\eta})$ an A_∞-map. Suppose that $C : (A, \partial_A) \to (B, \partial_B)$ is a chain map, homotopic to the linear part θ_1 of $\boldsymbol{\theta}$. Then C can be extended into an A_∞-map $\boldsymbol{C} = (C_1 = C, C_2, \ldots) : (A, \partial_A, \boldsymbol{\xi}) \to (B, \partial_B, \boldsymbol{\eta})$.*

Now observe that the linear part of the composition $\bar{\boldsymbol{\varphi}}\boldsymbol{\varphi}$ equals $1\!\!1_V$. Proposition 11 then implies the existence of an A_∞-map $\boldsymbol{C} = (1\!\!1_V, C_2, \ldots) : (V, \partial_V, \boldsymbol{\mu}) \to (V, \partial_V, \bar{\boldsymbol{\mu}})$ which is clearly an isomorphism. This finishes our proof of Proposition 10. Observe that the composition $\bar{\boldsymbol{\varphi}}\boldsymbol{\varphi}$ need not be an isomorphism, therefore the full force of Proposition 11 is necessary. □

Let us consider again chain homotopy equivalences f and g, with chain homotopies $h : gf \cong 1\!\!1_V$ and $l : fg \cong 1\!\!1_W$. Given an A_∞-structure $\boldsymbol{\mu} = (\mu_2, \mu_3, \ldots)$ on (V, ∂_V), let us construct, using formulas (1), an A_∞-structure $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots)$ on (W, ∂_W) and A_∞-maps $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ as before. A natural question is when such a situation gives rise to a "perfect" chain homotopy equivalence in the category of A_∞-algebras. The following proposition follows from the methods of [7].

Proposition 12. *The chain homotopy l can be extended into an A_∞-homotopy $\boldsymbol{L}$ between A_∞-maps $\boldsymbol{\varphi\psi}$ if the chain homotopy equivalence (f, g, h, l) extends into a strong homotopy equivalence in the sense of* [7, Definition 1]. *This, according to* [7, Theorem 11], *happens if and only if*

$$(9)\qquad \begin{aligned} &[fh - lf] = 0 \quad \text{in} \quad H_1\big(\mathrm{Hom}(V, W)\big) \quad \text{or, equivalently,}\\ &[gl - hg] = 0 \quad \text{in} \quad H_1\big(\mathrm{Hom}(W, V)\big)\,. \end{aligned}$$

If the A_∞ structure $\boldsymbol{\mu} = (\mu_2, \mu_3, \ldots)$ on (V, ∂_V) is generic enough, then the vanishing of the obstruction classes in (9) is also necessary for the existence of an extension of l into $\boldsymbol{L}$.

7. Two observations

Transfers and polyhedra. The formulas for $\boldsymbol{\nu}$, $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ and $\boldsymbol{H}$ given in (1) are summations of monomials in the "initial data" $\mu_2, \mu_3, \cdots, f, g, h$ with coefficients ± 1. Ezra Getzler conjectured that these monomials might in fact correspond to cells of certain cell decompositions of the polyhedra governing our algebraic structures – Stasheff's associahedra [8, page 9] K_n, $n \geq 2$, and the multiplihedra [8, page 113] L_n, $n \geq 2$. For K_n, the decomposition induced by the p-kernel $\boldsymbol{p}_n$ is given by taking the tubular

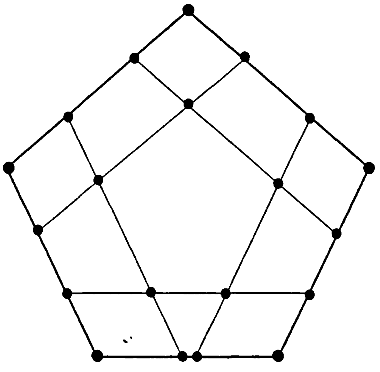


FIGURE 8. The decomposition of the associahedron K_4 induced by $\boldsymbol{p}_4$. It consists of 10 squares and one pentagon. The squares adjacent to the vertices of K_4 correspond to the five trees of P_4 with two interior edges, the squares adjacent to the edges of K_4 correspond to the five trees of P_4 with one interior edge. The pentagon in the center of K_4 corresponds to the corolla (tree with no interior edge) in P_4.

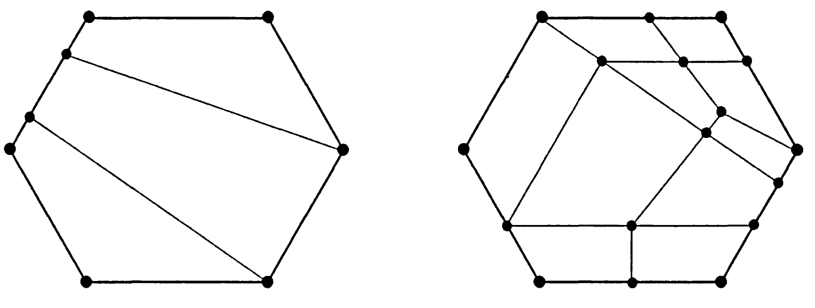


FIGURE 9. Decompositions of the multiplihedron L_3. The left picture shows the decomposition of L_3 into 3 squares corresponding to the terms of $\boldsymbol{p}_3$. The right picture shows the decomposition of the same multiplihedron into 10 squares corresponding to the terms of $\boldsymbol{q}_3$.

neighborhood of ∂K_n in the manifold-with-corners K_n, as illustrated for $n = 4$ in Figure 8. We do not know a similar simple rule for the multiplihedra, see also Figure 9.

Minimal models. The material of this subsection is well-known to specialists. Recall that an A_∞-algebra $(W, \partial_W, \mu_2, \mu_3, \ldots)$ is *minimal* if $\partial_W = 0$. Methods developed in the previous sections can be used to construct *minimal models* of A_∞-algebras as follows.

Let $A = (V, \partial_V, \mu_2, \mu_3, \ldots)$ be an A_∞-algebra and $W := H(V, \partial_V)$ the cohomology of its underlying chain complex. Let $Z := \mathit{Ker}(\partial_V)$, $B := \mathit{Im}(\partial_V)$ and choose a "Hodge decomposition"

$$V \cong D \oplus W \oplus B, \quad \text{with} \quad Z \cong W \oplus B. \tag{10}$$

Observe that the composition $\omega := \partial_V \circ \iota_D : D \to B$, where $\iota_D : D \hookrightarrow V$ denotes the inclusion, is a degree -1 isomorphism of vector spaces. Let $f : V \to W$ be the projection and $g : W \to V$ the inclusion induced by (10). Finally, let $h : V \to V$ be the degree -1 map defined as the composition $\iota_D \circ \omega^{-1} \circ \pi_B$, where $\pi_B : V \to B$ is the projection induced by (10).

It is clear that $f : (V, \partial_V) \to (W, 0)$ and $g : (W, 0) \to (V, \partial_V)$ are chain maps and that h is a chain homotopy between gf and $1\!\!1_V$. Therefore the formula

$$\nu_n := f \circ \boldsymbol{p}_n \circ g^{\otimes n}, \tag{11}$$

where $\boldsymbol{p}_n$ is the p-kernel defined in Section 4, gives a minimal model $\mathcal{M}_A = (W, \partial_W = 0, \nu_2, \nu_3, \ldots)$ of the A_∞-algebra $A = (V, \partial_V, \mu_2, \mu_3, \ldots)$. This construction is functorial up to a choice of the Hodge decomposition (10).

More precisely, observe that decompositions (10) form a groupoid with morphisms given by chain endomorphisms of (V, ∂_V). We clearly have:

Proposition 13. *The minimal model $\mathcal{M}_A$ is a functor from the groupoid of Hodge decompositions* (10) *to the groupoid of minimal A_∞-algebras and their A_∞-isomorphisms.*

Observe that the "input data" f, g, h constructed out of the Hodge decomposition (10) satisfy the side conditions mentioned in Remark 4, therefore we could as well use the formulas of [1].

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