WSAA 6

Marián J. Fabián

Theory of Frechet cones and nonlinear analysis

In: Zdeněk Frolík (ed.): Abstracta. 6th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1978. pp. 25–27.

Persistent URL: http://dml.cz/dmlcz/701777

THEORY OF FRECHET CONES AND NONLINEAR ANALYSIS

M.Fabian, Praha

In nonlinear analysis, properties of mappings are often expressed with help of their Fréchet derivatives. It requires differentiability of mappings and so the set of the mappings in question is restricted considerably.

The aim of this short communication is to introduce a more general concept - the so called Fréchet cone of a mapping at a point and show that it behaves almost as well as Fréchet derivative. We thus obtain a possibility of generalizing many theorems from analysis in which Fréchet differentiability is required.

First we shall introduce some auxiliary concepts. Let Z be a normed linear space over the reals. A subset $C \subset Z$ is called a cone (in Z) if

$$C \neq \emptyset , \{0\} \text{ and } \forall \lambda \geq 0 \quad \forall c \in C \quad \lambda c \in C .$$

For $\varepsilon > 0$, the conic ε-neighbourhood of C is defined by

$$V_{\varepsilon}(C) = \{ z \in Z \mid \operatorname{dist}(z,C) \geq \varepsilon \|z\| \} .$$

Now, let X,Y be normed linear spaces, take $Z = X \times Y$ and endow it with, e.g., the maximum norm. In what follows, we shall not distinguish between a mapping and its graph. Let $F : X \to Y$ be a mapping, continuous at some $x_0 \in \operatorname{int} D(F)$.

For $r > 0$ define

$$C_r(F,x_0) = \{ \lambda ((x,Fx)-(x_0,Fx_0)) \mid x \in D(F) , \ \|((x,Fx)-(x_0,Fx_0))\| < r , \ \lambda \geq 0 \}$$

and set

$$T(F,x_0) = \bigcap_{r>0} \overline{C_r(F,x_0)} .$$

The last set, if it is $\neq \{0\}$, is said to be the tangent cone of F at x_0 . This cone has been studied by Flett, Nashed etc. If, in addition,

$$\forall \varepsilon > 0 \ \exists r > 0 \qquad C_r(F,x_o) \subset V_\varepsilon(T(F,x_o)) ,$$

then $T(F,x_o)$ is called the Frechet cone of F at x_o and is denoted by $C_o^f(F,x_o)$.

Fréchet cone is a natural generalization of Fréchet derivarive as the following theorem shows.

Theorem (Durdil CMUC 1974). Let $F : X \to Y$ be a mapping, continuous at some $x_o \in \operatorname{int} D(F)$. Then F is Fréchet differentiable at x_o if and only if $C_o^f(F,x_o)$ exists and belongs to $\mathcal{L}(X,Y)$. In this case

$$C_o^f(F,x_o) = dF(x_o) .$$

For computing with Frechet cones, statements similar to those from differential calculus can be formulated. Namely, under some assumptions, the following rules hold:

$$C_o^f(F^{-1},Fx_o) = \left[C_o^f(F,x_o)\right]^{-1} ,$$

$C_o^f(F+G,x_o) = C_o^f(F,x_o)+dG(x_o)$, for $G : X \to Y$ Fréchet differentiable at x_o , $C_o^f(H \circ F,x_o) = dH(Fx_o) \cdot C_o^f(F,x_o)$, for $H : Y \to U$ Fréchet differentiable at Fx_o . It should be noted that Fréchet differentiability in the above formulae cannot be replaced by the requirement of the existence of Fréchet cone. Further, with help of Theorem of Durdil, we can easily derive from the above rules the well known ones from differential calculus.

A mean value theorem also holds. First, for a cone $C \subset Z$, define

$$\|C\|_X = \sup\left\{ \frac{\|y\|}{\|x\|} \ \Big|\ (x,y) \in C \ , \ (x,y) \neq (0,0) \right\} .$$

We remark that, if $L \in \mathcal{L}(X,Y)$, then L is a cone in $X \times Y$ and $\|L\|_X$ is equal to the usual norm $\|L\|$.

Theorem (Mean value theorem). Let $\Omega \subset D(F)$ be a nonempty open convex, $\gamma \geq 0$, and assume that, for each $x \in \Omega$, F is continuous at x and

$$\|C_o^f(F,x)\|_X \leq \gamma .$$

Then F is Lipschitzian on Ω with the Lipschitz constant γ .

So we have built a theory of Fréchet cones and we can apply it in generalizing many theorems from analysis in which Fréchet differentiability is required. As illustration, let us formulate

Theorem (on homeomorphism). Let X,Y be Banach spaces. Let $F : X \to Y$ be continuous and have Fréchet cone at each point of a nonempty open convex set $\Omega \subset D(F)$. Furthermore, let there exist $L \in \mathrm{Isom}\,(X,Y)$ and $\varepsilon \in (0, \|L^{-1}\|^{-1})$ such that

$$\forall x \in \Omega \; \|C_0^f(F,x)-L\|_X \quad \|L^{-1}\|^{-1} - \varepsilon \; .$$

Then $F(\Omega)$ is open and the mappings F/Ω , $(F/\Omega)^{-1}$ are Lipschitzian.

Theorem (on global homeomorphism). Let X,Y be Banach spaces and $F : X \to Y$ a mapping, with $D(F) = X$, continuous and having Fréchet cone at each $x \in X$. Suppose that, for each $u \in X$, there exist a neighbourhood Ω_u of u , $L_u \in \mathrm{Isom}\,(X,Y)$, and $\varepsilon_u \in (0, \|L_u^{-1}\|^{-1})$ such that

$$\forall x \in \Omega_u \; \|C_0^f(F,x)-L_u\|_X \leq \|L_u^{-1}\|^{-1} - \varepsilon_u \; .$$

Finally, let $\sup\{\|[C_0^f(F,x)]^{-1}\|_Y \mid x \in X\} < +\infty$.

Then F maps X onto Y homeomorphically.

Also, we can define a Gâteaux cone and formulate statements analogical to those dealing with Fréchet cones.