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AN ALTERNATE APPROACH TO THE PIROGOV - SINAI THEORY, NOT EMPLOYING THE MODELS WITH A PARAMETER.

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This is a very short exposition of another approach to the Pirogov - Sinai theory of phase diagrams. The detailed version will be published [1] . We outline here the main features of our approach only, differing from the "classical" P.S. theory and also from the approach of Kotecký - Preiss published in this volume.

We do not introduce the contour models with a parameter. Instead, we define the contour functionals explicitly, by (4) and introduce the notion of a "truncated functional" (5), which is a τ-functional. This leads to the construction of "truncated" contour models.

These truncated models are equal to the usual contour models if the corresponding ground state is stable (7). We distinguish the stable and unstable ground states by using the "parameters of nonstability" a_q which appear, in our approach, in another, perhaps more transparent meaning (8). We use these parameters also for a classification of contours on "small" and "large" ones (the latter appearing in a nonstable phases only).

Some comments about the methods of proofs:
Our main technical tool is the method of cluster expansion, widely used in the literature. It is very suitable for the investigation of contour models such that the corresponding contour functional is a τ-functional.
For the investigation of the behaviour of "nonstable" phases we use the following method: by fixing the large contours (resp. external large contours) of a configuration and "sweeping out" the remaining contours, we define an auxilliary model, having a configurations with large (resp. external large) contours only, and an appropriately changed hamiltonian. We develop a simple method of study of such models (see [1] , § 2), a byproduct of the method of cluster expansion, too.

Basic notions.

We will follow the notation of [4] (and [2]), if possible. Given any configuration on $\mathbb{Z}^{\nu}$ we distinguish its correct (more precisely q-correct, for an appropriate q) points. We define the contours Γ^q and "contour hamiltonians" $\Phi(\Gamma^q)$ – see [4]. Formally, we write the hamiltonian as

$$\mathcal{H} = \sum_q h(\psi_q)|\Lambda_q| + \sum_q \sum_{\Gamma^q} \Phi(\Gamma^q) \qquad (1)$$

where Λ_q is the union of all q-correct points and $\{\Gamma^q\}$ is the family of all q-contours of a given configuration.

Assume that $\Phi(\Gamma^q)$ satisfy the Peierls condition

$$\Phi(\Gamma^q) > \tau \,|\mathrm{supp}\, \Gamma^q| \qquad (2)$$

for a sufficiently large τ.

Remind the notion of a "crystallic" partition function $Z(\Gamma^q, \mathcal{H})$. Denote further by $\Omega_q^{\mathrm{red}}(\Gamma^q)$ the set of all configurations on $C(\Gamma^q) = \mathrm{supp}\, \Gamma^q \cup \mathrm{int}\, \Gamma^q$ which satisfy the condition that, being extended to the whole $\mathbb{Z}^{\nu}$ by the constant value ψ_q, no contours of this extended configuration have a distance ≤ 1 from the set supp Γ^q (we assume that the ground states are constant configurations, for the simplicity). Put

$$Z^{\mathrm{red}}(\Gamma^q, \mathcal{H}) = \sum_{\varphi \in \Omega_q^{\mathrm{red}}(\Gamma^q)} \exp(-\mathcal{H}(\varphi \mid \psi_{q\,(\mathrm{Ext}\, \Gamma^q)})) \qquad (3)$$

and define the <u>contour functional associated with $\mathcal{H}$</u>

$$F_q(\Gamma^q) = \log Z^{\mathrm{red}}(\Gamma^q, \mathcal{H}) - \log Z(\Gamma^q, \mathcal{H}) \qquad (4)$$

("the work needed to install the contour Γ^q in place of previously q-correct configuration").

Define also the <u>truncated contour functional</u>

$$\bar{F}_q(\Gamma^q) = \max\left(F(\Gamma^q) ,\ \tfrac{1}{3}\tau\,|\mathrm{supp}\, \Gamma^q| \right) \qquad (5)$$

(the choice of $\frac{1}{3}$ is quite arbitrary).

Denote by $-s_q = -\lim |\Lambda|^{-1} \log Z_q(\Lambda, \bar{F}_q)$ the free energy of the contour model with a contour functional $\bar{F}_q$.

Put

$$h_q = h(\psi_q) - s_q . \qquad (6)$$

Say that a q_0-th ground state is <u>stable</u> if

$$h_{q_0} = \min_q \{h_q\} . \qquad (7)$$

Denote by a_q the "parameters of nonstability" defined by

$$a_q = h_q - h_{q_0} \quad . \tag{8}$$

Note. It is shown in [1] that h_{q_0} is the free energy of the original model. The parameters a_q are not in general (if $a_q \neq 0$) equal to the parameters b_q of [2] or [4].

Basic results.

Theorem.

For a stable q, $F_q = \bar{F}_q$ and there is a Gibbs state, which is a perturbation of the q-th ground state such that the probability distributions of its external contours are identical with the distributions of external contours in the contour model, defined by the contour functional F_q .

Proof. It is based on the following technical result :

Definition . A contour Γ^q is called stable if

$$F_q(\Gamma^q) = \bar{F}_q(\Gamma^q) \quad .$$

Lemma . The following estimates are true :

i) nonstable contours satisfy the inequality

$$a_q \,|\, C(\Gamma^q) \,| \geq \frac{\tau}{3} |\text{supp } \Gamma^q| \quad . \tag{9}$$

Especially, there are no unstable contours Γ^q if q is stable.

ii) $$Z_q(\Lambda, \mathcal{H}) \geq \exp(-h_q |\Lambda|) \exp(-C|\partial\Lambda|) \tag{10}$$

and

iii) $$Z_q(\Lambda, \mathcal{H}) \leq \exp(-h_{q_0} |\Lambda|) \exp(C|\partial\Lambda|) \tag{11}$$

for some universal constant $C = C(\tau)$.

Proof . See [1] .

The proof proceeds by induction, combining i), ii), iii).
The induction step for i) is an elementary consequence of ii), iii).
The relation (1o) is proved by standard methods of cluster expansion.

The only nontrivial step is the induction step in iii).
We introduce the following useful notion:

Definition.

A contour Γ^q is called small if it is stable and moreover there is no unstable $\tilde{\Gamma}^q$ such that $\text{supp } \tilde{\Gamma}^q \subset \text{int } \Gamma^q$.

It is proven in [1] that

$$Z_q(\Lambda, \mathcal{H}) \leq \sum_{\{\Gamma_i^q\}} \exp(-a_q |Ext|) \prod_i \exp(-\Phi(\Gamma_i^q) + C'|\mathrm{supp}\,\Gamma_i^q|) \times \exp(-h_{q_0}|\Lambda|) \exp(C|\partial\Lambda|) \qquad (12)$$

with the same C as in (10,11) and $C' = C'(C)$, the sum being taken over all possible famillies $\{\Gamma_i^q\}$ of external large contours in Λ and with $Ext = \Lambda \setminus \bigcup_i C(\Gamma_i^q)$.
It is shown in § 2 of [1] that (12) implies the desired estimate iii).

We omit the construction of phase diagram (see [1] , [2] , [3] , [4]). In the last part of this note we formulate a result concerning the completeness of the phase diagram constructed by Pirogov-Sinai.

There are some partial results of Martirosjan (see [5]). The full answer is given in [1] .
The completeness was also proved, independently, by another methods, by D.Preiss.
We formulate here a simplified version, containing, however, all the important details.

Definition.
Given any configuration $\varphi \in \Omega_q(\Lambda)$ denote by $\mathcal{L} = \mathcal{L}(\varphi)$ the system of all large contours which satisfy the condition that supp $\Gamma \subset$ int $\tilde{\Gamma}$ for no small contour $\tilde{\Gamma}$ of φ. Define a canonical configuration $\varphi^{\mathcal{L}}$ associated to φ such that $\mathcal{L}$ is exactly the contour system of $\varphi^{\mathcal{L}}$ (i.e. we remove all the remaining contours).
Define a stable domain of $\varphi \in \Omega_q(\Lambda)$ as the union of all the components of $\Lambda \setminus \bigcup_{\Gamma \in \mathcal{L}} \mathrm{supp}\,\Gamma$ which have a stable value in the canonical configuration $\varphi^{\mathcal{L}}$. The points of a stable domain will be called stable points of a configuration φ.

Proposition.
Denote by $\Omega_q^N(\Lambda)$ the subset of $\Omega_q(\Lambda)$ consisting of configurations with at least N unstable points. Denote by $Z_q^N(\Lambda, \mathcal{H})$ the corresponding partition function. Then the following estimate holds, with the same constant as in (lo,11) :

$$Z_{q}^{N}(\Lambda,\mathcal{H}) \leq \exp(-h_{q_0}|\Lambda|)\ \exp(-\tfrac{a}{2} N)\ \exp(C|\partial\Lambda|) \quad .(13)$$

where $a = \min \{ a_q : a_q \neq 0 \}$.

<u>Note</u>. Roughly speaking, if $|\Lambda| \gg |\partial\Lambda|$ then for a large fraction of points $t \in \Lambda$ there is a large probability that there is "stable component" C of $\Lambda \setminus \bigcup_{\Gamma \in \mathcal{X}} \operatorname{supp} \Gamma$ such that dist(t , C^c) is large. The statement on completeness now follows using the assumed translation invariancy of a Gibbs state. See again [1] for the details.

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