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HIGHER REIDEMEISTER TORSION AND PARAMETRIZED MORSE THEORY

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0. Introduction. This paper constitutes a summary of the author's Ph.D. thesis [K]. Proofs of the results cited here will appear elswhere. The first section is devoted to outlining a means of passing in a continuous way from the space of pairs (M, f), where M is a compact smooth manifold and f is a Morse function on M, into a moduli space for finite cell complexes.

In section two the results of section one are applied in special instances to construct a new invariant which is a parametrized analogue of Reidemeister torsion. This invariant takes values in a certain subquotient of higher algebraic K-groups of the complex numbers.

1. Manifold bundles and families of cell complexes. Suppose $p:E \longrightarrow B$ is a bundle over a finite CW complex having compact smooth manifold fibres. Consider a continuous function f on E whose restriction to each fibre of p is a smooth function with no degenerate critical points. Then f is a family of Morse functions parametrized by points of B.[1] In the unparametrized case, that is, when B is a point, a classical procedure [M] shows how to construct from f a finite cell complex Y having the homotopy type of E, in which the number of cells of Y is equal to the number of critical points of f.

However, the usual method of assigning the cell complex Y to the function f is ambiguous unless extra data is chosen (e.g. a Riemannian metric, local coordinates, deformation retractions etc.). It would therefore be natural to ask whether or not the set of cell complexes associated with a given Morse function forms a contractible space. Unfortunately, this is *not* true for the standard construction [M].

By a different, coordinate free approach we prove,

[1] It is not always true that such a function always exists on a given bundle $p:E \longrightarrow B$. A necessary, but not sufficient obstruction when B is simply connected is that there should be a section of p.

Theorem A. If N is a smooth manfold and $f:N \longrightarrow \mathbb{R}$ is a Morse function, then there is a contractible space C(f) which is intrinsically defined in terms of N and f and which has the property that each point of C(f) uniquely determines a cell complex Y arising from $f:N \longrightarrow \mathbb{R}$.

Now consider the case when B is any finite CW complex. We address the question of whether it is possible or not to naturally associate to a manifold bundle $p:E \longrightarrow B$ together with a fibre-wise Morse function $f:E \longrightarrow \mathbb{R}$, a *bundle of cell complexes* parametrized by points of B. By this, we mean a fibration $\pi:Y \longrightarrow B$ such that for each $b \in B$ the fibre $Y_b = \pi^{-1}(b)$ has the structure of a finite cell complex in such a way that the attaching maps for the cells of Y_b vary continuously with respect to $b \in B$. By the theory of classifying spaces, it turns out that it is sufficient to answer this question in the *universal* case, i.e., for the bundle,

$$p_u:(EG\times\mathfrak{M}(N))\times_G N \longrightarrow B^{\mathfrak{M}}G,$$

where N is the fibre over the basepoint of p, $G \subseteq \mathrm{Diff}(N)$ is a group of diffeomorphisms, EG is a free, contractable G-space, $\mathfrak{M}(N)$ is the space of Morse functions on N (with the Whitney C^∞ topology), and $B^{\mathfrak{M}}G$ denotes the Borel construction $EG\times_G\mathfrak{M}(N)$. Note that this bundle is equipped with a universal fibre-wise Morse function f_u defined by $f_u((x,f),n) \doteq f(n)$. Consequently, the space $B^{\mathfrak{M}}G$ may be viewed as a classifying space for bundles with Morse functions. A positive answer is then provided by the following:

Theorem B. There is a space B^CG and a forgetful map $F:B^CG \longrightarrow B^{\mathfrak{M}}G$ such that

(i) F is a weak homotopy equivalence, and

(ii) each point $b \in B^CG$ with $F(b) = (x,f) \in EG\times_G\mathfrak{M}(N) = B^{\mathfrak{M}}G$ determines a cell complex Y_b associated to the Morse function $f:N \longrightarrow \mathbb{R}$, and furthermore, Y_b varies continuously with respect to $b \in B^CG$.

The importance of theorem B is spelled out in the following corollary:

Corollary. If $f:E \longrightarrow \mathbb{R}$ is a fibre-wise Morse function on a manifold bundle $p:E \longrightarrow B$, then it is always possible to perturb f by a fibre-wise homotopy to yield a fibre-wise Morse function g on $p:E \longrightarrow B$ having the property that there exists a parametrized family of cell complexes $q:Y \longrightarrow B$ which is associated to g.

We sketch a rough outline of the proof of theorem B. The proof involves constructing a G-space C(N), together with a natural G-equivariant map $h:C(N) \longrightarrow \mathfrak{M}(N)$ which is also a weak equivalence, such that the points of C(N) determine cell complexes. We may then set B^cG equal to the Borel contruction $EG\times_G C(N)$. One might *a priori* guess that C(N) is constructed in a sheaf-like manner from the space of Morse functions $\mathfrak{M}(N)$ by defining the stalk of h over a Morse function f to be the space C(f) of theorem A. However, the assignment $f \longmapsto C(f)$ unfortunately has the property that the cell complexes in C(f) do not necessarily vary continuously with respect to the parameter $f \in \mathfrak{M}(N)$. The reason for the discontinuity is that cell complexes in C(f) arise in part by choosing a set of regular values $r_1,\ldots,r_k$ of f that separate the collection of critical values (these define level surfaces in N which separate the critical points). If f_t is a family of Morse functions, it is possible that the number of distinct critical values of f_t is different for different t. Hence the critical values may pass through each other. Consequently, it might not be possible to choose a continuously varying set of regular values $r_1(t),\ldots,r_k(t)$ that separate the critical values of f_t.

To resolve the discontinuity problem, Igusa's *stratification theorem* [I_2; chapter III] is applied to the space $\mathfrak{M}(N)$. Let $\psi:\mathfrak{M}(N) \longrightarrow \mathbb{N}$ be the function which assigns to a Morse function f the difference between its number of critical points and its number of critical values. Then ψ defines a stratification of $\mathfrak{M}(N)$ by setting $\mathfrak{M}(N)_{(i)} = \psi^{-1}(i)$. It is not difficult to see that the assignment $f \longmapsto C(f)$ varies continuously if we remain inside a single stratum, for, within a connected component of a stratum, the relative arrangement of the critical values of functions is fixed. In essence, the idea then is to modify the definition of C(f) so that the cell complexes vary continuously within the closure of a stratum in $\mathfrak{M}(N)$, and to then apply the stratification theorem to glue all of the strata together.

For a function f in the closure of a particular stratum, the aforesaid modification of C(f) is obtained by generalizing the concept of level surface. We choose a collection of oriented hypersurfaces H_α having the following properties:

(i) The H_α are transverse to the 1-form df,

(ii) The H_α partition the critical points into subsets so that if f is in the stratum itself, then the H_α separate critical points having different critical heights.

(iii) If f is in the boundary of a stratum $\mathfrak{S}$, we require that the hypersurfaces H_α are chosen in such a way that a perturbation f_t of f into the interior of $\mathfrak{S}$ can be extended to a perturbation $H_\alpha(t)$ of hypersurfaces satisfying the condition that $H_\alpha(t)$ separates the critical heights of f_t.

2. Higher Reidemeister torsions. My interest in associating families of cell complexes to bundles with fibre-wise Morse function arises from the problem of defining a parametrized analogue of the classical R-torsion invariant of Franz, Reidemeister, and de Rham. This question was first raised by Wagoner [Wa].[2]

Let $\rho:\pi \longrightarrow U_r(\mathbb{C})$ be a unitary representation of the fundamental group of a manifold M. We assume that the homology of M in the local system defined by ρ is entirely vanishing; we then say that M is *acyclic* with respect to ρ (cf. [Wa]). It is in this context that the classical R-torsion is defined, and it lives in a certain quotient of the first algebraic K-group of $\mathbb{C}$.

Let k_ρ be the kernel of ρ and set $\pi_\rho = \pi/k_\rho$. Let $M(\pi_\rho)$ denote the infinite monomial matrices with coefficients in π_ρ. Then $M(\pi_\rho)$ is a subgroup of the stabilized general linear group $GL(\mathbb{C}) = \lim GL_n(\mathbb{C})$. Taking classifying spaces and then plus constructions, we get a map $BM(\pi_\rho)^+ \longrightarrow BGL(\mathbb{C})^+$. Let $Wh^\rho{}_{i+1}(\mathbb{C})$ denote the i^{th}-homotopy group of the homotopy fibre this map. For i = 0, $Wh^\rho{}_{i+1}(\mathbb{C})$ is precisely the group in which the classical R-torsion lives.

Now suppose that a smooth manifold bundle $p:E \longrightarrow B$ is given whose fibres are ρ-acyclic. We assume the structure group of p has the property that its action on the fibres of p are basepoint preserving and furthermore has the property that the induced action on fundamental groups is trivial. Suppose that a fibre-wise Morse function f on p is given. Choose a Riemannian structure on the fibres of p. If $b \in B$, let $f_b:E_b \longrightarrow \mathbb{R}$ denote the

[2] Wagoner also provides a solution to this problem in the 1-parameter case.

restriction of f to the fibre of p over b. By a *framing* ϕ_b for f at a point $b \in B$, we mean an orthonormal framing for the negative eigenspace of $D^2 f_b$ along the singularities of f_b. We shall say that (f,ϕ) is a *framed fibre-wise Morse function* on $p:E \longrightarrow B$ if a framing ϕ_b for f_b at every point $b \in B$ is given which varies continuously with respect to the parameter space B.[3]

Consider the case of a ρ-acyclic bundle over the n-sphere: $p:E \longrightarrow S^n$. Suppose a framed fibre-wise Morse function (f,ϕ) is given on p.

Theorem C. *(Higher R-torsion for framed fibre-wise Morse functions).* The triple (p,f,ϕ) determines a well defined element $\tau^\rho(p,f,\phi) \in Wh^\rho{}_{n+1}(\mathbb{C})$.

The proof of theorem C is deduced from theorem B together with a construction called *linearization* which allows one to pass from an n-parameter family of cell complexes to an element of the group $Wh^\rho{}_{n+1}(\mathbb{C})$. In the unparametrized case $n = 0$, this construction coincides with the usual construction of the R-torsion from the cellular chain complex associated to the Morse function f (assuming that f is self indexing).

3. Reidemeister torsions for all ρ-acyclic bundles. A still unsettled question is whether it is possible to define invariants τ^ρ for *all* ρ-acyclic manifold bundles over the n-sphere. I will briefly mention how one might accomplish this. Let $\mathfrak{M}^{fr}(N)$ denote the space of framed Morse functions on N with the Whitney C^∞ topology. Then $\mathfrak{M}^{fr}(N)$ is a G-space where G is the group of diffeomorphisms of N which preserve the base-point and which induce the identity on fundamental groups. The proof of theorem C above follows from the construction of a homomorphism $\tau^\rho:\pi_n(B^{\mathfrak{M}^{fr}}G) \longrightarrow Wh^\rho{}_{n+1}(\mathbb{C})$. Consequently, higher R-torsions can be defined for all bundles over spheres if τ^ρ can be extended to $\pi_n(BG)$. Let $\mathfrak{L}(N)$ be the space of *framed functions* on N, i.e., functions having only Morse and birth–death singularities together with framings of their critical points ([I1]). Igusa's *framed function theorem* says that $\mathfrak{L}(N)$ has connectivity equal to $\dim(N) - 1$. Using the G-action on $\mathfrak{L}(N)$ defined by precomposition, form the Borel construction $B^{\mathfrak{L}}G = EG\times_G \mathfrak{L}(N)$. Then $B^{\mathfrak{M}^{fr}}G$ is a subspace of $B^{\mathfrak{L}}G$, and the forgetful map $B^{\mathfrak{L}}G \longrightarrow BG$ has

[3] In terms of cell complexes, a framed function defines an explicit Euclidean coordinatization of the cells.

connectivity $\geq \dim(N)$. Suppose that τ^ρ can be extended to $\pi_n(B^{\Omega}G)$. If the dimension of N is made large by replacing N with $N \times D^k$ for $k >> n$ *(stabilization)*, then the forgetful map $B^{\Omega}G \longrightarrow BG$ becomes highly connected and therefore τ^ρ is extendable to $\pi_n(BG)$ for all n. In order to construct an extension of τ^ρ to $\pi_n(B^{\Omega}G)$, it would be sufficient to show how a bundle $p:E \longrightarrow B$ with fibre-wise framed function (f,ϕ) gives rise to a family of cell complexes $q:Y \longrightarrow B$, where it is now necessary to include Whitehead *elementary collapses* and *elementary expansions* in the transition between the fibres of $p:E \longrightarrow B$, as is reflected by the existence of birth-death singularities in the fibre-wise restrictions of f. In a future paper with K. Igusa [I-K], this program will be undertaken.[4]

4. A remark on the machinery employed. The proofs in [K] apply the language of Waldhausen's theory of *categories with cofibrations and weak equivalences* [W], which enlarges the class of categories for which one can define K-theory. We use a particular model of Waldhausen's of algebraic K-theory of a space called *the expansion space*, which is a type of moduli space for cell complexes [I-W]. The primary issue in the proof of theorem C is to pass continuously from the moduli space of Manifold, framed Morse function pairs (i.e. $B^{\mathfrak{M}^{fr}}G$) into the expansion space. We then show how to linearize from the expansion space into the algebraic K-theory of the complex numbers. The higher R-torsion is, by definition, the composition of these two constructions.

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[4] added remark: K. Igusa and I have recently shown by other methods how define higher torsion invariants for all ρ-acyclic manifold bundles.