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ASPECTS OF PARABOLIC INVARIANT THEORY

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ABSTRACT. These lectures include a brief discussion of parabolic geometries in general but are concerned primarily with conformal and CR structures. Motivated by the problems of constructing invariant operators on tensor bundles and constructing polynomial invariants of such structures, the lectures will describe basic invariant operators for each of these structures which in a certain sense are analogues of the Levi-Civita connection of Riemannian geometry. Some applications of these to the problems mentioned will also be treated. This work was presented as a series of three lectures at the 18th Winter School on Geometry and Physics, Srní, Czech Republic, January 1998.

1. INTRODUCTION

In the following lectures we will focus almost exclusively on conformal and CR structures. Conformal structures were also discussed by Michael G. Eastwood at the 15th Winter School [10]. Although there is a small overlap in coverage for the most part Eastwood's notes are complementary to the discussion here and indeed the interested reader is encouraged to review those notes.

Although I accept full responsibility for the lectures presented here much of the discussion is based on ongoing joint work with Michael G. Eastwood and C. Robin Graham and has developed from joint work with Toby N. Bailey and Michael Eastwood [1]. Other input is indicated below.

2. CONFORMAL STRUCTURES AND PARABOLIC GEOMETRIES

A Riemannian geometry consists of a smooth n-manifold M equipped with a positive definite metric g. Recall that any such metric is a smooth positive definite section of $\odot^2 T^*M$, the symmetric tensor product of the cotangent bundle to M. A (Riemannian) *conformal* n-manifold is a pair $(M, [g])$ where M is a smooth n-manifold and $[g]$ is an equivalence class of Riemannian metrics where any two metrics g and $\widehat{g}$ are said to be equivalent if $\widehat{g}$ is a positive scalar function multiple of g. It is clear that conformal manifolds are equipped with a well defined notion of angle but not length. We will consider only conformal and Riemannian structures of dimension $n \geq 3$.

We will write $\mathcal{E}^a$ and $\mathcal{E}_a$ for, respectively, the tangent and cotangent bundles to M. Tensor products of these bundles will be indicated by adorning the symbol $\mathcal{E}$ with appropriate indices. For example in this notation $\odot^2 T^*M$ is written $\mathcal{E}_{(ab)}$, where the $(\cdots)$ indicates that the enclosed indices have been symmetrised. A choice of metric is a section of $\mathcal{E}_{(ab)}$ and so will usually be written g_{ab} rather than just g. The pairing of vectors with their duals and the generalisation of this to tensors will be indicated by

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repeated indices in the usual fashion. For example $v^a u_a$ indicates a scalar field resulting from the *contraction* of the tangent vector field v^a with the 1-form field u_b. Mostly we will regard the indices as *abstract indices* and then this notation is in the framework of Penrose's abstract index notation [35]. In case a frame is chosen and the indices are concrete then this notation is according to Einstein's summation convention. Given a choice of metric, indices will be raised and lowered using this without mention.

Recall that a Riemannian manifold is naturally endowed with the Levi-Civita connection ∇_a. This is the unique torsion free connection on the tangent space and its tensor powers which preserves the metric. The curvature $R_{ab}{}^c{}_d$ of this is known as the Riemannian curvature and is defined by

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) v^c = R_{ab}{}^c{}_d v^d.$$

It is well known that this curvature is a (local) invariant of the Riemannian structure. It depends only on the underlying metric and is independent of any choice of frame or coordinates. This follows immediately from the fact that the Levi-Civita connection is a Riemannian invariant differential operator. Other invariants can be constructed using this and its covariant derivatives. For example

$$R_{abcd}R^{abcd}, \ (\nabla_a R_{bcde})(\nabla^a R^{bcde}) \ \text{ and } R_{bd} := R_{ab}{}^a{}_d$$

are other invariants. The first two are scalar invariants and the last, which is the Ricci tensor, is a tensor valued invariant. It is a classical result that all local invariants, taking values in tensor bundles (and polynomial in the jets of the metric and its inverse), arise in this way, i.e. from contractions of covariant derivatives of the curvature.

If the metric g_{ab} is replaced by $\widehat{g}_{ab} = \Omega^2 g_{ab}$, where Ω is a smooth non-vanishing function, then the connection ∇_a is replaced by the connection $\widehat{\nabla}_a$ where

$$\widehat{\nabla}_a v^b = \nabla_a v^b + \Upsilon_a v^b - \Upsilon^b v_a + \Upsilon^c v_c \delta_a^b, \tag{1}$$

with δ_a^b the Kronecker delta and $\Upsilon_a := \Omega^{-1}\nabla_a \Omega$. This transformation of g_{ab} will be described as a *conformal rescaling*. It is helpful to define line bundles $\mathcal{E}[w]$ on M as follows. The bundle whose smooth sections are metrics from the conformal class is a ray subbundle (i.e. a fibre subbundle with fibre $\mathbb{R}_+$) of $\mathcal{E}_{ab}$. We identify this subbundle with a ray bundle of scalars, which we denote $\mathcal{E}_+[-2]$. For each $w \in \mathbb{R}$ the ray bundle $\mathcal{E}_+[w]$ is then defined to be the $\left(-\frac{w}{2}\right)^{\text{th}}$ power of $\mathcal{E}_+[-2]$. Finally, for each w, $\mathcal{E}[w]$ is defined to be the canonical extension of $\mathcal{E}_+[w]$ to a line bundle. Under a conformal rescaling as above we have

$$\widehat{\nabla}_a \phi = \nabla_a \phi + w \Upsilon_a \phi,$$

for $\phi \in \mathcal{E}[w]$.

The Riemannian curvature is not invariant under conformal rescaling and the above Riemannian invariants do not give meaningful information about the conformal structure. It is useful to observe that R_{abcd} can be decomposed into the totally trace-free Weyl curvature C_{abcd} and a remaining part described by the symmetric *Rho-tensor* P_{ab} according to

$$R_{abcd} = C_{abcd} + 2g_{c[a}\mathsf{P}_{b]d} + 2g_{d[b}\mathsf{P}_{a]c},$$

where $[\cdots]$ indicates the skew part over the enclosed indices. The Rho-tensor is a trace modification of the Ricci tensor. It is easily verified explicitly that, in fact, C_{abcd} is invariant under conformal rescaling. However, while this is a useful observation the essential problem remains since, for example, $\nabla_a C_{bcde}$ is not conformally invariant and so it is not clear, at this point, how to construct higher order invariants of the conformal structure.

A related problem is the construction of linear invariant differential operators acting between tensor bundles. Again in the Riemannian case such operators can all be described in terms of the Levi-Civita connection. The key here is that, given a tensor field f, the list $f, \nabla_a f, \nabla_a \nabla_b f, \cdots$ is essentially equivalent to the infinite jet of f at each point of M. Thus we may as well decree at the outset that we are interested in operators which are linear in these ∇ derivatives of f. Each tensor $\nabla_a \cdots \nabla_b f$ may be regarded as a function on the principal bundle of orthonormal frames taking values in a finite dimensional representation of $\mathrm{O}(n)$. Since $\mathrm{O}(n)$ is reductive such representations are completely decomposable and thus all possible linear operators can be described explicitly.

This breaks down in the conformal case since the operators $\nabla_a \cdots \nabla_b f$ are not in general conformally invariant. Clearly it would be desirable if there were some conformally invariant analogue of the Levi-Civita connection which packaged the jet information of tensor fields into representations of a reductive Lie group. In fact such an operator exists and there are analogous operators for CR structures and these will be described explicitly below. The definitions of these operators are motivated by the flat models of the structures concerned so let us briefly review the flat model for conformal structures.

Let $\mathbb{T}$ denote $\mathbb{R}^{n+2}$ equipped with a symmetric bilinear form h of signature $(n+1,1)$, given in block form by

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathrm{Id}_{n\times n} & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and coordinates X^A. We use h to raise and lower the indices of $\mathbb{T}$, for example $X_A = h_{AB}X^A$. The space of generators of the null cone of h is the n-sphere S^n – this is our candidate for the flat model. Since the null cone is given by the vanishing of $Q := h_{AB}X^A X^B$ then on the null cone $\frac{1}{2}\partial/\partial X^A Q = X_A$ is orthogonal to the null cone. Thus tangent vectors to the null cone can be identified with vectors v^A in $\mathbb{T}$ such that $X_A v^A = 0$. Functions on S^n may be identified with functions on the null cone homogeneous of degree 0, that is functions f such that $f(\lambda X^A) = f(X^A)$. Since, as an operator on such functions, $\partial_A := \partial/\partial X^A$ lowers the homogeneity by 1, vector fields on S^n may be represented by vector fields v^A, on the null cone where v^A is homogeneous of degree 1 (and $v^A X_A = 0$). In fact we are free add multiples of X^A to v^A since the Euler vector field, $X^A\partial_A$, annihilates the functions of S^n. Write $\mathcal{E}^A$ for the tangent bundle to $\mathbb{T}$ restricted to the null cone and $\mathcal{E}^A(1)$ for this tensored with functions homogeneous of weight 1, $\mathcal{E}(1)$. Then in summary,

$$\mathcal{E}^a_{S^n} \cong \{v^A \in \mathcal{E}^A(1) \ : \ v^A X_A = 0\}/\sim,$$

where $\sim$ indicates the freedom to add to each v^A multiples fX^A for $f \in \mathcal{E}(0)$. Using this, h_{AB} determines a conformal metric on S^n by

$$v^a \mapsto h_{AB}v^A v^B.$$

It is easily verified that the right hand side is well defined. Since $h_{AB}v^Av^B$ is homogeneous of degree 2 a metric is determined by choosing a non-vanishing section of the bundle of functions homogeneous of degree -2, $\xi \in \Gamma\mathcal{E}(-2)$, and taking the quadratic form given by

$$v^a \mapsto \xi h_{AB}v^Av^B.$$

We may also deduce from this that, as a bundle on S^n, $\mathcal{E}(-2)$ is the same as $\mathcal{E}[-2]$.

It is readily verified that such a metric is conformally flat (i.e. locally there is a conformal transformation which takes this to the flat metric). S^n equipped with a conformal metric in this way is the *standard flat model* for a conformal structure. Since each point of the the null cone determines a metric for the corresponding point of S^n, the null cone (with origin removed) is essentially the total space of the bundle of metrics over S^n. (More precisely the future null cone gives a square root of this.)

Let G denote the identity connected component of $\mathrm{O}(h)$. This acts transitively on the rays of the null cone, so the conformal S^n may be identified with G/P where P is the parabolic subgroup of G consisting of all elements of the block form

$$\begin{pmatrix} \lambda^{-1} & 0 & 0 \\ r & m & 0 \\ -\lambda r^t r/2 & -\lambda r^t m & \lambda \end{pmatrix} \tag{2}$$

with $r \in \mathbb{R}^n$, $m \in \mathrm{SO}(n)$ and where r^t denotes the transpose of r. P stabilises the null ray through the point

$$e = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

on the null cone. Since the flat model is a homogeneous space let us review some properties of these and and consider their generalisations.

3. Homogeneous structures and Cartan Connections

Let G be a Lie group and H any subgroup. Recall that as a vector space the Lie algebra $\mathfrak{g}$ of G is just the tangent space to G at the identity, $\mathfrak{g} = T_eG$. This tangent space is then identified with the space of left invariant vector fields via

$$(\zeta_X f)(g) := \left[\frac{d}{dt}f(ge^{tX})\right]_{t=0},$$

for differentiable functions f on G. The space of left invariant vector fields is closed under commutation and the Lie bracket on $\mathfrak{g}$ is defined to agree with the commutator of the elements regarded as left invariant vector fields. This is consistent with the adjoint action of G on $\mathfrak{g}$.

Functions on G/H may clearly be identified with functions on G that are constant up the fibres of $G \to G/H$. That is, functions annihilated by $Y \in \mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of H identified with the appropriate subalgebra of $\mathfrak{g}$. For such a function and $h \in H$

$$\left[\frac{d}{dt}f(ge^{tX})\right]_{t=0} = \left[\frac{d}{dt}f(ghe^{t\mathrm{Ad}(h^{-1})X})\right]_{t=0}$$

It follows then that the total space of the tangent bundle $T(G/H)$ to G/H may be identified with the quotient space,

$$G \times_H \frac{\mathfrak{g}}{\mathfrak{h}} := G \times \frac{\mathfrak{g}}{\mathfrak{h}} / \sim,$$

where the equivalence relation is

$$(g, X) = (gh, Ad(h^{-1})X)$$

and Ad now indicates the representation of H on $\mathfrak{g}/\mathfrak{h}$ determined by the adjoint representation of G on $\mathfrak{g}$.

More generally corresponding to each representation μ of H on a vector space V one obtains a homogeneous bundle $\mathcal{V} = (G \times_H V \to G/H)$ the total space of which is $G \times_H V$, that is $G \times V$ factored by the equivalence relation

$$(g, v) = (gh, \mu(h^{-1})v).$$

Sections of $\mathcal{V}$ are functions

$$f : G \to V$$

satisfying the homogeneity condition

$$f(gh) = \mu(h^{-1})f(g).$$

We will use the notation $\mathcal{E}\mathcal{V}$, or $\mathcal{E}(\mathcal{V})$, to mean the sheaf of germs of smooth sections of $\mathcal{V}$. By a slight abuse of notation we will also use this notation to mean simply local sections.

If W carries a representation ρ of G then the homogeneous bundle

$$\mathcal{W} := G \times_H W$$

is trivial. The mapping giving

$$(G/H) \times W \cong G \times_H W$$

is

$$(gH, \tilde{w}) \leftrightarrow (g, w) \text{ where } \tilde{w} = \rho(g)w.$$

It is easily checked that this is well defined. It follows that local sections can be identified,

$$\mathcal{E}((G/H) \times W) \cong \mathcal{E}(G \times_H W), \tag{3}$$

by

$$\mathcal{E}((G/H) \times W) \ni \tilde{w} \leftrightarrow w \in \mathcal{E}(G \times_H W)$$

where $\tilde{w}(g) = \rho(g)w(g)$.

Since $G \times_H W$ is trivial it admits a flat connection determined by differentiation with the left invariant vector fields on its trivialisation $(G/H) \times W$,

$$(\nabla_X w)(g) := \rho(g^{-1})\zeta_{X_g}\tilde{w}(g). \tag{4}$$

On the right hand side we have chosen a function $X_{(\cdot)} : G \to \mathfrak{g}$ satisfying $X_{gh} = \mathrm{Ad}(h^{-1})X_g$ to represent the tangent vector field X (on G/H). Note that it is immediate from the definition of $\nabla_X w$ that

(a) $\nabla_X w = 0$ if X takes values in $\mathfrak{h}$.

– so $\nabla_X w$ is independent of the function $X_{(\cdot)}$ chosen to represent the tangent section X. Observe also that

(b) $(\nabla_X w)(gh) = \rho(h^{-1})(\nabla_X w)(g)$
– so $(\nabla_X w)$ is a section of $G \times_H W$. This property (b) is verified by the following calculation:

$$\begin{aligned}
\text{l.h.s} &= \rho((gh)^{-1}) \left[\frac{d}{dt}\tilde{w}(ghe^{tX_{gh}})\right]_{t=0} \\
&= \rho(h^{-1})\rho(g^{-1}) \left[\frac{d}{dt}\tilde{w}(ghe^{t\mathrm{Ad}(h^{-1})X_g})\right]_{t=0} \\
&= \rho(h^{-1})\rho(g^{-1}) \left[\frac{d}{dt}\tilde{w}(ge^{tX_g})\right]_{t=0} \\
&= \text{r.h.s.}
\end{aligned}$$

These properties show that ∇ is a well defined first order differential operator taking values in 1-forms. Finally note that it also follows from the definition that, for f a function on G/H,
(c) $\nabla f w = f \nabla w + df \otimes w$
and so ∇ is a connection.

Using that $\tilde{w}(g) = \rho(g)w(g)$ we may expand out the right hand side of (4) to obtain

$$(\nabla_X w)(g) = \zeta_{X_g} w(g) + \rho(X_g)w(g). \tag{5}$$

Of course we can take this as the definition of ∇ and verify properties (a),(b) and (c) directly using this. One then observes that we do not require the triviality of the structure to obtain these properties. In fact one just needs:
(a) $\zeta_X f(g) = \left[\frac{d}{dt} f(ge^{tX})\right]_{t=0}$ if $X \in \mathfrak{h}$ (i.e. that ζ_X is a fundamental vector field of $G \to G/H$).
(b) $\zeta_{X_{gh}} = \mathrm{Ad}(h^{-1})X_g$ for all $h \in H$.
(c) $\zeta_{X_g} : \mathfrak{g} \xrightarrow{\cong} T_g G$ as vector spaces.
and that ρ is a $(\mathfrak{g}, H)$-representation. Dually the only properties of the Maurer-Cartan form ω (recall this is the canonical form such that $\omega(\zeta_X) = X$ for all $X \in \mathfrak{g}$) that are used are
(a) $\omega(\zeta_X) = X$ for $X \in \mathfrak{h}$,
(b) $(R_h)^*\omega = \mathrm{Ad}(h^{-1})\omega$ for all $h \in H$,
(c) For all $g \in G$, $\omega : T_g G \xrightarrow{\cong} \mathfrak{g}$ as vector spaces.

It follows then that if, more generally, we have a manifold M and a bundle $\mathcal{G} \to M$ with fibre H and a $\mathfrak{g}$ valued 1-form ω satisfying (a),(b) and (c) (except that G is replaced with $\mathcal{G}$ in (c)) then (5) gives a connection on the induced bundle $\mathcal{G} \times_H W$. W here is a $(\mathfrak{g}, H)$-module. In this case ω is called a *Cartan connection* and ∇ is the corresponding induced connection on $\mathcal{G} \times_H W$. We will term $\mathcal{G}$ the *Cartan bundle* for M. Of course one gets an associated bundle $\mathcal{V}$ (but not in general a connection) for any representation ρ of H on a space V. Sections of $\mathcal{V}$ are functions

$$v : \mathcal{G} \to V$$

satisfying the homogeneity condition

$$v(gh) = \rho(h^{-1})v(g). \tag{6}$$

These observations lead us the idea of specifying $\mathcal{G} \to M$ and its Cartan connection ω as a means of describing geometries which are deformations of homogeneous (or *flat*) structures G/P. In this picture the *parabolic geometries* of interest in the general programme are geometries with a canonical Cartan connection ω taking values in $\mathfrak{g}$ where $\mathfrak{g}$ is the Lie algebra of a simple Lie group G and where the fibre of $\mathcal{G} \to M$ is a parabolic subgroup P of G (that is the complexification of the Lie algebra $\mathfrak{p}$ of P is a parabolic subalgebra of the complexification of $\mathfrak{g}$). This includes a large class of structures (see, for example, [8, 9] and, in particular, [7] for details of a large subclass) and the canonical connections are called normal Cartan connections. In these lectures we shall be concerned only with conformal and CR geometries. In the case of conformal structures the normal Cartan connection has been described explicitly for some time [31]. For CR structures it was independently discovered by Tanaka [38], and Chern and Moser [6].

4. Invariant Calculus for Conformal Structures

The above observations suggest that the invariant theory of parabolic geometries is simplified if we deal with bundles induced from representations of the appropriate simple group G, or at least $(\mathfrak{g}, P)$-representations, rather than say irreducible representations of the parabolic structure group P. Then, as observed above, the corresponding associated bundles have a connection induced from the Cartan connection of $\mathcal{G}$. From an algebraic point of view it is also clearly an advantage to work with G-modules as the theory of finite dimensional representations of simple groups is understood. Thus we will proceed with such a programme for conformal structures. However, we must of course bear in mind that in the end we must learn to deal with a large class of bundles which are induced from P-modules that do not extend to $(\mathfrak{g}, P)$-modules. The tangent bundle is one example.

We will not deal here with the details of how the Cartan bundle $\mathcal{G}$, for a conformal manifold M, is constructed (see [31] or [9] for the details of this). It is sufficient for our purposes to know of its existence. We will deal in more detail with an induced bundle which we now describe. Recall $\mathbb{T}$ was introduced above to denote $\mathbb{R}^{n+2}$ equipped with an $(n+1,1)$ bilinear form h. Clearly $\mathbb{T}$ is a representation space for G, the identity connected component of $\mathrm{O}(h)$. Let us write $\mathcal{E}^A$, which is called the *tractor bundle*, for the induced bundle $\mathcal{G} \times_P \mathbb{T}$ and $\mathcal{E}_A$ for the dual co-tractor bundle. Since sections of $\mathcal{E}^A$ are functions from $\mathcal{G}$ to $\mathbb{T}$, satisfying (6) (where now $H = P$ as in (2)), it is clear that h determines a *tractor metric*, which we will also denote by h. This is given by $h(v,w)(u) = h(v(u), w(u))$ for $v, w \in \Gamma(\mathcal{E}^A)$ and $u \in \mathcal{G}$ or, adorning the sections with abstract indices, we would write $h_{AB}v^A w^A$. We will describe sections of tensor products of the tractor bundle and its dual, such as h_{AB}, as tractor fields. These may be weighted, for example $\mathcal{E}^A{}_B[2] = \mathcal{E}^A \otimes \mathcal{E}_B \otimes \mathcal{E}[2]$.

Since P preserves the subspace E of $\mathbb{T}$ spanned by e it follows that there is a corresponding line subbundle of $\mathcal{E}^A$ which, by some elementary representation theory, is naturally identified with $\mathcal{E}[-1]$ and we write X^A for the preferred section of $\mathcal{E}^A[1]$ which gives the injecting morphism

$$\mathcal{E}[-1] \to \mathcal{E}^A$$

by
$$\rho \mapsto \rho X^A.$$
Similarly since P preserves the subspace of elements of the form
$$\begin{pmatrix} 0 \\ a \\ \vdots \\ b \end{pmatrix},$$
for $a, \cdots, b \in \mathbb{R}$, there is a natural bundle surjection
$$\mathcal{E}^A \to \mathcal{E}[1]$$
given by
$$U^A \mapsto U^A X_A$$
where $X_A := h_{AB} X^B$. (We will use h_{AB} and its inverse to raise and lower indices in this fashion without further mention.) In fact there is a composition series
$$\mathcal{E}^A = \mathcal{E}[1] + \mathcal{E}^a[-1] + \mathcal{E}[-1]. \tag{7}$$
If one chooses a section B_A of $\mathcal{E}_A$, such that $B_A B^A = 0$ and $\xi := B_A X^A$ is non-vanishing, then this composition series splits since $\xi_A := \xi^{-1} B_A$ may be used to reverse the above maps in the obvious way. For example $\mathcal{E}[1] \to \mathcal{E}^A$ by $U \mapsto U\xi^A$. It is an elementary exercise to show that such a choice of tractor field B_A is precisely equivalent to a choice of metric from the equivalence class (cf. the discussion of the flat model above). Thus we write
$$[\mathcal{E}^A]_g = \mathcal{E}[1] \oplus \mathcal{E}^a[-1] \oplus \mathcal{E}[-1] \tag{8}$$
where the $[\cdot]_g$ indicates that the enclosed bundle has been split by the choice of the metric g. We may use this notation for sections also
$$[U^A]_g = \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix} \in [\Gamma \mathcal{E}^A]_g$$
If the metric is given and understood then we may drop the $[\cdot]_g$. Under conformal rescaling the quantities above transform according to
$$\text{If } [U^A]_g = \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix}$$
then
$$[U^A]_{\widehat{g}} = \begin{pmatrix} \widehat{\sigma} \\ \widehat{\mu}^a \\ \widehat{\rho} \end{pmatrix} = \begin{pmatrix} \sigma \\ \mu^a + \Upsilon^a \sigma \\ \rho - \Upsilon_b \mu^b - \frac{1}{2}\Upsilon^b \Upsilon_b \sigma \end{pmatrix}. \tag{9}$$
Note that
$$[X^A]_g = [X^A]_{\widehat{g}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$
One can use (8) and (9) as the definition of the tractor bundle. This is the point of view in [1] for example. In any case these formulae enable explicit computations in

terms of the tangent bundle and Levi-Civita connection and so forth. For example the *tractor connection* ∇, that is the connection on $\mathcal{E}^A$ induced from the Cartan connection on $\mathcal{G}$, is given explicitly by the formula

$$\nabla_j \begin{pmatrix} \sigma \\ \mu^i \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_j \sigma - \mu_j \\ \nabla_j \mu^i + \delta_j{}^i \rho + \mathsf{P}_j{}^i \sigma \\ \nabla_j \rho - \mathsf{P}_{ji} \mu^i \end{pmatrix}.$$

One may readily check directly that this is conformally invariant by using (9) and (1). This calculation is given explicitly in [10]. This definition, due to T.Y. Thomas [39], was discovered independently although slightly later than Cartan's conformal connection on $\mathcal{G}$.

Of course the tractor connection ∇ extends to be a connection on tensor products of the tractor and co-tractor bundles by requiring ∇ to satisfy a Leibniz rule over tensor products. It is a useful exercise to verify that ∇ preserves the tractor metric. Note however that we cannot use ∇ repeatedly since, for example, if $U^A \in \Gamma\mathcal{E}^A$ then $\nabla_a U^A$ is not itself a section of a tractor bundle. Of course this problem would be resolved if $\nabla_a U^A$ could be identified with a section of some appropriate tractor bundle. Another 'problem' with ∇ is that it is not invariant on tractor bundles of weight $w \neq 0$. In fact if f is a tractor field of weight w then one has (under conformal rescaling) $\widehat{\nabla}_a f = \nabla_a f + w\Upsilon_a f$ since, on $\mathcal{E}[w]$, ∇_a is just the Levi-Civita connection. As we shall see we can deal with both of these points.

For each choice of metric g from the conformal class define an operator $\tilde{D}_A$ by

$$[\tilde{D}^A f]_g := \begin{pmatrix} wf \\ \nabla^a f \\ 0 \end{pmatrix}$$

where f is a function of weight w, or is a section of a weight w tractor bundle but has indices suppressed. Let us compare $\tilde{D}_A$ for the metric g with $\widehat{\tilde{D}}_A$ for the metric $\widehat{g}$.

$$\begin{aligned} [\widehat{\tilde{D}^A f}]_{\widehat{g}} &:= \begin{pmatrix} wf \\ \widehat{\nabla}^a f \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} wf \\ \nabla^a f + w\Upsilon^a f \\ 0 \end{pmatrix} \end{aligned}$$

On the other hand

$$\begin{aligned} [\tilde{D}^A f]_{\widehat{g}} &= \begin{pmatrix} wf \\ \nabla^a + w\Upsilon^a f \\ -\Upsilon_b \nabla^b f - \frac{1}{2} w \Upsilon_b \Upsilon^b f \end{pmatrix} \\ &= [\widehat{\tilde{D}}^A f - X^A(\Upsilon_b \nabla^b f + \tfrac{1}{2} w \Upsilon_b \Upsilon^b f)]_{\widehat{g}}. \end{aligned}$$

So

$$\widehat{\tilde{D}_A f} = \tilde{D}_A f + X_A(\Upsilon^i \nabla_i f + \frac{w}{2} \Upsilon^i \Upsilon_i f).$$

Thus although this shows that the operator $\tilde{D}_A$ is not invariant it follows immediately from this transformation formula that the operator

$$D_{AP} : \mathcal{E}^*[w] \to \mathcal{E}_{[AP]} \otimes \mathcal{E}^*[w]$$

given by,

$$D_{AP}f := 2X_{[P}\tilde{D}_{A]}f \qquad (10)$$

is invariant. Here the $*$ indicates any tractor indices, so f in (10) is a weighted tractor. For a given choice of metric this may be expressed

$$D^{AP}f = \begin{array}{c} \leftarrow \quad\quad P \quad\quad \rightarrow \\ \begin{pmatrix} 0 & 0 & wf \\ 0 & 0 & \nabla^a f \\ -wf & -\nabla^p f & 0 \end{pmatrix} \end{array}. \qquad (11)$$

The operator D_{AP} is a reasonable candidate for a conformal analogue of the Levi-Civita connection. As observed it is invariant on weighted tractors. It is a first order operator and, furthermore, it is clear from (11) that, at any point and, for f of any weight, the list $f, D_{AP}f, D_{AP}D_{BQ}f, \cdots$ has all the information of the infinite jet of f. Also D_{AP} satisfies a Leibniz rule. To show this it is clearly sufficient to show that $\tilde{D}_A$ behaves as a derivative. If f_1 and f_2 are section of tractor bundles of respective weights w_1 and w_2 then

$$\begin{aligned} \tilde{D}^A(f_1f_2) &= \begin{pmatrix} (w_1+w_2)f_1f_2 \\ \nabla^a(f_1f_2) \\ 0 \end{pmatrix} = \begin{pmatrix} (w_1f_1)f_2 + f_1(w_2f_2) \\ f_2\nabla^a f_1 + f_1\nabla^a f_2 \\ 0 \end{pmatrix} \\ &= f_2\begin{pmatrix} w_1f_1 \\ \nabla^a f_1 \\ 0 \end{pmatrix} + f_1\begin{pmatrix} w_2f_2 \\ \nabla^a f_2 \\ 0 \end{pmatrix} \\ &= f_1\tilde{D}^A f_2 + f_2\tilde{D}^A f_1, \end{aligned}$$

as required. Finally note that it is easily verified directly that D_{AP} preserves the tractor metric, $D_{AP}h_{BC} = 0$, thus the action of D_{AP} commutes with raising and lowering of indices. These properties are each analogous to properties of the Riemannian metric connection. On the other hand there remains the problem of employing this operator on non-tractor bundles and extracting operators which take values in irreducible bundles. These are essentially algebra problems and dealing with them in general involves understanding how certain P-modules arise in the composition series of the G-modules inducing the tractor bundles. In general this is a hard problem and not one we will deal with directly. Rather we will describe some general procedures for extracting invariants and invariant operators from the tractor calculus without a direct confrontation with the representation theory.

4.1. Some tractor calculus. As an elementary example of using the invariant operator D_{AP} to construct other operators and objects consider $h^{AB}D_{A(Q}D_{|B|P)_0}f$ for f some weighted tractor, where $(\cdots)_0$ indicates the symmetric trace-free part with respect to the enclosed indices. A straightforward calculation using the definitions above gives

$$D_{AP}X_B = 2X_{[P}h_{A]B}.$$

Using this and expanding out the above using the definition (10) of D_{AP} we see that

$$h^{AB}D_{A(Q}D_{|B|P)_0}f = -X_{(Q}D_{P)_0}f \tag{12}$$

where $D_P : \mathcal{E}^*[w] \to \mathcal{E}^*[w-1]$ is some differential operator which is clearly also invariant on weighted tractors.

An explicit formula for D_P is easily extracted from (12),

$$D_A f = (n+2w-2)\tilde{D}_A f - X_A \Box f, \tag{13}$$

where,

$$\Box f := \tilde{D}_P \tilde{D}^P f = \nabla_p \nabla^p f + w\mathsf{P} f \tag{14}$$

and P is the trace of P_{ab}. From this formula it is easily verified that D_A is in fact precisely Thomas's D-operator as in [1, 40]. Note the useful identities

$$X^A D_A f = w(n+2w-2)f$$

and

$$D_A X^A f = (n+2w+2)(n+w)f \tag{15}$$

for f a weighted tractor of weight w.

The commutators of the various invariant operators above produce invariant curvature objects. The curvature of the tractor connection ∇ on $\mathcal{E}^C$, $\Omega_{ab}{}^C{}_L$, is defined by

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)U^C = \Omega_{ab}{}^C{}_L U^L,$$

and, given a choice of conformal scale, $\Omega_{ab}{}^K{}_L$ is represented by

$$\begin{pmatrix} 0 & 0 & 0 \\ 2\nabla_{[a}P_{b]}{}^k & C_{ab}{}^k{}_l & 0 \\ 0 & -2\nabla_{[a}P_{b]l} & 0 \end{pmatrix}.$$

The vanishing of this is precisely equivalent to the structure being locally equivalent to the flat model. Using this it is straightforward to show that *on flat structures* $[D_A, D_P] = 0$ as an operator on any weighted tractor bundle. Furthermore in the general case (that is on non-flat or curved structures) $[D_A, D_P]f = 0$ if f has no indices, i.e.,

$$[D_A, D_P]f = 0 \quad \text{for} \quad f \in \Gamma\mathcal{E}[w]. \tag{16}$$

Given a choice of metric write $\Omega_{AB}{}^C{}_L$ for the tractor object

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \Omega_{ab}{}^C{}_L & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Of course this is not invariant but, in view of the composition series (7), it follows that $X_{[A}\Omega_{BC]}{}^K{}_L$ is invariant and therefore so is

$$W_{AB}{}^C{}_L := \frac{3}{n-2} D^K X_{[K}\Omega_{AB]}{}^C{}_L.$$

Explicitly this is given by

$$W_{AB}{}^{C}{}_{L} = (n-4)\Omega_{AB}{}^{C}{}_{L} + 2X_{[A}\tilde{D}^{K}\Omega_{B]K}{}^{C}{}_{L}, \tag{17}$$

so, for $n \neq 4$ this extends $\Omega_{ab}{}^{K}{}_{L}$ to an invariant tractor object. This also turns up in the commutator of the tractor D,

$$[D_A, D_B]v^C = (n-2)W_{AB}{}^{C}{}_{L}v^L + 4(n-2)X_{[A}\Omega_{B]K}{}^{C}{}_{L}\tilde{D}^K v^L,$$

for $v^C \in \Gamma\mathcal{E}^C[0]$, from which one can also deduce its invariance.

4.2. **Invariant powers of the Laplacian.** *Joint work with Michael Eastwood.* The construction of linear conformally invariant differential operators is an area of considerable interest [3, 4, 16, 20, 28, 32, 37, 41]. Some uses of the tractor D operator to manufacture such operators were described in Eastwood's notes [10], in particular the role of D_A in the powerful *curved translation procedure* of Eastwood and Rice [16] was indicated there. In the following we will focus on a more elementary use of D_A which directly exploits the tractor calculus.

It is immediately clear from (13) that if f is a tractor of weight $1-\frac{n}{2}$ then $\Box f$ is invariant. In the case that f is just a weighted function this is the usual conformally invariant Laplacian or Yamabe operator. However the ∇_p in (14) means the coupled tractor-Levi-Civita connection and our derivation of this operator from D_{AP} proves that $\Box f$ is invariant even if f has tractor indices. For example, if $f_{AB} \in \Gamma\mathcal{E}_{AB}[1-\frac{n}{2}]$ then $\Box f_{AB}$ is invariant. Following Eastwood [10] we will thus say $\Box$ is *strongly* invariant since it remains invariant when acting on tractor fields rather than just weighted scalar functions.

Now consider $D_A D_B f$ where $f \in \Gamma\mathcal{E}[2-\frac{n}{2}]$. Since $[D_A, D_B]f = 0$ we have $D_{[A}D_{B]}f = 0$ (where, as usual, $[\cdots]$ indicates the skew part over the enclosed indices). On the other hand $D_B f$ has weight $1-\frac{n}{2}$. So

$$0 = D_{[A}D_{B]}f = X_{[A}\Box D_{B]}f$$

and it follows immediately that

$$D_A D_B f = X_A X_B \Box_2 f \tag{18}$$

where $\Box_2 f$ is an invariant differential operator. We can deduce immediately from (13) that this is fourth order with leading term $\Delta^2 f$. Of course $\Box_2 f$ is the well known operator which was apparently first introduced by Paneitz [34], but also independently discovered by Riegert [36] and Eastwood and Singer [17],

$$\begin{aligned}\Box_2 f &= \Delta^2 f + 4\mathrm{P}^{ab}\nabla_a\nabla_b f - (n-2)\mathrm{P}\Delta f - (n-6)(\nabla^a\mathrm{P})\nabla_a f \\ &\quad + (n-4)(\frac{n}{4}\mathrm{P}^2 - \mathrm{P}^{ab}\mathrm{P}_{ab} - \frac{1}{2}\Delta\mathrm{P})f.\end{aligned}$$

This argument does not directly generalise to the higher order powers of the Laplacian. Note however that our calculations have shown that

$$\Box D_B f = -X_B \Box_2 f.$$

Thus using (15)

$$D^B \Box D_B f = (n-4)\Box_2 f.$$

Thus the formula on the left recovers $\Box_2$ except when $n = 4$. This suggests using

$$D^I \cdots D^B \Box \underbrace{D_B \cdots D_I}_{k-1} f$$

to recover the higher order analogues of the $\Box_2$ operator. If $f \in \Gamma\mathcal{E}[k - \frac{n}{2}]$ this is certainly invariant, since, recall each D_P lowers the weight by 1, so $D_B \cdots D_I f$ has weight $1 - \frac{n}{2}$. Now let us consider the case that the structure is (conformally) flat. If the tractor connection is flat, then

$$X_{[A}\Box D_{|B} \cdots D_{E|}D_{F]}D_G \cdots D_I f = D_{[A}D_{|B} \cdots D_{E|}D_{F]}D_G \cdots D_I f = 0,$$

since the tractor D operators commute in the flat case. Thus, in this case

$$\Box D_B \cdots D_I f = X_B \cdots X_I(\Delta^k f + \text{lower order terms}).$$

It then follows easily by repeated use of (15) that

$$D^I \cdots D^B \Box D_B \cdots D_I f = \Pi_{i=1}^{k-1}(i-k)(n-2i-2)\Delta^k f + \text{lower order terms}.$$

However the latter result must also hold in the curved case since an explicit calculation to verify this would be the same in the curved case except that curvature terms could arise upon commuting covariant derivatives (and it is easy to see these curvature terms must be of lower order in f). Thus $D^I \cdots D^B \Box D_B \cdots D_I f$ gives higher order analogues of $\Box_2$ except when n is even and $n \leq 2k$. This gives these operators as explicit formulae. (However to expand these formulae in terms of the Levi-Civita connection and curvature terms is extremely tedious.) The observation that the formulae $D^I \cdots D^B \Box D_B \cdots D_I f$ would recover these invariant operators was first made by Eastwood [11]. Eastwood also observed that it is immediately clear from these formulae that the operators recovered are strongly invariant. This is because the operators D_A are invariant on weighted tractors and, as observed above, $\Box$ is invariant on tractor fields of weight $1 - \frac{n}{2}$. The existence of these invariant operators (for $k \geq 2$) is due to C.R. Graham, R. Jenne, L. Mason and G. Sparling [25]. In fact they also show the existence of another in the series, namely an operator of the form $\Delta^k f +$ lower order terms for f a (weight 0) function and $2k = n$ the dimension of the manifold. In fact one of these was recovered above. Viz $\Box_2 f$ in dimension 4. Note from the recovery of this as in (18) we *cannot* conclude that $\Box_2$ is strongly invariant since the argument there used that $[D_A, D_B]f = 0$ which is not the case if f is allowed to take values in a weighted tractor bundle. In fact $\Box_2$ is not strongly invariant in dimension 4, see [10] for an explicit proof of this. It seems likely that an adaption of the argument which led to (18) will produce the conformally invariant operators $\Delta^{\frac{n}{2}} f +$ lower order terms [13]. If so this would enable the family of operators to be put in a special self-adjoint form that would have applications in spectral theoretic questions [5] (See also [12] for some progress on this problem).

It is worth observing at this point that C.R. Graham has shown that in dimension there is no conformally invariant operator with principal part $\Delta^3 f$ [26] (even though there is on the conformally flat structures). It seems likely that in even dimensions $n = 2k$ there will in general be no conformally invariant linear differential operators with principal part Δ^ℓ for $\ell > k$.

4.3. Invariants of conformal structures and polynomial invariants. There has been considerable interest in the programme of constructing invariants of parabolic structures since Fefferman initiated the programme by attempting to describe the local scalar invariants of CR structures [18]. The programme was expanded in [19] to include conformal geometries. These both involve embedding the structure in a higher dimensional structure which is equipped with a metric. Then invariants of the original parabolic structures can be obtained as linear combinations of "complete contractions" of the curvature tensor, and its covariant derivatives, of the ambient classical structure. For even dimensional conformal structures and CR structures this ambient metric construction is obstructed at finite order. For the cases where this works there remains the algebraic problem of determining to what extent all invariants arise via these complete contractions. This problem was essentially solved by Bailey, Eastwood and Graham in [2]. It follows from this, for example, that all invariants of odd dimensional conformal structures arise from the complete contractions alluded to. The tractor calculus offers an avenue to avoid the problem of the obstruction to the ambient metric constructions. In [21] is described a complete invariant theory for projective geometries via tractor calculus techniques. However this is not a true test case in the sense that were one to treat the projective structures along the lines of the Fefferman-Graham approach then there would be no obstruction to deal with. Nevertheless one can use the tractor calculus to produce invariants "beyond the obstruction" [22]. Let us sketch here some of the ideas involved.

Just as ∇_a and the Riemannian curvature R_{abcd} may be used to construct Riemannian invariants, analogous complete or partial contractions involving the operator D_A and the tractor object W_{ABCD}, introduced above, may be used to construct invariants of a conformal structure. For example corresponding to the Riemannian invariant $\nabla_i\nabla_j(R^{cdik}R_{cd}{}^j{}_k)$ we may write the conformal invariant

$$D_I D_J(W^{CDIK}W_{CD}{}^J{}_K).$$

In fact using (17) and a calculation by Graham [27, 1], one obtains that this is of the form

$$\frac{1}{2}(n-8)(n-4)^2((n-6)\mathrm{FG} + \text{constant} \times C^{ab}{}_{de}C^{fg}{}_{ab}C^{de}_{fg}),$$

where FG indicates an invariant that Fefferman and Graham obtained in [19]. Note that although, by construction, $D_I D_J(W^{CDIK}W_{CD}{}^J{}_K)$ is clearly invariant in all dimensions, it goes wrong in dimensions 4,6 and 8 in the sense that the order of the invariant drops in these dimensions. This is no surprise as invariants constructed in this way using just D_A and W_{ABCD} are closely related to the invariants obtained by the Fefferman-Graham construction [19, 2]. However, as mentioned above, the latter is obstructed at finite order in even dimensions.

This problem can be circumvented, at least partially, by direct use of the operator D_{AP} to construct *quasi-Weyl* invariants. It is awkward to discuss these in the context of invariants of structure so let us consider the construction of invariants of sections of $\mathcal{E}[1-\frac{n}{2}]$. This problem in the case of conformally flat structures is an interesting model problem since the general problem is not amenable to treatment by "harmonic extension" as in [14, 2]. For our current discussion there is no need, at this point, to imagine that the structure is flat. For $f \in \mathcal{E}[1-\frac{n}{2}]$ consider the expression

$D^P D^Q g^{AB} f^2 D_{A(P} D_{|B|Q)_0} f$. It is straightforward to show that

$$D^P D^Q g^{AB} f^2 D_{A(P} D_{|B|Q)_0} f = (2-n)^2(3-n)(4-n) f^2 \Box f, \tag{19}$$

where, recall,

$$\Box f := \nabla_a \nabla^a f + (1 - \frac{n}{2}) \mathsf{P} f.$$

Note that (19) vanishes in dimension 3 and 4. We can improve on this result. By an easy calculation one obtains that, in any dimension,

$$g^{AB} f^2 D_{A(P} D_{|B|Q)_0} f = X_P X_Q f^2 \Box f.$$

So $f^2 \Box f$ is an invariant for all n. However this last result depends crucially on the fact that, for each n, the weight of f is $1 - n/2$. In contrast using (12) we have that

$$g^{AB} f^2 D_{A(P} D_{|B|Q)_0} f = X_{(Q} J_{P)_0} \tag{20}$$

for f *of any weight*. Since we know in advance that the left hand side has this form we may as well "remove" the X_Q and form $D^P J_P$. Now for $f \in \mathcal{E}(1 - \frac{n}{2})$ we have

$$D^P J_P = (2-n)^2 f^2 \Box f$$

which compares favourably to (19). The observation (20) which allowed this improvement is typical of a general result. It is beyond the scope of these lectures to discuss this is in detail but let us consider another example. Suppose we begin with the expression $g^{ac} g^{be} (\nabla_a \nabla_b f) \nabla_c \nabla_e f$, where for the moment we assume nothing about the weight of f. Now formally replace each g^{ac} with h^{AC}, each ∇_a with D_{AP} (note the upper case A corresponds to the lower case a). To obtain

$$h^{AC} h^{BE} (D_{AP} D_{BQ} f) D_{CR} D_{ES} f,$$

or $(D_{AP} D_{BQ} f) D^A{}_R D^B{}_S f$ Now take the trace-free symmetric part of this

$$J_{PQRS} := (D_{A(P} D_{|B|Q} f) D^A{}_R D^B{}_{S)_0} f.$$

By general elementary arguments one can show that in such expressions half the indices arise from free (i.e. uncontracted) X's. That is J_{PQRS} is necessarily of the form

$$J_{PQRS} = X_{(P} X_Q J_{RS)_0},$$

where we will assume that J_{RS} is symmetric and trace-free. One can easily show directly that the map $\mathcal{E}_{(Q \cdots V)_0}[w] \to \mathcal{E}_{(PQ \cdots V)_0}[w+1]$, given by $f_{Q \cdots V} \mapsto X_{(P} f_{Q \cdots V)_0}$, is injective and it is straightforward to describe the inverting map by an explicit formula. Thus the valence 2 tractor field J_{RS} must also be invariant. It follows then that

$$J := D^R D^S J_{RS}$$

is also conformally invariant. We describe this as the quasi-Weyl invariant corresponding to the original Riemannian invariant expression $g^{ac} g^{be} (\nabla_a \nabla_b f) \nabla_c \nabla_e f$ and J is an explicit formula which is clearly invariant for f of all weights and in all dimensions. Unfortunately it is an extremely tedious calculation to describe such an invariant J in terms of ∇_a derivatives of f, and so establish, for example, that the invariant is non-vanishing. However the story is different if one starts off with an expression for a conformal invariant rather than just any Riemannian invariant. More precisely we want now to start with an expression which is non-trivial and invariant on conformally flat structures. Suppose, for example, that this time we begin with the expression $I = g^{ab}g^{ce}(\nabla_a\nabla_b f)\nabla_c\nabla_e f$. As above, if we formally replace each g^{ac} with h^{AC}, each ∇_a with D_{AP} and then take the trace-free symmetric part then we obtain an expression I_{PQRS} which by general considerations (and without using any information about the weight of f) we know to be of the form $I_{PQRS} = X_{(P}X_Q I_{RS)_0}$. Thus the trace free symmetric object I_{RS} is invariant and the corresponding quasi-Weyl invariant is $D^R D^S I_{RS}$. One can conclude that this does not vanish when the weight of f is $1 - \frac{n}{2}$ since it does not vanish on the conformally flat structures. To see this we must note that for this weight $I = g^{ab}g^{ce}(\nabla_a\nabla_b f)\nabla_c\nabla_e f$ is invariant on flat structures (i.e. conformally flat structures with metrics such that $\mathsf{P}_{ab} = 0$). Exploiting a general argument [22] which uses this invariance of I, one can deduce that, at least on the flat structures,

$$I_{RS} = X_R X_S I$$

thus

$$D^R D^S I_{RS} = 2n(n+2)I.$$

Again this is typical of a general result and using the general results called upon above one can show that for $f \in \mathcal{E}[1 - \frac{n}{2}]$ all curved analogues of (scalar) invariants which exist in the conformally flat case arise as linear combinations of a countable set of basic quasi-Weyl invariants and the two invariants $\Box f$ and $f\Box f$. The details are in [22].

The corresponding treatment of conformal structure invariants yields similar results for invariants of the curvature, again the details are in [22].

5. CR STRUCTURES

with C. Robin Graham

Here we wish to give a brief description of the basic tractor calculus for CR structures. The discussion here is based primarily on joint work with C. Robin Graham. Conversations with Kengo Hirachi, Michael Eastwood and John Lee have also been extremely useful. Where possible the conventions for pseudohermitian structures as in Jerison-Lee and Lee's articles [30, 33] have been followed. Where the notation varies from Lee's it is so the formulae are as formally similar as possible to the analogous conformal formulae above.

For M a smooth $(2n+1)$-dimensional orientable smooth manifold a *CR-structure* on M is an n-$\dim_{\mathbb{C}}$ complex subbundle $T^{1,0} \subset \mathbb{C}TM$ s.t. $T^{1,0} \cap T^{0,1} = \{0\}$, where $T^{0,1} := \overline{T^{1,0}}$. We will assume this is *integrable* i.e. $[T^{1,0}, T^{1,0}] \subseteq T^{1,0}$.

A q-form is said to be of type $(q, 0)$ if it vanishes upon contraction with a vector in $T^{0,1}$ and of type $(0, q)$ if it vanishes upon contraction with a vector in $T^{1,0}$. We write $\Omega^{q,0}$ and $\Omega^{0,q}$ for respectively the bundle (or sheaf of germs thereof) of $(q, 0)$-forms and $(0, q)$-forms. The *canonical bundle* $\Omega^{n+1,0}$ has fibres of complex dimension 1. We will assume this bundle admits an $(n+2)^{\text{th}}$ root and denote by $\mathcal{E}(1, 0)$ the bundle which is the $-1/(n+2)$ power of $\Omega^{n+1,0}$. For $w - w' \in \mathbb{Z}$ the bundle of (w, w')-*densities*, $\mathcal{E}(w, w')$, is defined to be $(\mathcal{E}(1,0))^w \otimes (\overline{\mathcal{E}(1,0)})^{w'}$ (these conventions are consistent with Eastwood and Graham [15]).

Set

$$H = \mathrm{Re}(T^{1,0} \oplus T^{0,1}).$$

This is a 2n $\dim_{\mathbb{R}}$ subbundle of TM. H carries a natural complex structure map given by $J(v+\bar{v}) = i(v-\bar{v})$ for $v \in T^{1,0}$. Since M and H are orientable M admits a *pseudohermitian structure*, viz a non-vanishing real form θ s.t. $H = \ker\theta$. Associated with this is the *Levi-form*:

$$g(v,w) = d\theta(v,Jw) \text{ for } v,w \in H \ (or\ \in \mathbb{C}H).$$

We will assume M is *strictly pseudoconvex*, that is we can choose θ so that g is positive definite, and we will only work with such θ. Given a pseudohermitian structure θ define $T(=T_\theta)$ to be the unique vector field on M satisfying

$$T \lrcorner\, \theta = 1 \text{ and } T \lrcorner\, d\theta = 1.$$

We usually work with an *admissible coframe*, that is a set of $(1,0)$-forms $\{\theta^a\}$, $a = 1,\cdots,n$, satisfying $\theta^a(T) = 0$ and such that, upon restriction to $T^{1,0}$, these form a basis for $(T^{1,0})^*$. Indices $a, b, \bar{a}$, and so on, below refer to such a frame, the conjugate frame or the dual to either of these. In such a frame the components of the Levi form are given by $g_{a\bar{b}}$ and are used to raise and lower indices.

A choice of pseudohermitian structure determines a connection the *(Tanaka)-Webster-Stanton* connection via:

$$\begin{aligned}\nabla\theta &= 0\\ \nabla g &= 0\\ [\nabla, J] &= 0\end{aligned}$$

and the torsion equations,

$$\begin{aligned}[\nabla_a, \nabla_T]f &= A_a^{\bar{b}}\nabla_{\bar{b}}f\\ [\nabla_{\bar{a}}, \nabla_T]f &= A_{\bar{a}}^{b}\nabla_{b}f\\ [\nabla_a, \nabla_{\bar{b}}]f &= -ig_{a\bar{b}}\nabla_T f\end{aligned}$$

where $A_{ab} = A_{(ab)} = \overline{A_{\bar{a}\bar{b}}}$. The curvature of ∇ is determined by a tensor $R_{a\bar{b}c\bar{d}}$ (see [33]), A_{ab} and $\nabla_{\bar{c}}A_{ab}$ and barred versions of last two. The Webster-Ricci tensor is defined

$$R_{a\bar{b}} = R_c{}^c{}_{a\bar{b}}$$

and the Webster scalar curvature

$$R = R_a{}^a.$$

From these one can define the (CR) *Rho-tensor*:

$$\mathsf{P}_{a\bar{b}} := \frac{1}{n+2}\left(R_{a\bar{b}} - \frac{1}{2(n+1)}Rg_{a\bar{b}}\right)$$

Choosing a new scale for θ

$$\theta \mapsto \hat{\theta} = \Omega\theta = e^{\Upsilon}\theta$$

results in

$$g \mapsto \Omega g.$$

We write Υ_a for $\nabla_a\Upsilon$ and so forth.

Each choice of g from the conformal class may be identified with a section of the line bundle $\mathcal{E}(-1,-1)$. Henceforth use g to denote the section of $\mathcal{E}_{a\bar{b}}(1,1)$ which gives this isomorphism and for each choice of *CR-scale* $0 < U \in \mathcal{E}(1,1)$, $U = \overline{U}$, write $g^{(U)}$ for the form determined by U:

$$g^{(U)} = U^{-1}g.$$

Similarly write θ for the $\mathcal{E}(1,1)$ valued 1-form and write $\theta^{(U)}$ for the form determined by a choice of CR-scale. If ∇ is the connection determined by $\theta^{(U)}$ then $\nabla U = 0 = \nabla g$.

A change in CR-scale $U \mapsto \hat{U} = e^{-\Upsilon}U$ induces transformations to $\nabla \mapsto \hat{\nabla}$ and to the curvature and so forth (see [33]). For example, for an unweighted co-vector v_b

$$\begin{aligned}
\widehat{\nabla_a v_b} &= \nabla_a v_b - v_a \Upsilon_b - v_b \Upsilon_a \\
\widehat{\nabla_{\bar{a}} v_b} &= \nabla_{\bar{a}} v_b + g_{b\bar{a}} \Upsilon^c v_c \\
\widehat{\nabla_0 v_b} &= \nabla_0 v_b + i\Upsilon^{\bar{a}} \nabla_{\bar{a}} v_b - i\Upsilon^a \nabla_a v_b - i v_c (\nabla_b \Upsilon^c - \Upsilon^c \Upsilon_b)
\end{aligned}$$

(where ∇_0 is the line bundle valued version of ∇_T) and for $f \in \mathcal{E}(w,w')$

$$\widehat{\nabla_{\bar{a}} f} = \nabla_{\bar{a}} f + w' \Upsilon_{\bar{a}} f$$

and so on.

5.1. **THE CR TRACTOR BUNDLE.** For a given choice of CR-scale $\mathcal{E}_I$ is identified with a direct sum

$$[\mathcal{E}_A]_{(U)} = \mathcal{E}(1,0) \oplus \mathcal{E}_a(1,0) \oplus \mathcal{E}(0,-1)$$

and under change of CR-scale (as above)

$$v_A = \begin{pmatrix} \sigma \\ \tau_a \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} \sigma \\ \tau_a + \Upsilon_a \sigma \\ \rho - \Upsilon^b \tau_b - \frac{1}{2}(\Upsilon^b \Upsilon_b + i\Upsilon_0)\sigma \end{pmatrix}.$$

Write Z^I for the section of $\mathcal{E}^I(1,0)$ giving the map $Z^I : v_I \to \sigma$. There is a tractor Hermitian form $h_{I\bar{J}}$ given by

$$h_{I\bar{J}} U^I V^{\bar{J}} = g_{a\bar{b}} \mu^a \nu^{\bar{b}} + \sigma\gamma + \rho\alpha$$

for

$$U^I = \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix}, \quad V^{\bar{J}} = \begin{pmatrix} a \\ \nu^{\bar{b}} \\ c \end{pmatrix}.$$

Define

$$(n+2)Q_a = \nabla_a \mathsf{P} - i\nabla^b A_{ab}$$

where $\mathsf{P} := \mathsf{P}_b^b = \frac{1}{2(n+1)} R$. Let

$$S = -\frac{1}{n}(\nabla^a Q_a + \nabla^{\bar{a}} Q_{\bar{a}} + \mathsf{P}_{a\bar{b}} \mathsf{P}^{a\bar{b}} - A_{ab} A^{ab}).$$

Then on an unweighted tractor

$$v_A = \begin{pmatrix} \sigma \\ \tau_a \\ \rho \end{pmatrix}$$

there is a CR invariant tractor connection given by

$$\nabla_b v_A = \begin{pmatrix} \nabla_b \sigma - \tau_b \\ \nabla_b \tau_a + iA_{ab}\sigma \\ \nabla_b \rho - P_b^a \tau_a + Q_b \sigma \end{pmatrix},$$

$$\nabla_{\bar{b}} v_A = \begin{pmatrix} \nabla_{\bar{b}}\sigma \\ \nabla_{\bar{b}}\tau_a + h_{a\bar{b}}\rho + P_{a\bar{b}}\sigma \\ \nabla_{\bar{b}}\rho + iA^a_{\bar{b}}\tau_a - Q_{\bar{b}}\sigma \end{pmatrix},$$

and

$$\nabla_0 v_A = \begin{pmatrix} \nabla_0\sigma + \frac{i}{n+2}P\sigma - i\rho \\ \nabla_0\tau_a - iP_a^b\tau_b + \frac{i}{n+2}P\tau_a + 2iQ_a\sigma \\ \nabla_0\rho + \frac{i}{n+2}P\rho + 2iQ^a\tau_a + iS\sigma \end{pmatrix}$$

For a given choice of CR scale we define a first order differential operator $\tilde{D}^{\bar{A}}$: $\mathcal{E}^*(w,w') \to \mathcal{E}^{\bar{A}} \otimes \mathcal{E}^*(w-1,w')$ by

$$\tilde{D}^{\bar{A}} f := \begin{pmatrix} wf \\ \nabla^{\bar{a}} f \\ 0 \end{pmatrix}$$

where $\mathcal{E}^*(w,w')$ indicates a tractor bundle of arbitrary valence and weight (w,w'). Calculating the transformation of this under a change of CR scale one obtains

$$\widehat{\tilde{D}_A f} = \tilde{D}_A f + Z_A(\Upsilon^b\nabla_b f + \frac{w}{2}\Upsilon^b\Upsilon_b f + i\Upsilon_0).$$

So, in analogy with the conformal case, we have that the operator defined by

$$D_{AP} := 2Z_{[P}\tilde{D}_{A]} \tag{21}$$

is invariant on weighted tractor bundles. In the CR case this operator is only part of the story. Toward understanding this let us digress to the CR flat model and interpret D_{AP} in that context.

5.2. **The flat model.** In this subsection we will reuse some of the symbols introduced above (such as, for example, Z^A) to denote special objects related to the in the flat structure. This should cause no confusion as it will turn out that the new usage is in fact consistent with the use of these symbols in the general case.

Let W denote $\mathbb{C}^{n+2}$ with complex coordinates

$$Z^A = \begin{pmatrix} Z^0 \\ Z^a \\ Z^{n+1} \end{pmatrix}, \ a = 1, \cdots, n$$

and let $h_{I\bar{J}}$ denote the $(n+1,1)$ Hermitian form on W given by

$$h_{A\bar{B}}Z^A Z^{\bar{B}} = 2\mathrm{Re} Z^0 \overline{Z}^{n+1} + h_{a\bar{b}}Z^a Z^{\bar{b}},$$

where $h_{a\bar{b}}$ is the standard positive definite Hermitian form on $\mathbb{C}^n$. Let us write Q for this as a function on $\mathbb{C}^{n+2}$, i.e., $Q := h_{A\bar{B}}Z^A Z^{\bar{B}}$. The null cone is given by the vanishing of Q. We write $\mathcal{N}$ for this in $\mathbb{C}^{n+2} - \{0\}$, $\mathcal{N} := \{Z^A \neq 0 \ : \ Q = 0\}$. The flat model for CR geometry is the image $\mathcal{Q}$ of this null cone under the map $\mathbb{C}^{n+2} - \{0\} \to \mathbb{CP}_{n+1}$. Thus $\mathcal{Q}$ is also given by the vanishing of Q but now where regard Q as a function on $\mathbb{CP}_{n+1}$.

In this picture sections f of $\mathcal{E}(w,w')$ on $\mathcal{Q}$ may be identified with functions on $\mathcal{N}$ which are homogeneous in the sense that

$$f(\lambda Z) = \lambda^w \bar{\lambda}^{w'} f(Z). \tag{22}$$

We will imagine that such functions are in fact (arbitrarily) extended to functions on $\mathbb{C}^{n+2}$ that are homogeneous in the sense of (22). In terms of such homogeneous functions it is easily verified that the operator D_{AP} on $\mathcal{Q}$ is given by

$$D_{AP}f = 2Z_{[P}\partial_{A]}f,$$

where $\partial_A := \partial/\partial Z^A$. This operator is clearly covariant under the (parabolic) subgroup P of $SU(n+1,1)$ which stabilises the null ray through the point Z^A. Note also that it is *intrinsic* to $\mathcal{Q}$ in the sense that, on $\mathcal{Q}$, it is independent of how functions are extended off $\mathcal{Q}$. To see this suppose that f and $\tilde{f}$ are functions on $\mathbb{C}^{n+2}$ satisfying (22) and where

$$\tilde{f} = f + Qf_1$$

for some smooth homogeneous function f_1. Thus f and $\tilde{f}$ agree on $\mathcal{N}$. Then

$$2Z_{[P}\partial_{A]}\tilde{f}|_{\mathcal{Q}} = 2Z_{[P}\partial_{A]}f|_{\mathcal{Q}}$$

since

$$\begin{aligned} 2Z_{[P}\partial_{A]}Qf_1 &= Z_{[P}h_{A]\bar{B}}Z^{\bar{B}}f_1 + Q2Z_{[P}\partial_{A]}f_1 \\ &= Z_{[P}Z_{A]}f_1 + Q2Z_{[P}\partial_{A]}f_1 \\ &= Q2Z_{[P}\partial_{A]}f_1 \end{aligned}$$

which vanishes on $\mathcal{N}$.

Thus, in this setting, we can understand D_{AP} as an invariant and intrinsic operator on $\mathcal{Q}$ embedded in the ambient $\mathbb{CP}_{n+1}$. This might inspire us to look for other such operators. Indeed consider $2Z_{[\bar{B}}\partial_{A]}f$, meaning $(Z_{\bar{B}}\partial_A - Z_A\partial_{\bar{B}})f$. Note that

$$(Z_{\bar{B}}\partial_A - Z_A\partial_{\bar{B}})Q = Z_{\bar{B}}Z_A - Z_AZ_{\bar{B}} = 0.$$

Thus by a similar argument to the one just above we can quickly deduce that $D_{A\bar{B}}f := 2Z_{[\bar{B}}\partial_{A]}f$ is another invariant operator which is intrinsic to $\mathcal{Q}$.

5.3. **The basic invariant operators.** Since the operator $D_{A\bar{B}}f$ just observed on the flat model is first order one would expect it to extend to the general curved setting. This is the case and it is easily verified directly that

$$D_{A\bar{B}}f := Z_{\bar{B}}\tilde{D}_Af - Z_A\tilde{D}_{\bar{B}}f - Z_AZ_{\bar{B}}[i\nabla_0 f + \frac{(w'-w)}{n+2}\mathsf{P}f]$$

defines an invariant operator on weighted tractor bundles.

The pair D_{AP} and $D_{A\bar{B}}$ are basic first order CR invariant operators which between them correspond to the operator D_{AP} of conformal structures. It requires both of these to recover an analogue of the conformal tractor D operator:
For f a tractor field of weight (w, w'), the CR invariant operator

$$D_Bf$$

may be defined by

$$-Z_{(A}D_{B)}f = h^{P\bar{Q}}D_{(A|P|}D_{B)\bar{Q}}f. \tag{23}$$

This definition manifestly gives $D_B : \mathcal{E}^*(w,w') \to \mathcal{E}^*(w-1,w')$ as a strongly invariant operator. More explicitly

$$D_Bf = (n+w+w')\tilde{D}_Bf - Z_B\Box f, \tag{24}$$

where here

$$\Box f := [\nabla^a\nabla_a + iw\nabla_0 + w(1 + \frac{(w'-w)}{n+2})P]f,$$

and the invariance of D_B is readily verified directly using this expression. The operators D_{AP} and $D_{A\bar{B}}$ can be recovered from D_B and its complex conjugate via

$$(n + w + w')D_{AP}f = 2Z_{[P}D_{A]}f$$

and

$$(n + w + w')D_{A\bar{B}}f = 2Z_{[\bar{B}}D_{A]}f = Z_{\bar{B}}D_A f - Z_A D_{\bar{B}} f. \tag{25}$$

It is straightforward to interpret D_A in the flat case. Suppose that f is homogeneous on $\mathcal{N}$ as in (22). Let $\tilde{f}$ be a "harmonic" extension of this to $\mathbb{C}^{n+2}$ in the sense that $\tilde{f}$ is homogeneous on $\mathbb{C}^{n+2}$, as in (22), and satisfies

$$\tilde{f}|_{\mathcal{N}} = f \quad \text{and} \quad h^{A\bar{B}}\partial_A\partial_{\bar{B}}\tilde{f}|_{\mathcal{N}} = 0.$$

Then it follows easily from (23) that $D_B f$ is given by

$$D_B f = (n + w + w')\partial_B \tilde{f}|_{\mathcal{N}}.$$

If $n + w + w' \neq 0$ one can always find such an extension and so for such weights this determines $D_B f$ in the flat case. In fact it seems that this "ambient" description of D_A, as well as the similar descriptions of D_{AP} and $D_{A\bar{B}}$, can be generalised to a curved setting via the Fefferman ambient metric construction [18]. This is work in progress with Kengo Hirachi. (Indeed it was in this joint work that we first constructed the operator $D_{A\bar{B}}$.) Results along these lines for the conformal case have already been obtained by Graham [27].

5.4. **Invariant Powers of the Sub-Laplacian.** In analogy with the construction of the conformally invariant powers of the Laplacian we can use the tractor calculus to construct powers of the so called sub-Laplacian.

Note that if f is a tractor field of weight (w, w') such that $n+w+w' = 0$ then from (24)

$$D_A f = Z_A \Box f.$$

Clearly then $\Box$ is a CR invariant differential operator,

$$\Box : \mathcal{E}^*(w, -n-w) \to \mathcal{E}^*(w-1, -n-w-1),$$

where $*$ indicates any tractor indices. This is the sub-Laplacian of Jerison-Lee [29] (in fact a slight generalisation in the sense that [29] deal only with the case that $w = w' = -n/2$). One might imagine that the operator $\overline{\Box} f$, which arises similarly from $D_{\bar{A}}$, would give another CR invariant differential operator on $f \in \mathcal{E}^*(w, -n-w)$. In fact

$$(\Box - \overline{\Box})f = (n + w + w')[i\nabla_0 f + \frac{(w'-w)}{n+2}Pf].$$

Thus on tractors f of weight (w, w') such that $n + w + w' = 0$ the operators $\Box$ and $\overline{\Box}$ agree.

In contrast the operator $D^A\Box D_A f$ is not in general the same as $D_A\Box D^A f$ so it seems natural to take $\frac{1}{2}(D^A\Box D_A + D_A\Box D^A)f$ as the definition of the corresponding fourth order operator for f of weight $(w, 1-n-w)$. It is straightforward to show this is non-vanishing except when $n = 1$ and $w = w' = 0$. Detailed results on the higher order analogues of these operators will appear in [23].

5.5. **Concluding Remarks.** For CR geometries one can define a notion of quasi-Weyl invariants essentially by analogy with the conformal definition only now we have two operators, D_{AP} and $D_{A\bar{B}}$, available to use in such constructions. It is clear that by an appropriate adaption of the treatment of the conformal case (as in [22]) one can produce invariants that are "beyond the obstruction" to the Fefferman ambient construction. However at this point a detailed study of this case has not been made.

The tractor calculus discussed above for conformal structures is based on the standard (or defining) representation of $\mathrm{SO}(n+1, 1)$. It is also possible to instead develop a *local twistor calculus* based on the fundamental representation of $\mathrm{Spin}(n+1, 1)$. In joint work with Jan Slovak [24] this has been described for four dimensional geometries and their generalisation to n-dimensional almost-Grassmannian (or paraconformal) structures.

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