Herrmann Haase

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Measures on Self-Similar Sets

H. HAASE

Greifswald*)

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We consider different measures on self-similar sets in the sense of Hutchinson and study some of their relations.

A set $K \subseteq \mathbb{R}^n$ is called self-similar iff

$$K = \bigcup_{i=1}^{m} f_i(K) \tag{1}$$

where the f_i's are similitudes $(|f_i(x) - f_i(y)| = r_i|x - y|,\ x, y \in \mathbb{R}^n)$ with factor r_i such that $0 < r_i < 1$ for $i = 1, \ldots, m$ [3]. To speak about self-similarity it is necessary that the components $f_i(K)$ do not overlap too much. The condition $f_i(K) \cap f_j(K) = \emptyset$ for $i \neq j$ is too strong since only the case of Cantor sets K would be covered. The so called open set condition, namely there exists an open set $V \subset \mathbb{R}^n$ such that $f_i(V) \subset V$ for all $i = 1, \ldots, m$ and $f_i(V) \cap f_j(V) = \emptyset$ for $i \neq j$ ensures a wide range of examples. There exists a unique real number s such that

$$\sum_{i=1}^{m} r_i^s = 1 . \tag{2}$$

This number s is exactly the Hausdorff dimension of K and for the Hausdorff measure m^s, $E \subseteq \mathbb{R}^n$, defined by the formula

$$m^s(E) = \sup_{\varepsilon > 0} \inf \{\sum_k (\operatorname{diam} E_k)^s;\ E \subseteq \bigcup_k E_k,\ \operatorname{diam} E_k \leqq \varepsilon\} , \tag{3}$$

there are valid

$$0 < m^s(K) < \infty , \quad m^s(f_i(K)) = r_i^s\, m^s(K) , \quad m^s(f_i(K) \cap f_j(K)) = 0 \quad \text{for} \quad i \neq j . \tag{4}$$

An arbitrary Borel measure μ on $\mathbb{R}^n$ is called similarly invariant of dimension s iff

*) Sektion Mathematik, E.-M.-Arndt Universität, F.-L.-Jahn-Str. 15a, Greifswald, DDR-2200.

$\mu(f(A)) = r^s \mu(A)$ for $A \subseteq \mathbb{R}^n$ and each similitude f with factor r. In the case $f_i(K) \cap f_j(K) = \emptyset$ for $i \neq j$ such a measure is unique up to a constant factor.

Other usefull measure constructions for self-similar sets are the centered covering measure c^s and the packing measure p^s. Following [4] they are defined by pre-measures C^s and P^s in the following way

(5) $$C^s(E) = \lim_{\varepsilon \to 0} [\inf \{\sum_k (2r_k)^s ;$$

$$(B_{n_k}(x_k)) \text{ is a covering of } E \text{ by closed balls centered in } E \text{ with } r_k \leqq \varepsilon\}]$$

and

(6) $$P^s(E) = \lim_{\varepsilon \to 0} [\sup \{\sum_k (2r_k)^s ;$$

$$(B_{r_k}(x_k) \text{ is a packing of } E \text{ by closed balls centered in } E \text{ with } r_k \leqq \varepsilon\}]$$

for all sets $E \subseteq \mathbb{R}^n$. Remark that $(B_{r_k}(x_k))$ is a packing iff $B_{r_k}(x_k) \cap B_{r_j}(x_j) = \emptyset$ for $j \neq k$. The centered covering measure c^s is defined as

(7) $$c^s(E) = \sup \{C^s(F);\ F \subseteq E\}$$

and the packing measure p^s as

(8) $$p^s(E) = \inf \{\sum_k P^s(E_k);\ E \subseteq \bigcup_k E_k\} .$$

Since it is easy to prove that $p^s(K) < \infty$ and using the relation

(9) $$\frac{1}{2^s} c^s(E) \leqq m^s(E) \leqq c^s(E) \leqq p^s(E) \quad [4]$$

all properties concerning m^s and K described above carry over to these both new measures. In fact, they are further examples of similarity invariant measures of dimension s. For c^s we have for an arbitrary bounded Borel measure μ on R^n and a Borel set $E \subseteq R^n$ that if $c^s(E) < \infty$

(10) $$c^s(E) \inf_{x \in E} \limsup_{r \to 0} \frac{\mu(B_r(x)}{(2r)^s} \leqq \mu(E) \leqq c^s(E) \sup_{x \in E} \limsup_{r \to 0} \frac{\mu(B_r(x))}{(2r)^s} \quad [4]$$

and the counterpart if $p^s(E) < \infty$

(11) $$p^s(E) \inf_{x \in E} \liminf_{r \to 0} \frac{\mu(B_r(x))}{(2r)^s} \leqq \mu(E) \leqq p^s(E) \sup_{x \in E} \liminf_{r \to 0} \frac{\mu(B_r(x))}{(2r)^s} \quad [1, 4] .$$

Now let $p_1, \ldots, p_m$ be real positive numbers such that $\sum_{i=1}^m p_i = 1$. Let $(\Omega, \mathscr{A}, P)$ be a probability space, $(A_l)_{l \in N}$ a sequence of random similitudes ($A_l: \Omega \to \mathscr{S}$ is a measurable map where $\mathscr{S}$ denotes the space of all similitudes of $\mathbb{R}^n$ and $\mathscr{S}$ is equipped with some suitable algebra of Borel sets) which are independent and identically distributed as $P([A_l = f_i]) = p_i$ for $i = 1, \ldots, m$. For fixed $x \in R^n$ and $\omega \in \Omega$ $((A_1(\omega) \circ A_2(\omega) \circ \ldots \circ A_l(\omega))(x))_{l \in N}$ is convergent to some point of K. The distribution measure μ of this random limit point satisfies

$$\mu(f_{i_1} \circ \ldots \circ f_{i_k}(K)) = p_{i_1} p_{i_2} \ldots p_{i_k} \tag{12}$$

where $i_1, \ldots, i_k \in \{1, \ldots, m\}$.

Theorem. Suppose that the open set V of the open set condition satisfies $V \cap K \neq \emptyset$. Then the measure μ has an integral representation w.r.t. p^s if and only if $p_i = r_i^s$ for $i = 1, \ldots, m$.

We introduce a modified packing measure $\tilde{p}^s$ by taking as packing elements all sets $f_{i_1} \circ \ldots \circ f_{i_k}(K)$, $i_1, \ldots, i_k \in \{1, \ldots, m\}$. In this case we call a sequence $(B_l)_{l \in I}$ a packing for E iff $E \cap B_l \neq \emptyset$ for all $l \in I$ and $B_l \cap B_k = \emptyset$ for $l \neq k$ (I is some index set). Instead of $(2r_k)^s$ we take then $(\text{diam } B_k)^s$. As proved in [2] an analogue density theorem for $\tilde{p}^s$ as for p^s holds (see the formula below). The condition $V \cap K \neq \emptyset$ ensures that the problem can be reduced to $\tilde{p}^s$, i.e. there are constants c_1 and $c_2 > 0$ such that

$$c_1 \tilde{p}^s(E) \leqq p^s(E) \leqq c_2 \tilde{p}^s(E) \tag{13}$$

for $E \subseteq \mathbb{R}^n$.

Proof of Theorem: μ has an integral representation w.r.t. p^s iff

$$\mu\left(\left\{x \in K;\ 0 < \sup_{\varepsilon > 0} \inf \left\{\frac{\mu(f_{i_1} \circ \ldots \circ f_{i_k}(K))}{(\text{diam} f_{i_1} \ldots f_{i_k}(K))^s};\ x \in f_{i_1} \circ \ldots \circ f_{i_k}(K),\right.\right.\right.$$
$$\left.\left.\left.\text{diam} f_{i_1} \circ \ldots \circ f_{i_k}(K) \leqq \varepsilon\right\} < \infty\right\}\right) = 1\,.$$

Let $(X_j)_{j \in N}$ now a sequence of i.i.d. real random variables on the probability space $(\Omega, \mathscr{A}, P)$ such that

$$P([X_j = \log p_i - s \log r_i]) = p_i \quad \text{for} \quad i = 1, \ldots, m\,.$$

Then μ has a representation w.r.t. $\tilde{p}^s$ iff

$$P([\liminf_{k \to \infty} |\sum_{j=1}^{k} X_j| < \infty]) = 1\,.$$

Applying the strong law of large numbers and the law of iterated logarithm this is equivalent to the fact that expected value and variance of the X_j's vanish. A direct calculation gives

$$s = \frac{\sum_{i=1}^{m} p_i \log p_i}{\sum_{i=1}^{m} p_i \log r_i} \quad \text{and} \quad s = \frac{\log p_i}{\log r_i}, \quad \text{hence} \quad p_i = r_i^s \quad \text{for all} \quad i\,.$$

Remarks. The idea for this proof is due to [5] where the case $\mathbb{R}$ with $f_1(x) = r_1 x$, $f_2(x) = r_2 x + (1 - r_2)$ $(r_i > 0,\ r_1 + r_2 < 1,\ i = 1, 2)$ and with Hausdorff measure $m^s(r_1^s + r_2^s = 1)$ was considered. The main result of [5] shows that there is no hope to obtain integral representations in the general setting for μ with more general measure functions. Related problems were also considered in [6, 7].

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