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A Note on the Prime Ideal Theorem

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The Prime Ideal Theorem is shown to be equivalent with the following two statements (1) Any compact nontrivial quantale has a prime element, and (2) Any compact normal complete nontrivial distributive lattice has a maximal element. Some another equivalents of the Prime Ideal Theorem are given.

The lattice-theoretical investigations of complete lattices equipped with an additional binary operation • which distributes over arbitrary joins in each variable can be traced back to Ward and Dilworth [14]. Such a gadget is called quantale (following C. J. Mulvey). Some topological properties of quantales were obtained by Borceux [4].

The original motivation for this paper was the question whether the Prime Ideal Theorem (every nontrivial distributive lattice has a prime ideal) is strong enough to ensure the existence of a prime element in arbitrary compact quantale, rather than just in compact frames (see [3], [7]). We present some new relationships between this principle and quantale-theoretic conditions, e.g. the existence of a maximal element in arbitrary compact normal frame.

All unexplained facts concerning frames and quantales can be found in [7], [9] or [10]. Detailed accounts of choice principles appear in [6].

§ 1. m-prime ideals yields prime elements

1.1. Definition. (i) A (weak) m-semilattice is a \(\lor\)-semilattice \(S\) with the top element 1 and the bottom element 0 equipped with an associative operation • so that • distributes over finite (finite nonempty) joins in each variable and \(1 \cdot x = x\) for all \(x \in S\). A morphism of weak m-semilattices is a mapping preserving finite joins, • and 1.

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(ii) An element $a$ of a weak $m$-semilattice $S$ is called

1. 2-sided if $a \cdot 1 \leq a$,
2. idempotent if $a \cdot a = a$.

The set of all 2-sided (idempotent) elements will be denoted by $\mathcal{S}(E(S))$. It is easy to check that $\mathcal{S}$ is a weak $m$-semilattice.

A weak $m$-semilattice $S$ is said to be nontrivial if it has at least two 2-sided elements.

(iii) A weak $m$-semilattice $S$ is called 2-sided (idempotent) if any of its element is 2-sided (idempotent).

The following is well known (see [4], [5]).

1.2. Lemma. For any elements $a, b, c$ in a weak $m$-semilattice $S$ we have the following

i. $b \leq c$ implies $a \cdot b \leq a \cdot c$,
ii. $a \leq b$ implies $a \cdot c \leq b \cdot c$,
iii. $a \cdot 0 = 0$,
iv. $a$ is 2-sided implies $a \cdot b \leq a$,
(v) $a \in \mathcal{S} \cap E(S)$ implies $a \lor (b \cdot c) = (a \lor b) \cdot (a \lor c)$,
(vi) $a \in \mathcal{S} \cap E(S)$ implies $a \cdot 1 = a$.

1.3. Corollary. Let $S$ be a weak $m$-semilattice. If $E(S) = S$ then $\mathcal{S} \cap E(S)$ constitutes a distributive lattice. Moreover, a weak $m$-semilattice is a distributive lattice iff $S = E(S) = \mathcal{S}$.

1.4. Definition. Let $S$ be a weak $m$-semilattice. (i) An ideal of $S$ will be just an ideal of the $\lor$-semilattice $S$. An ideal $I$ is called $m$-prime if $x \cdot y \in I$ implies $x \in I$ or $y \in I$ for all $x, y \in S$.

(ii) An element $p \neq 1$ of $S$ is called prime if $x \cdot y \leq p$ implies $x \leq p$ or $y \leq p$ for all $x, y \in S$. The set of all prime (maximal) elements of $S$ will be denoted by $P(S)$ ($D(S)$).

As for weak $m$-semilattices, we shall introduce the notion of a weak quantale.

1.5. Definition. A (weak) quantale $K$ is a complete (weak) $m$-semilattice $K$ in which $\cdot$ distributes over arbitrary (nonempty) joins in each variable. Congruences on (weak) quantales are congruences with respect to $\cdot$ and $\lor$. A frame $K$ is a quantale $K$ satisfying $K = E(K) = \mathcal{R}$.

Let us recall that a complete lattice $L$ is said to be compact if $E \subseteq L$, $\forall E = 1$ implies there is $F \subseteq E$, $F$ finite such that $\forall F = 1$.

The following is well known for $m$-semilattices (see [10]).

1.6. Proposition. Let $Id(S)$ be the compact weak quantale of all ideals of a weak $m$-semilattice $S$ ($I \cdot J$ being generated by $\{x \cdot y; x \in I, y \in J\}$). Then the prime elements of $Id(S)$ are precisely the $m$-prime ideals of $S$. 

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1.7. Remark. Let $K$ be a weak quantale. We put $a C b$ if and only if $(a . 1 \lor c = 1 \Rightarrow b . 1 \lor c = 1$ for any $c \in K)$ for all $a, b \in K$.

1.8. Proposition. Let $K$ be a compact weak quantale. Then $C$ is a congruence on $K$.

Proof. Let $x C y$, $u C v$, $x . u . 1 \lor c = 1$. Then $x . 1 \lor c = u . 1 \lor c = 1$ i.e. $y . 1 \lor c = 1 = v . 1 \lor c$. Now, we have $y . v . 1 \lor c = 1$.

We have to show that $x . 1 \lor c = 1$. Then by compactness of $K$ there is $F \subseteq I$, $F$ finite such that $1 = c \lor \bigvee\{x_i; i \in F\}$. The symmetry argument concludes the proof.

For a compact weak quantale $K$, we denote $K/e = K/\sim$. It is easy to check that $K/\sim$ is a conjunctive frame (see [11], [13]).

1.9. Lemma. Let $K$ be a weak compact quantale. Then $K/e$ is a compact frame.

Proof. Let $a_i \in K_e$, $i \in I$, $K_e \lor a_i = 1$. Clearly, $\{a_i; i \in I\} \subseteq K$. Since $K$ is compact we have $\lor a_i = 1$ i.e. $1 = (\lor a_i) . 1 \leq \lor a_i$. Consequently, we have $\lor a_i = 1$ and the rest follows from the compactness of $K$.

1.10. Theorem. The following statements are equivalent.

(i) Any nontrivial compact frame has a prime element.

(ii) Any nontrivial compact (weak) quantale has a prime element.

(iii) Any nontrivial (weak) $m$-semilattice has an $m$-prime ideal.

(iv) The Prime Ideal Theorem.

Proof. (i) $\Rightarrow$ (ii) Let $K$ be a compact (weak) quantale. From 1.9 and (i) we have that $K/e$ has a prime element $p$ which is clearly prime and 2-sided in $K$.

(ii) $\Rightarrow$ (iii) It follows from 1.6.

(iii) $\Rightarrow$ (iv) It is evident.

(iv) $\Rightarrow$ (i) It results from [1].

1.11. Lemma. Let $S$ be a weak $m$-semilattice (weak quantale), $a \in S$. Then the $\lor$-semilattice (complete lattice) $\uparrow(a) = \{x \in S; a \leq x\}$ is a weak $m$-semilattice (weak quantale) with respect to a multiplication $\cdot$ defined by

$$x . y = x . y \lor a \quad \text{for all} \quad x, y \in \uparrow(a).$$

Moreover, if $a \in S \cap E(S)$ then $\uparrow(a)$ is an $m$-semilattice (quantale).

Proof. It is immediate.

The following proposition is a generalization of the result of Banaschewski [2, 3].

1.12. Proposition. The following are equivalent:

(i) The Prime Ideal Theorem.
(ii) Any nontrivial compact complete m-semilattice such that its 2-sided elements constitute a complete sublattice has a prime element.

(iii) Any nontrivial compact complete weak m-semilattice such that its 2-sided elements constitute a complete sublattice has a prime element.

Proof. (i) ⇒ (ii) Let $K$ be a nontrivial compact complete m-semilattice, $\hat{K}$ a complete sublattice of $K$. Clearly, $\hat{K}$ is a compact complete m-semilattice. We can suppose that $\hat{K} \neq \{1\}$ i.e. the coproduct $M$ of 2-sided m-semilattices $\uparrow(a) = \{x \in \hat{K}; a \leq x\}$, $a \in \hat{K}-\{1\}$ is a nontrivial m-semilattice $M$ with coproduct maps $h_a; \uparrow(a) \to M$. For any $a \in \hat{K}-\{1\}$ there is an $m$-prime ideal $P_a = h_a^{-1}(P)$ of $\uparrow(a)$, $P$ is an $m$-prime ideal of $M$. We may define a map $\sigma: \hat{K}-\{1\} \to \hat{K}-\{1\}$ putting $\sigma(a) = \bigvee P_a$. It is easy to check that the definition of $\sigma$ is correct and the assumptions of Bourbaki's fix point lemma are satisfied. Now, we have that $\sigma$ has a fix point, say $c$. Without any difficulties we see that $c$ is prime in $K$.

(ii) ⇒ (iii) Let $K$ be a nontrivial compact complete weak m-semilattice, $\hat{K}$ complete sublattice of $K$. Clearly, $Q = \uparrow(a)$ is a nontrivial compact m-semilattice satisfying assumptions of (ii); here $a = 0 \cdot 1 \in \hat{K} \cap E(K)$, $a \neq 1$. Since $\emptyset \neq P(Q) \subseteq P(K)$, we are ready.

(iii) ⇒ (i) It is evident.

§ 2. Normality yields the Maximal Ideal Theorem

2.1. Definition. A weak m-semilattice $S$ is said to be normal if, given $a, b \in S$ with $a \lor b = 1$, we can find $d, c \in S$ with $d \cdot c = 0$, $d \lor a = 1 = b \lor c$.

The following result is well known for m-semilattices (see [10], Theorem 4.5).

2.2. Proposition. Let $S$ be a weak m-semilattice. Then the following conditions are equivalent:

(i) $S$ is normal.

(ii) $Id(S)$ is normal.

Note that 2-sided maximal means maximal with respect to the weak m-semilattice of all 2-sided elements.

2.3. Theorem. The following statements are equivalent:

(i) Any compact normal nontrivial (weak) quantale has a 2-sided maximal element.

(ii) Any compact normal nontrivial frame has a maximal element.

(iii) Any normal nontrivial (weak) m-semilattice has a 2-sided maximal ideal.

(iv) The Maximal Ideal Theorem for normal distributive lattices.
(v) Any compact normal nontrivial complete distributive lattice has a maximal element.

(vi) Any compact regular nontrivial frame has a maximal element i.e. it is spatial.

(vii) The Prime Ideal Theorem.

**Proof.** (i) ⇒ (ii) It is evident.

(i) ⇒ (iii) It follows immediately from 2.2.

(ii) ⇒ (iv) Using 2.2 for distributive lattices.

(iii) ⇒ (iv) It is evident.

(iv) ⇒ (v) Let $K$ be a compact normal nontrivial complete distributive lattice. Then there is a maximal ideal $M$ of $K$. Clearly, by compactness of $K$ we have that $M$ is principal i.e. $M = \downarrow (m) = \{ x \in K; x \leq m \}$ for some $m \in M$. It is easy to check that $m$ is a maximal element of $K$.

(v) ⇒ (vi) Using the fact that any compact regular frame is normal (see [7]).

(vi) ⇒ (vii) It is well known (see [7], [12]).

(vii) ⇒ (i) Let $K$ be a compact normal nontrivial (weak) quantale. Then $K_e$ is exactly the compact regular coreflection of $K$ i.e. there is a maximal element $m$ of $\hat{K}$. because any prime element of a regular frame is maximal. The rest follows from the fact that there is not any 2-sided element $\neq 1$ which is greater than $m$.

Next we shall consider a particular class of weak $m$-semilattices (for distributive lattices see [8]).

2.4. Definition. We shall say that a weak $m$-semilattice $S$ is semi-normal if, whenever $a \vee b = 1$, we can find elements $c, d \in S$ with $a \vee d = 1 = c \vee b$ and $\downarrow (d \cdot c) \subseteq \{ 0 \}$ in $\text{Id}(S)$. Clearly, any normal weak $m$-semilattice is semi-normal.

2.5. Proposition. Let $K$ be a quantale so that $C$ is a congruence on $K$. Then the following are equivalent:

(i) $K$ is semi-normal.

(ii) $a \vee b = 1$ implies there are $c, d \in K$ such that $a \vee d = 1 = c \vee b$ and $d \cdot c \in C$ in $K$.

**Proof.** It is enough to verify that $\downarrow (x) \subseteq \{ 0 \}$ in $\text{Id}(K)$ if and only if $x \in C$ in $K$. But this is an immediate reformulation of 1.7.

2.6. Corollary. Let $S$ be a weak $m$-semilattice. Then the following conditions are equivalent:

(i) $S$ is semi-normal.

(ii) $\text{Id}(S)$ is semi-normal.

**Proof.** The proof is analogous to [10], Theorem 4.5.
2.7. Theorem. The following statements are equivalent:

(i) Any compact semi-normal nontrivial (weak) quantale has a 2-sided maximal element.
(ii) Any compact semi-normal nontrivial frame has a maximal element.
(iii) Any semi-normal nontrivial (weak) m-semilattice has a 2-sided maximal ideal.
(iv) The Maximal Ideal Theorem for semi-normal distributive lattices.
(v) Any compact semi-normal nontrivial complete distributive lattice has a maximal element.
(vi) Any compact normal nontrivial complete distributive lattice has a maximal element.

Proof. (i) ⇒ (ii), (i) ⇒ (iii) ⇒ (iv), (v) ⇒ (vi) It is transparent.
(ii) ⇒ (iv) Using 2.6 for distributive lattices.
(iv) ⇒ (v) Let $K$ be a compact semi-normal nontrivial complete distributive lattice. Then there is a maximal ideal $M$ of $K$. Clearly, by compactness of $K$ we have that $M$ is principal i.e. $M = \downarrow(m) = \{x \in K; x \leq m\}$ for some $m \in M$. It is easy to check that $m$ is a maximal element of $K$.
(vi) ⇒ (i) Let $K$ be a nontrivial compact semi-normal (weak) quantale. Clearly, $Q = \uparrow(a)$ is a nontrivial compact normal weak quantale satisfying assumptions of 2.3(i); here $a = \bigvee\{x; x \leq 0\}, a \in \bar{K}$, $a \neq 1$. Since $\emptyset \neq D(\bar{Q}) \subseteq D(\bar{K})$ we are ready.

References