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On Non-Solid Köthe Space of Δ_2 Type

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0. Introduction

By Köthe normed spaces one understand linear normed subspaces of measurable real functions ordered in a natural way and such that $\|f\| \leqq \|g\|$ if $|f(t)| \leqq |g(t)|$ for almost all $t \in T$. So those spaces are normed lattices. But there are function spaces of vector functions which are not ordered, e.g. the Lebesgue-Bochner spaces L_X^p which are the sets of all measurable vector functions for which

$$\int_T \|f(t)\|^p \, d\mu < \infty ,$$

i.e. the functions $t \mapsto \|f(t)\|$ belong to $L_{\mathbb{R}}^p$. Analogously we define Orlicz-Bochner spaces. More general, if L is a Köthe space of real functions and X is a Banach space, then by $L(X)$ we understand the family of all measurable X-valued functions for which the applications $t \to \|f(t)\|$ belong to L. These spaces are always solid spaces i.e. spaces in which the following condition is satisfied:

$$\|f(t)\|_X \leqq \|g(t)\|_X \quad \text{for almost all} \quad t \in T \quad \text{implies} \quad \|f\|_L \leqq \|g\|_L .$$

In the case $X = \mathbb{R}$ or $X = \mathbb{C}$ these spaces are called ideal spaces and they were considered mainly by Zabrejko P. P. (cf. [15]). However there have been also examined vector functions spaces which have not been of the type $L(X)$, e.g. various generalizations of Orlicz spaces (see [7], [8], [11], [14]). This has been a motivation for introducing and study here some general vector function spaces, called Köthe spaces. This kind of Köthe space, in more particular case, was defined by C. Castaing and A. Kamińska in [3]. Our purpose is to consider a functional H which will enable the introduction of the conception of Condition Δ_2 to the Köthe space, Next, we will show some applications of this condition.

1. Preliminaries

Let (T, Σ, μ) be a measure space, where T is an abstract set, Σ is a σ-algebra of subsets of T and μ is a non-negative, complete, atomless and σ-finite measure on Σ.

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$(X, \|\cdot\|)$ will denote a separable Banach space. By $\mathscr{M}(T, X)$ we will denote the linear space of all μ-equivalence classes of strongly measurable functions $f: T \to X$. Let χ_A be the characteristic function of A. By $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ we will denote sets of natural, real and complex numbers, respectively

1.1. Definition. A Banach space $(L, \|\cdot\|_L) \subset \mathscr{M}(T, X)$ is said to be a Köthe space if it possesses the following properties:

a) $\chi_A f \in L$ for all $A \in \Sigma$ and $f \in L$,

b) if $A, B \in \Sigma$ and $A \subset B$, then

$$\|\chi_A f\|_L \leqq \|\chi_B f\|_L$$

for all $f \in L$.

In the next definition we introduce an important subspace of L.

1.2. Definition. A linear subspace E of L is called a subspace of continuous elements, if E contains all functions from L with continuous norm i.e.

$$f \in E \quad \text{iff} \quad \|\chi_{C_n} f\|_L \downarrow 0 \quad \text{as} \quad n \to \infty$$

for every decreasing sequence of measurable sets $C_n \subset T$ $(n = 1, 2, \ldots)$ such that $\mu(\bigcap_{n=1}^{\infty} C_n) = 0$.

2. Functional H

Consider the functional defined on L by the following formula

$$H(f) = \inf \left\{ \sum_{i=1}^{n} \|\chi_{A_i} f\|_L : \|\chi_{A_i} f\|_L \leqq 1 \ (i = 1, 2, \ldots, n) \right\},$$

where the infimum is taken over all partitions into measurable subsets $\{A_1, A_2, \ldots, A_n\}$ of T with n variable (cf. [2]). For given $f_0 \in L$ and $r > 0$, we denote

$$K(f_0, r) = \left\{ f \in L : H\left(\frac{f - f_0}{r}\right) < \infty \right\}.$$

2.1. Lemma. The functional H has the following properties:

a) $H(f) = \|f\|_L$ for $\|f\|_L \leqq 1$ and $H(f) \geqq \|f\|_L$ for $\|f\|_L > 1$;

b) $H((1 - \alpha) f + \alpha g) \leqq (1 - \alpha) H(f) + \alpha H(g) + 2H(f) H(g)$ for $f, g \in L$ and $0 < \alpha < 1$; in particular, the set $K(f_0, r)$ is convex for every $f_0 \in L$ and $r > 0$;

c) H is monotone with respect to multiplication by characteristic function, i.e.

$$A \subset B \ (A, B \in \Sigma) \quad \text{implies that} \quad H(\chi_A f) \leqq H(\chi_B f)$$

for every $f \in L$;

d) $H(f + g) \leqq H(f) + H(g)$ for $f, g \in L$ with

$$\mu(\operatorname{supp} f \cap \operatorname{supp} g) = 0 .$$

2.2. Example. Let $L^p(1 \leqq p < \infty)$ be Lebesgue-Bochner space. One can prove the formula

$$H(f) = \|f\|_p([\|f\|_p^p + 1])^{1/q},$$

where $1/p + 1/q = 1$ and $[c]$ denotes the greatest integer less than c.

2.3. Example. Let L^∞ be the space of all essentialy bounded strongly measurable functions from T into X with the norm

$$\|f\|_\infty = \operatorname*{supess}_{t \in T} \|f(t)\| .$$

It is obvious that

$$H(f) = \begin{cases} \|f\|_\infty & \text{if } \|f\|_\infty \leqq 1 \\ \infty & \text{if } \|f\|_\infty > 1 \end{cases}$$

The explicit calculation of the functional H in other concrete Köthe spaces of vector-valued functions becomes much difficult. We will give a two-sides estimation of the functional H for Musielak-Orlicz spaces of vector-valued functions. Such estimation for solid Musielak-Orlicz spaces of complex-valued functions was presented also in [2]. In our case, we will define Musielak-Orlicz space as follows:

2.4. Definition. A function $M: X \times T \to [0, \infty)$ is said to be an N-function iff

a) M is $\mathscr{B} \times \Sigma$-measurable, where $\mathscr{B}$ denotes the σ-algebra of Borel subsets of X,

b) $M(\cdot, t)$ is even, convex, lower semicontinuous, not identically equal to 0, continuous at zero and $M(0, t) = 0$ for a.a. $t \in T$.

By Musielak-Orlicz space L_M we mean

$$\{f \in \mathscr{M}(T, X) : I_M(af) = \textstyle\int_T M(af(t), t) \, d\mu < \infty \text{ for some } a > 0\},$$

equipped with the so called Luxemburg norm

$$\|f\|_M = \inf\{a > 0 : I_M(a^{-1}f) \leqq 1\} .$$

If $M(x, t) = M(x, s)$ for every $t, s \in T$ and $x \in X$, then the space L_M is called a generalized Orlicz space.

By E_M we denote a subspace of finite elements i.e.

$$E_M = \{f \in \mathscr{M}(T, X) : I_M(af) < \infty \text{ for every } a > 0\} .$$

Obviously, $E_M \subset L_M$. Moreover, E_M is nontrivial if the N-function M satisfies so called Condition (B) (cf. [8]), which for separable Banach space X can be also formulated in the following simple form (see [14])

Condition (B). There exists a set T_0 of measure zero such that for every natural number n and for every $t \notin T_0$

$$\sup_{\|x\| \leqq n} M(x, t) < \infty .$$

If the Condition (B) is satisfied, then the space E_M equals to the space of all $f \in L_M$

possessing absolutely continuous norms (see [11] Th. 1.2). This means that E_M is the subspace of continuous elements.

2.5. Lemma. The inequality

$$I_M(f) \leqq H(f) \leqq I_M(f) + 1$$

holds for every $f \in L_M$.

2.6. Corollary. If $I_M(f)$ is a natural number, then $H(f) = I_M(f)$.

3. Δ_2-Space

In this section we will use the functional H for more precise characterization of Köthe spaces. From Lemma 2.5 follows that in the case of Musielak-Orlicz space the functional H is "nearby" to modular. Moreover, we will show that using this functional one can be defined some properties of the Köthe space – properties which in the case of the Musielak-Orlicz space are defined by modular.

Define the ball

$$B(f_0, r) = \{f \in L: \|f - f_0\|_L \leqq r\}.$$

3.1. Definition. The Köthe space L is said to be a Δ_2-space if the functional H is bounded on some ball $B(0, r)$ with radius $r > 1$, where 0 denotes the function identically equal to zero.

3.2. Theorem. The following statements are equivalent:

a) L is a Δ_2-space;

b) the functional H is bounded on each ball $B(0, r)$ with radius $r > 1$;

c) the functional H is of polynomial growth, i.e. it satisfies the inequality

$$H(f) \leqq c(1 + \|f\|_L^k)$$

with some positive constant c and k.

The conception of the Δ_2-space in the case of Musielak-Orlicz spaces reduces to the fact that N-function M satisfies the following so-called Condition Δ_2:

there exist a set T_0 of measure zero, a number $K \geqq 1$ and an integrable function $h: T \to [0, \infty)$ such that

$$M(2x, t) \leqq K\, M(x, t) + h(t)$$

for all $x \in X$ and $t \in T \setminus T_0$.

The next theorem shows that fact.

3.3. Theorem. Assume that M satisfies Condition (B). Then the following statements are equivalent:

a) N-function M satisfies Condition Δ_2;

b) L_M is a Δ_2-space;

c) E_M is a Δ_2-space.

4. Applications

An introduction of the conception of Δ_2-space for Köthe spaces makes the consideration of many problems possible. For example, we will investigate boundedness and analyticity of the superposition operator.

4.1. Definition. Suppose the function $F\colon T \times X \to X$ satisfies the Carathéodory conditions, i.e. it is continuous in $x \in X$ for a.a. $t \in T$ and measurable for every $x \in X$. The operator $\bar{F}$ defined by the formula

$$[\bar{F}f](t) = F(t, f(t)),$$

where $f \in \mathcal{M}(T, X)$, is called a superposition operator.

4.2. Definition. Let Λ be a Köthe space. We say that the superposition operator $\bar{F}: L \to \Lambda$ is locally bounded at the point $f_0 \in L$ iff

$$\underset{r>0}{\exists}\ \underset{c>0}{\exists}\ \underset{f\in L}{\forall}\ \|f - f_0\|_L < r \Rightarrow \|\bar{F}f - \bar{F}f_0\|_\Lambda < c.$$

4.3. Theorem. Assume that $\mu(T) < \infty$ and the superposition operator $\bar{F}$ acts from Köthe space L into the subspace E of continuous elements of another Köthe space Λ. Then

a) $\bar{F}$ is locally bounded for every $f \in L$;

b) $\bar{F}$ is bounded on any ball $B(0, r) \subset L$, provided L is a Δ_2-space.

The assumption that L is a Δ_2-space can not be omitted. This fact in ideal spaces was shown by J. Appel and E. De Pascale in [1].

For the study analyticity properties of the superposition operator we will suppose that $X = \mathbb{C}$. In this case $\mathcal{M}(T, \mathbb{C})$ denotes the set of all complex-measurable functions on T. Our definition of Köthe spaces is more general then the definition of ideal spaces included in the paper [2] even in the case of complex-valued functions. The following natural example shows this fact.

4.4. Example. Let $T = (0, 1)$ and μ be the Lebesgue measure. Define $M\colon \mathbb{C} \to [0, \infty)$ by the formula

$$M(z) = x^2 + y^4, \quad \text{where} \quad z = x + iy.$$

It is easy to verify that M is an N-function satisfying Condition Δ_2. Let f_1 and f_2 be two functions defined by the following formulas:

$$f_1(t) = t^{-1/3} + it \quad \text{and} \quad f_2(t) = t + it^{-1/4} \quad \text{for} \quad t \in (0, 1).$$

We have $|f_1(t)| > |f_2(t)|$ for every $t \in (0, 1)$. On the other hand $I_M(f_1) = \frac{16}{5}$ and $I_M(f_2) = \infty$, therefore $f_1 \in L_M$ and $f_2 \notin L_M$. Thus the space L_M can not be an ideal space. Obviously, L_M is a Köthe space.

4.5. Definition. A superposition operator $\bar{F}: L \to \Lambda$ is called analytic at a point $f_0 \in L$ if its increment at f_0 can be written in the form

$$\bar{F}(f_0 + h) - \bar{F}(f_0) = \sum_{n=1}^{\infty} A_n h^n ,$$

where the right-hand side is convergent for small h.

In the above definition A_n $(n = 1, 2, \ldots)$ are monomial operators, i.e. of the form $A_n h^n = \tilde{A}_n(h, h, \ldots, h)$, where $\tilde{A}$ is an n-linear operator from L into Λ. The series $\sum_{n=1}^{\infty} A_n h^n$ is uniformly convergent in the interior of the ball $B(f_0, \varrho_u)$, where

$$\varrho_u = \left(\lim_{n\to\infty} \|A_n\|^{1/n}\right)^{-1} .$$

4.6. Theorem. Suppose that the Köthe space L is Δ_2-space and the superposition operator $\bar{F}$ from L into Λ is analytic in some open set $G \subset L$. Then operator $\bar{F}$ is a polynomial.

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