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Topologies in Atomic Quantum Logics

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We show that in an atomic quantum logic we can introduce a completely regular T_1 (= Tychonoff) topology compatible with a totally bounded uniformity (i.e. a completion of the uniformity is compact). In a case of (o)-continuous quantum logic we use the properties of such topology for the proof of some properties of the logic and the existence of a separating set of outer valuations for the logic. Finally we show connections this topology to some others topologies in the logic.

1. Definitions and preliminary results

Let $(L, 0, 1, \perp, \vee, \wedge)$ be an *quantum logic* (or *logic*, for brevity) i.e. an orthomodular lattice (see [B], [K], [S] for the details). A *measure* on L is a map $m: L \rightarrow \langle 0, \infty)$ such that $m(a \vee b) = m(a) + m(b)$ for any $a \leqq b^{\perp}$, $a, b \in L$. A set M of measures on a logic L is:

(i) *separating* for L if $a \in L$, $a \neq 0 \Rightarrow$ there exists $m \in M$ such that $m(a) \neq 0$.

(ii) *weakly separating* for L if $a \neq b$, $a, b \in L \Rightarrow$ there exists $m \in M$, $x \in L$ such that either $m(a \vee x) \neq m(b \vee x)$ or $m(a \wedge x) \neq m(b \wedge x)$. It is clear that $M = \{m\}$ is separating iff m is faithful, i.e. $m(a) = 0$ iff $a = 0$.

Let M be a set of measures on a logic L. Denote $\mathcal{U}_{D(M)}$ the uniformity generated by the system $D(M)$ of pseudo-metrics on L, where

$$D(M) = \{\varrho_{mx\vee} \mid m \in M,\ x \in L\} \cup \{\varrho_{mx\wedge} \mid m \in M,\ x \in L\}$$

and for any $m \in M$, $x \in L$

$$\varrho_{mx\vee}(a, b) = |m(a \vee x) - m(b \vee x)|$$

$$\varrho_{mx\wedge}(a, b) = |m(a \wedge x) - m(b \wedge x)| .$$

The topology in L compatible with the uniformity $\mathcal{U}_{D(M)}$ is denoted by τ_M. Obviously, the topology τ_M is completely regular and the uniformity $\mathcal{U}_{D(M)}$ is totally bounded,

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since $\mathscr{U}_{D(M)}$ is generated by the family of bounded functions on L (see [C], 4.2.13, p. 168). Hence the completion of $(L, \mathscr{U}_{D(M)})$ is compact (see [C], 6.3.31, p. 257).

It is clear that for any net $(a_\alpha)_\alpha$ of elements of L it holds

$$a_\alpha \to^{\tau_M} a \quad \text{iff} \quad \forall m \in M\ \forall x \in L\colon m(a_\alpha \vee x) \to m(a \vee x) \quad \text{and}$$
$$m(a_\alpha \wedge x) \to m(a \wedge x)\,.$$

Hence the topology τ_M is T_2 iff M is weakly separating and then τ_M is Tychonoff (see [N]). The topology τ_M if $M = \{m\}$ has been introduced in [R]. In [P – R] the topology τ_M has been compared with the order topology τ_0 in L.

An element $a \in L$ is called the *atom* if $a \neq 0$ and $0 \leqq b \leqq a$ implies $b = 0$ or $b = a$. A logic L is *atomic* if every nonzero element in L contains an atom. If L is atomic then every element in L is the supremum of all atoms lying under it.

Let L be an atomic logic and denote $A = \{a \in L \mid a \text{ is an atom in } L\}$. We define for every $a \in A$ functions $f_a\colon L \to \{0, 1\}$, $f_{a^\perp}\colon L \to \{0, 1\}$ as follows:

$$f_a(x) = 1 \quad \text{if} \quad a \leqq x\,, \quad \text{or} \quad f_a(x) = 0 \quad \text{if} \quad a \not\leqq x\,, \quad x \in L$$
$$f_{a^\perp}(x) = 1 \quad \text{if} \quad x \leqq a^\perp\,, \quad \text{or} \quad f_{a^\perp}(x) = 0 \quad \text{if} \quad x \not\leq a^\perp\,, \quad x \in L\,.$$

Denote $\Phi = \{f_a \mid a \in A\}$, $\Phi^\perp = \{f_{a^\perp} \mid a \in A\}$, $\Psi = \Phi \cup \Phi^\perp$. The uniformities and the topologies induced by the function families Φ, $\Phi^\perp$, Ψ are denoted by $\mathscr{U}_\Phi$, $\mathscr{U}_{\Phi^\perp}$, $\mathscr{U}_\Psi$ and τ_Φ, $\tau_{\Phi^\perp}$, τ_Φ respectively (see [C], p. 168). Clearly for any net $(x_\alpha)_\alpha \subseteq L$ and any $x \in L$

$$x_\alpha \to^{\tau_\Phi} x \quad \text{iff} \quad \forall a \in A\colon f_a(x_\alpha) \to f_a(x)$$
$$x_\alpha \to^{\tau_\Phi^\perp} x \quad \text{iff} \quad \forall a \in A\colon f_{a^\perp}(x_\alpha) \to f_{a^\perp}(x)$$
$$x_\alpha \to^{\tau_\Psi} x \quad \text{iff} \quad x_\alpha \to^{\tau_\Phi} x \quad \text{and} \quad x_\alpha \to^{\tau_\Phi^\perp} x$$

In view of this observation it is easy to prove the following Lemma. We denote by τ_i the *interval topology* in L (i.e. the intervals $\langle a, b\rangle = \{x \in L \mid a \leqq x \leqq b\}$, $a, b \in L$ forms a closed subbasis for τ_i).

Lemma 1.1. *Let L be an atomic logic. Then*

(i) $\tau_\Psi \supseteq \tau_\Phi \supseteq \tau_i$, $\tau_\Psi \supseteq \tau_{\Phi^\perp} \supseteq \tau_i$, *the topologies* $\tau_\Phi, \tau_{\Phi^\perp}, \tau_\Psi$ *are Tychonoff and the uniformities* $\mathscr{U}_\Phi$, $\mathscr{U}_{\Phi^\perp}$, $\mathscr{U}_\Psi$ *are totally bounded.*

(ii) $x_\alpha \to^{\tau_\Phi} x$ *iff* $x_\alpha^\perp \to^{\tau_\Phi^\perp} x^\perp$.

(iii) $x_\alpha \to^{\tau_\Psi} x$ *iff* $x_\alpha^\perp \to^{\tau_\Psi} x^\perp$.

(iv) $x_\alpha \to^{\tau_\Phi} x$, $y_\alpha \to^{\tau_\Phi} y \Rightarrow x_\alpha \wedge y_\alpha \to^{\tau_\Phi} x \wedge y$.

(v) $x_\alpha \to^{\tau_\Phi^\perp} x$, $y_\alpha \to^{\tau_\Phi^\perp} y \Rightarrow x_\alpha \vee y_\alpha \to^{\tau_\Phi^\perp} x \vee y$.

(vi) *If* $\tau_\Phi = \tau_{\Phi^\perp} = \tau_\Psi$ *then* $x_\alpha \to^{\tau_\Psi} x$, $y_\alpha \to^{\tau_\Psi} y \Rightarrow x_\alpha \vee y_\alpha \to^{\tau_\Psi} x \vee y$, $x_\alpha \wedge \wedge\, y_\alpha \to^{\tau_\Psi} x \wedge y$.

Recall that a net $(a_\alpha)_\alpha \subseteq L$ (o)-converges to $a \in L$ (denote $a_\alpha \to^{(0)} a$) if there are nets $(b_\alpha)_\alpha, (c_\alpha)_\alpha \subseteq L$ such that $b_\alpha \leqq a_\alpha \leqq c_\alpha$ for every α and $b_\alpha \uparrow a$, $c_\alpha \downarrow a$. The order topology in L is the strongest (finest) topology such that the (o)-convergence implies the topological convergence. A logic L is *(o)-continuous* if $a_\alpha \uparrow a$ implies $a_\alpha \wedge b \uparrow a \wedge b$ for every $b \in L$ (dually $a_\alpha \downarrow a$ implies $a_\alpha \vee b \downarrow a \vee b$ for every $b \in L$). A measure m on L is (o)-continuous if $a_\alpha \downarrow a$ implies $m(a_\alpha) \to m(a)$.

Lemma 1.2. *Let L be an (o)-continuous atomic logic. Then*

(i) $\forall a \in A$: $\langle a, 1\rangle$ *is a clopen set in* τ_Φ.

(ii) $\forall a \in A$: $\langle 0, a^\perp\rangle$ *is a clopen set in* $\tau_{\Phi^\perp}$.

(iii) $\forall a \in A$: $\langle a, 1\rangle$, $\langle 0, a^\perp\rangle$ *are clopen sets in* τ_Ψ.

(iv) *For every $x \in L$ the neighbourhood filter $\mathscr{U}(x)$ in τ_0 has a base of intervals in L which are clopen sets in τ_Ψ.*

(v) $\tau_0 = \tau_\Psi$.

(vi) *If M is a separating set of (o)-continuous measures on L then $\tau_M = \tau_\Psi$.*

Proof. (i) Let a be an atom in L. Since $\tau_\Phi \supseteq \tau_i$ the interval $\langle a, 1\rangle$ is a closed set in τ_Φ. Let $(x_\alpha)_\alpha$ be a net in L such that $x_\alpha \to^{\tau_\Phi} x \in \langle a, 1\rangle$. Then $f_a(x_\alpha) \to f_a(x) = 1$ and hence there exists α_0 such that for every $\alpha \geqq \alpha_0$: $f_\alpha(x_\alpha) = 1$ which implies $x_\alpha \in \langle a, 1\rangle$. Thus $\langle a, 1\rangle$ is also an open set in τ_Φ.

(ii) follows by arguments quite similar to that of previous case.

(iii) follows from (i) and (ii) and from the fact that $\tau_\Psi \supseteq \tau_\Phi$, $\tau_\Psi \supseteq \tau_{\Phi^\perp}$.

(iv) Let $x \in L$, $x \neq 0$, $x \neq 1$. Let $V(x)$ be any open neighbourhood of x in τ_0 As L is atomic, there are sets of atoms $\{a_\alpha \mid \alpha \in A\}$, $\{b_\beta \mid \beta \in B\}$ such that $x = \bigvee_{\alpha\in A} a_\alpha$. $x^\perp = \bigvee_{\beta\in B} b_\beta$. Put $C = \{\gamma \subseteq A \cup B \mid \gamma \cap A \neq \emptyset, \gamma \cap B \neq \emptyset, \gamma \text{ is finite}\}$ and let $\gamma_1 \leqq \gamma_2$ iff $\gamma_1 \subseteq \gamma_2$. For every $\gamma \in C$ put $x_\gamma = \bigvee_{\alpha\in\gamma\cap A} a_\alpha$, $y_\gamma = \bigwedge_{\beta\in\gamma\cap B} b_\beta^\perp$. Then $x_\gamma \uparrow x$, $y_\gamma \downarrow x$. In view of (iii) $\bigcap_{\alpha\in\gamma\cap A} \langle a_\alpha, 1\rangle = \langle x_\gamma, 1\rangle$, $\bigcap_{\beta\in\gamma\cap B} \langle 0, b_\beta^\perp\rangle = \langle 0, y_\gamma\rangle$ and $\langle x_\gamma, y_\gamma\rangle$ are clopen sets in τ_Ψ.

Suppose that for every $\gamma \in C$ there is $z_\gamma \in \langle x_\gamma, y_\gamma\rangle$ such that $z_\gamma \notin V(x)$. Since $x_\gamma \leqq z_\gamma \leqq y_\gamma$ for every $\gamma \in C$ we get $z_\gamma \to^{(o)} x$. As $L \setminus V(x)$ is closed in τ_0 we obtain $x \in L \setminus V(x)$, a contradiction. Thus $\{\langle x_\gamma, y_\gamma\rangle \mid \gamma \in C\}$ is a base of the neighbourhood filter of x in τ_0 By the similar way we obtain that for $x = 1$ the collection $\{\langle x_\gamma, 1\rangle \mid \gamma \in C\}$ and for $x = 0$ the collection $\{\langle 0, y_\gamma\rangle \mid \gamma \in C\}$ are bases of the neighbourhood filters in τ_0 of $x = 1$ and $x = 0$ respectively.

(v) If $x_\alpha \to^{(o)} x$ then the (o)-continuity of L results that $x_\alpha \to^{\tau_\Phi} x$ and $x_\alpha \to^{\tau_{\Phi^\perp}} x$. Hence $\tau_0 \supseteq \tau_\Phi$ and $\tau_0 \supseteq \tau_{\Phi^\perp}$. Thus $\tau_0 \supseteq \tau_\Psi$ and in view of (iv) $\tau_0 = \tau_\Psi$.

(vi) If $x_\alpha \to^{(o)} x$ then the (o)-continuity of L implies that for every $y \in L$ $x_\alpha \vee y \to^{(o)} x \vee y$, $x_\alpha \wedge y \to^{(o)} x \wedge y$ and in view of the (o)-continuity of every $m \in M$ we obtain $x_\alpha \to^{\tau_M} x$ and thus $\tau_0 \supseteq \tau_M$. The relation $\tau_M \supseteq \tau_0$ has been proved in [P–R], Theorem 3. Now, using (v) we get $\tau_\Psi = \tau_0 = \tau_M$.

2. Complete atomic logic

A logic L is called *complete* if it is a complete lattice. It is known that in a lattice L the interval topology τ_i is compact iff L is a complete lattice.

Lemma 2.1. *Let L be an atomic logic. Then*

(i) *If $(L, \mathcal{U}_\Psi)$ is a complete uniform space then L is the complete logic.*

(ii) *If L is a complete logic and the interval topology τ_i in L is T_2 then $\tau_\Psi \supseteq \tau_0 = \tau_i$.*

Proof. (i) Assume that $(L, \mathcal{U}_\Psi)$ is a complete uniform space. Then since $\mathcal{U}_\Psi$ is totally bounded, it is also compact. Since $\tau_\Psi \supseteq \tau_i$ we conclude that τ_i is compact and L is the complete logic.

(ii) If L is a complete logic and the interval topology τ_i in L is T_2 then $\tau_0 = \tau$. by [E–W] p. 809 and by (i) of Lemma 1.1 $\tau_\Psi \supseteq \tau_0$.

We note that the deviating definition of (o)-convergence and τ_0-topology in partially ordered set in terms of filters is for example in [E] and [E–W]. One can show that in lattice (but not in all poset) this two definitions of order topology coincide. Moreover, in complete lattice L we can show that a net $(x_\alpha)_\alpha \subseteq L$ (o)-converges to $x \in L$ iff the filter $\mathfrak{F}$ derived from the net $(x_\alpha)_\alpha$ (o)-converges to x (see [E] and [E–W] for the definitions).

In [S] have been studied uniform logics. Recall that *a complete logic L with the T_2 uniformity $\mathcal{U}$ on L is called a uniform logic if*

(i) *the map $x \to x^\perp$ is uniformly continuous,*

(ii) *the map $(x, y) \to x \vee y$ is uniformly continuous,*

(iii) *$x_\alpha \downarrow x$, $x_\alpha, x \in L$ implies $x_\alpha \to^\tau x$, where τ is the topology compatible with $\mathcal{U}$.*

A map $m: L \to \langle 0, \infty)$ on a logic L is called *outer valuation* on L if $m(0) = 0$, $m(x) \leqq m(y)$ for all $x \leqq y$, $x, y \in L$ and $m(x \vee y) \leqq m(x) + m(y)$ for all $x, y \in L$.

An outer valuation on L is called the *outer $\mathbb{R}$-valuation* if $m(a_\alpha \Delta b_\alpha) \to 0$, $m(c_\alpha \Delta d_\alpha) \to 0$ implies $m[(a_\alpha \vee c_\alpha) \Delta (b_\alpha \vee d_\alpha)] \to 0$, $a_\alpha, b_\alpha, c_\alpha, d_\alpha \in L$. The symbol $a \Delta b$ will denote the symmetric difference, i.e. $a \Delta b = (a \wedge (a \wedge b)^\perp) \vee (b \wedge \wedge (a \wedge b)^\perp)$.

If $m: L \to \langle 0, \infty)$ is an outer valuation on a logic L then the map $\varrho_m: L \times L \to \to \langle 0, \infty)$

$$\varrho_m(x, y) = m(x \Delta y), \quad x, y \in L$$

is a pseudo-metric on L. Note that if $m: L \to \langle 0, \infty)$ is a measure on L then ϱ_m need not be pseudo-metric and ϱ_m is a pseudo-metric in L iff m is subadditive (i.e. $m(x \vee y) \leqq m(x) + m(y)$, $x, y \in L$). If m is a subadditive measure on L then for the topology $\tau_{\{m\}}$ compatible with the uniformity $\mathcal{U}_{D(\{m\})}$ defined in § 1 and the topology τ_{ϱ_m} compatible with the uniformity $\mathcal{U}_{\varrho_m}$ induced by ϱ_m it holds $\tau_{\{m\}} = \tau_{\varrho_m}$ (but $\mathcal{U}_{\{m\}} \neq \mathcal{U}_{\varrho_m}$ in general, since $\mathcal{U}_{\varrho_m}$ need not be totally bounded) (see [R]). If M^* is a set of outer valuation on a logic L, we denote $\mathcal{U}_{R(M^*)}$ the uniformity on L induced.

by collection of pseudo-metrics $R(M^*) = \{\varrho_m \mid m \in M^*\}$ and τ_{M^*} the topology compatible with $\mathscr{U}_{R(M^*)}$.

Finally recall that a logic L is called *separable* if any set of mutually orthogonal nonzero elements in L is at most countable.

Theorem 2.2. *Let L be a complete* (o)*-continuous logic and the interval topology τ in L is T_2. Then*

(i) $\tau_0 = \tau_\Psi = \tau_\Phi = \tau_{\Phi^\perp} = \tau_i$ *is a compact totally disconnected, completely regular T_2 topology and $\mathscr{U}_\Psi$ is complete and only one uniformity compatible with τ_0.*

(ii) $x_\alpha \to^{(o)} x$ *iff* $x_\alpha \to^{\tau_0} x$ *iff* $x_\alpha \Delta x \to^{(o)} 0$, *and* $x_\alpha \to^{\tau_0} x$, $y_\alpha \to^{\tau_0} y$ *implies* $x_\alpha \vee y_\alpha \to^{\tau_0} x \vee y$, $x_\alpha \wedge y_\alpha \to^{\tau_0} x \wedge y$, $x_\alpha y_\alpha, x, y \in L$.

(iii) $(L, \mathscr{U}_\Psi)$ *is a uniform logic.*

(iv) *There exists a separating set M^* of* (o)*-continuous outer* $\mathbb{R}$*-valuations on L and* $\tau_{M^*} = \tau_0$, $\mathscr{U}_{R(M^*)} = \mathscr{U}_\Psi$. *Moreover any $m \in M^*$ is uniformly continuous on* $(L, \mathscr{U}_{R(M^*)})$.

(v) *If M is any separating set of* (o)*-continuous measures on L then* $\tau_M = \tau_{M^*}$, $\mathscr{U}_{D(M)} = \mathscr{U}_{R(M^*)}$ *and every $m \in M$ is uniformly continuous on* $(L, \mathscr{U}_{D(M)})$.

(vi) *If* $f: L \to (-\infty, \infty)$ *is a τ_0-continuous function such that $f(a) = 0$ iff $a = 0$ then* $f(a_\alpha) \to 0$ *iff* $a_\alpha \to^{(o)} 0$.

(vii) *If* $m: L \to \langle 0, \infty)$ *is an* (o)*-continuous faithful measure on L then* $x_\alpha \to^{\tau(m)} x$ *iff* $m(x_\alpha \Delta x) \to 0$.

(viii) *L is separable iff τ_0 is metrizable and in this case L contains a τ_0-dense countable subset.*

(ix) *L is separable iff there exists an* (o)*-continuous faithful outer* $\mathbb{R}$*-valuation m on L and then for any* (o)*-continuous faithful measure ω on L it holds*

$$\forall \varepsilon > 0 \; \forall x \in L \; \exists \delta > 0: m(b) < \delta \Rightarrow \omega(b \vee x) < \omega(x) + \varepsilon$$

(*for the* $x = 0$ *we get* $\omega \ll_\varepsilon m$).

Proof. (i), (ii) The facts that L is complete and τ_i is T_2 imply that L is atomic (see [S] p. 75) and τ_i is compact (see [B], p. 326). In view of lemmas 1.1 and 1.2 (v) we get $\tau_0 = \tau_\Psi \supseteq \tau_i$. Using the results of [E—W], p. 817 and the assertion (iv) of Lemma 1.2 we obtain $\tau_0 = \tau_\Psi = \tau_i$ and also that $x_\alpha \to^{(o)} x$ iff $x_\alpha \to^{\tau_0} x$, for any net $(x_\alpha)_\alpha \subseteq L$. Now (ii) follows from the (o)-continuity of L. Since τ_Ψ is compact complete regular topology then τ_Ψ has one and only one uniformity compatible with the topology (see [N], Theorem VI.17, p. 290) and $\mathscr{U}_\Psi$ is complte.

(iii) In view of (ii) and the fact that $\tau_0 = \tau_\Psi$ is compact we obtain that the orthocomplementation and the lattice operations are uniformly continuous and we conclude that $(L, \mathscr{U}_\Psi)$ is the uniform logic.

(iv) The existence of M^* and the fact that $(L, \mathscr{U}_{R(M^*)})$ is a uniform logic follows from [S], Theorem 4, p. 59. Hence $\mathscr{U}_{R(M^*)} = \mathscr{U}_\Psi$ (see [S], Theorem 3, p. 56). Any

$m \in M^*$ is uniformly continuous on $(L, \mathscr{U}_{R(M^*)})$ since m is τ_0-continuous and $\tau_0 = \tau_{\mathscr{U}R(M^*)}$ is compact.

(v) In view of (vi) of Lemma 1.2 we have $\tau_M = \tau_0$. Now we use (i) and (iv) of this Theorem.

(vi) Let $f: L \to (-\infty, \infty)$ be a τ_0-continuous function such that $f(a) = 0$ iff $a = 0$. Let for a net $(x_\alpha)_\alpha \subseteq L, f(x_\alpha) \to 0$. τ_0 is compact and hence from any subnet of the net $(x_\alpha)_\alpha$ there exists a subnet $x_\gamma \to^{\tau_0} c$. Since $f(a) = 0$ iff $a = 0$ and $f(x_\gamma) \to f(c) = 0$ we get $c = 0$ and thus $x_\alpha \to^{\tau_0} 0$. Now from (ii) we have $x_\alpha \to^{(o)} 0$.

(vii) If $m: L \to \langle 0, \infty)$ is an (o)-continuous faithful measure on L then $M = \{m\}$ is the separating set of measures for L. Using (vi) of Lemma 1.2 we get $\tau_{\{m\}} = \tau_0$. Now the assertion is immediate from (ii) and (vi).

(viii) $(L, \mathscr{U}_\Psi)$ is a uniform logic and hence L is separable iff $\mathscr{U}_\Psi$ is metrizable (see [S], Theorem 2, p. 55). But in this case (L, τ_0) is a totally bounded metric space and hence it is a separable metric space, i.e. L contains a countable τ_0-dense subset in L (see [C], p. 103).

(ix) The assertion that L is separable iff there exists an (o)-continuous faithful outer $\mathbb{R}$-valuation m, follows from (iii) and Corollary 3, [S] p. 61. By (iv) $\tau_{\varrho_m} = \tau_0$ and $\mathscr{U}_{\varrho_m} = \mathscr{U}_\Psi$. Let ω be an (o)-continuous faithful measure on L. Then $\tau_{\{\omega\}} = \tau_0 = \tau_{\varrho_m}$ and $\mathscr{U}_{D(\{\omega\})} = \mathscr{U}_{\varrho_m}$.

Let $\varepsilon > 0$, $x \in L$. Denote $U_{\varepsilon,x} = \{(a, b) \in L \times L \mid |\omega(a \vee x) - \omega(b \vee x)| < \varepsilon\}$. Then $U_{\varepsilon,x} \in \mathscr{U}_{D(\{\omega\})}$ and

$$U_{\varepsilon,x}[0] = \{b \in L \mid (0, b) \in U_{\varepsilon,x}\} = \{b \in L \mid |\omega(b \vee x) - \omega(x)| < \varepsilon\} = \\ = \{b \in L \mid \omega(b \vee x) < \omega(x) + \varepsilon\}.$$

$U_{\varepsilon,x}[0]$ is a neighbourhood of a point 0 in $\tau_{(\omega)} = \tau_{\varrho_m}$ and hence there exists $\delta > 0$ such that $\{b \in L \mid \varrho_m(b, 0) \langle \delta\} \subseteq U_{\varepsilon,x}[0]$. We obtain $\{b \in L \mid m(b \,\Delta\, 0) < \delta\} \subseteq \{b \in L \mid \omega(b \vee x) < \omega(x) + \varepsilon\}$ and hence $\omega(b) < \delta \Rightarrow \omega(b \vee x) < \omega(x) + \varepsilon$. This completes the proof.

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