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C_4-Saturated Graphs of Minimum Size

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We consider simple undirected graphs, with bo loops or multiple edges. Standard terminology of graph theory is used; undefined notions can be found e.g. in [1].

Let F be a given graph. Call a graph G *F-saturated* if F is not a subgraph of G, but a subgraph isomorphic to F appears whenever a new edge is added to G. Denoting by $V(G)$ and $E(G)$ the set of vertices and edges, respectively, of G, define

$$\text{sat}(n, F) = \min\{|E(G)| : |V(G)| = n,\ G \text{ is } F\text{-saturated}\},$$

the minimum number of edges in an F-saturated graph on n vertices. Now the problem is to determine sat (n, F) for given F and n (possibly when n is large), and to describe the graphs G with n vertices and sat (n, F) edges, that are F-saturated. Note that for $n < |V(F)|$ the complete graph is the unique F-saturated one.

The first result of this type was published in 1964 (Erdös, Hajnal and Moon [2]), but it took two decades until the first general upper bound on sat (n, F) appeared (Kászonyi and Tuza [3]). A survey of results is given in [5], where also hypergraphs and weakened conditions are discussed.

It is surprising how difficult the determination of sat (n, F) is even in case of very small F. For instance, denoting by C_k the cycle on k vertices, the value of sat (n, C_5) is not known. Perhaps it is $3n/2 + 0(1)$ as n tends to infinity. For C_4, Ollman [4] proved the following result.

Theorem 1. For $n \geqq 5$, sat $(n, C_4) = [(3n - 5)/2]$. Moreover, if G is a C_4-saturated graph with n vertices and $[(3n - 5)/2]$ edges, then G has some of the structures shown in Fig. 1; namely, if n is even, then G has a 'central' triangle, each of whose vertices are adjacent to precisely one vertex of degree one, and the remaining vertices of G are in adjacent pairs, each of them joined to a vertex of the central triangle; if n is odd, then G either is obtained from the previous construction by deleting one vertex of degree one, or consists of a C_5, two consecutive vertices of which are joined to arbitrary numbers of adjacent pairs.

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The original proof of Theorem 1 in [6] is about 20 typewritten pages long. In the present note we give a shorter argument which still is not a very simple one. Perhaps the difficulty is related with the fact that in case of n odd we have two entirely different types of extremal structures.

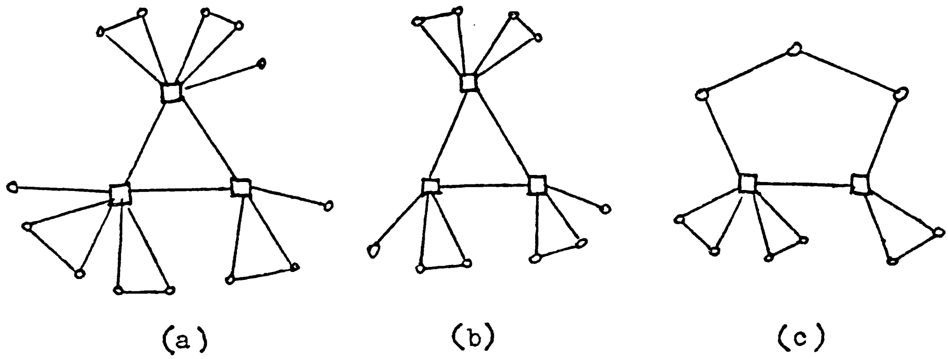

Figure 1. C_4 — saturated graphs with $[(3n - 5)/2]$ edges.
(a); n even; (b) and (c): n odd.

Before restricting our investigations to C_4-saturated graphs, let us formulate some simple properties that hold under more general assumptions. Recall that a graph is said to be k-vertex-connected (k-edge-connected) if it cannot be made disconnected by the deletion of less than k vertices (edges), i.e. deleting fewer vertices (edges) we always find paths between any two vertices. For $k = 0$ we have no restriction, and in case of $k = 1$ we simply say that the graph is connected. For a subgraph G' of G, $G \setminus G'$ denotes the subgraph of G induced by $V(G) \setminus V(G')$.

Proposition 2. (a) If F is k-vertex-connected, other than the complete graph on k vertices, then every F-saturated G is $(k - 1)$-vertex-connected.

(b) If F is k-edge-connected, then every F-saturated G is $(k - 1)$-edge-connected.

Proof. Let G be an F-saturated graph. To prove (a), suppose to the contrary that a set X of at most $k - 2$ vertices disconnects G. Assuming that V_1 and V_2 are the vertex sets of two components of $G \setminus X$, add an edge (x_1x_2) to G, $x_i \in V_i$ $(i = 1, 2)$. Then a subgraph F must appear. Without loss of generality we can assume that this F has at least two vertices outside $X \setminus V_1$. In this case, however, $X \cup \{x_2\}$ would be a cut-set of cardinality at most $k - 1$ in F, a contradiction.

To prove (b), suppose that there are at most $k - 2$ edges between the vertex sets V_1 and V_2, $V_1 \cup V_2 = V(G)$. Since F and hence G as well has at least k vertices, there are $x_i \in V_i$ $(i = 1, 2)$ such that (x_1x_2) is not an edge of G. Adding (x_1x_2) to $E(G)$, F has to occur as a subgraph. In this F, however, its two parts contained in V_1 and V_2, respectively, would be separated by at most $k - 1$ edges, a contradiction. □

For saturated graphs whose connectivity is as small as possible, we have the following.

Proposition 3. (a) Let F be a k-vertex-connected graph, and let G be an F-saturated graph with a set X of $k - 1$ vertices such that $G \setminus X$ is disconnected. Denote by $G_1, \ldots, G_t$ the connected components of $G \setminus X$. If any two vertices of X are adjacent, then
(a1) $G \setminus G_i$ is F-saturated for $1 \leqq i \leqq t$;
(a2) $G_i \cup X$ induces an F-saturated graph $(1 \leqq i \leqq t)$.

(b) Let F be a k-edge-connected graph, and suppose that a graph G has a partition $V_1 \cup V_2 = V(G)$ such that there are just $k - 1$ edges between V_1 and V_2. If G is F-saturated, then the subgraph induced by V_i $(i = 1, 2)$ is F-saturated, too.

Proof. If a new edge is contained in $X \cup V(G_i)$ or in $V(G_i) \cup V(G_j)$, then the subgraph isomorphic to F must appear in $X \cup V(G_i)$ or in $V(G_i) \cup V(G_j) \cup X$, respectively, otherwise X would be a cut-set of F. This proves (a). Similarly, if a new edge is contained in V_i, then F cannot have a vertex in V_{3-i}, otherwise the deletion of the $k - 1$ edges joining V_1 with V_2 would disconnect F. This proves (b). □

In particular, if F is connected and G is an F-saturated graph, then every connected component of G is F-saturated. Another important case is when F is 2-vertex-connected (*2-connected*, for short). Define a *block* of a graph as a 2-connected subgraph maximal under inclusion.

Corollary 4. Let F be a 2-connected graph. If G is an F-saturated graph, then every block of G is F-saturated.

Proof. Apply induction on the number b of blocks. For $b = 1$ we have nothing to prove. Moreover, we can assume that G is connected, by putting $X = \emptyset$ in Theorem 3(a2). Then, if G is not 2-connected, it contains a vertex x such that $G_i \cup \{x\}$ induces a block of G, for some connected component G_i of G. By Theorem 3, $G_i \cup \{x\}$ is F-saturated, as well as $G \setminus G_i$. Since every block other than $G_i \cup \{x\}$ is a block of $G \setminus G_i$, too, the statement follows by induction since $G \setminus G_i$ has fewer blocks than G. □

The *distance* between two vertices x and y of a connected graph is the number of edges in the shortest $x - y$ path. The *diameter* of G is the largest distance between any two vertices $x, y \in V(G)$.

Proposition 5. Let F be a 2-connected graph having no cycle of more than s vertices. If G is an F-saturated graph, then G has diameter at most $s - 1$.

Proof. Deleting any edge from F, the graph obtained has diameter at most $s - 1$ (otherwise the deleted edge would be contained in a cycle longer than s). Adding an edge (xy) to G, a subgraph isomorphic to F and containing (xy) occurs, so that the distance of x and y cannot be larger in G than in $F - \{(xy)\}$. □

Let us summarize the consequences of the above observations for C_4-free graphs.

Lemma. Let G be a C_4-saturated graph. Then
(6.1) G is connected,
(6.2) G has diameter at most 3.

Moreover, if G contains a cut-vertex x, and $G_1, \ldots, G_t$ are the connected components of $G \setminus \{x\}$, then

(6.3) every $G_i \cup \{x\}$ induces a C_4-saturated graph,

(6.4) $G \setminus G_i$ is C_4-saturated for $1 \leqq i \leqq t$,

(6.5) all vertices not adjacent to x belong to the same component G_i,

(6.6) all the other G_j are disjoint edges that form triangles with x, with possibly one exception which is a single vertex adjacent to x. □

Proof of Theorem 1. One can see that the graphs shown in Fig. 1 are C_4-saturated, independently of the number of triangles attached to the vertices indicated by squares (of course the number may be zero as well). Thus, for every $n \geqq 5$, $\text{sat}(n, C_4) \leqq [(3n - 5)/2]$.

To prove the lower bound, let G be a C_4-saturated graph with a minimum number of edges on n vertices. Putting $f(n) = [(3n - 5)/2]$, for $n = 5$ and 6 we have $f(n) = n$. In these cases, if G had less than $f(n)$ edges, then either G would be disconnected (which is impossible by (6.1)), or it would be a tree. In the latter case, however, it would contain two non-adjacent vertices x and y such that either the common neighbour of x and y has degree 2, or x and y both have degree 1 and they are adjacent to the same vertex. Anyway, adding (xy) to G we obtain a triangle but not a C_4, contradicing the assumption that G is C_4-saturated. Hence, the statement is true for $n = 5, 6$.

I. Suppose first that G has a cut-vertex x. By (6, 6), the connected components $G_1, \ldots, G_t$ of $G \setminus x$ are isolated edges, with possibly two exceptions. If, say, G_1 is an edge then deleting G_1 from G we obtain a graph of $n - 2$ vertices and $f(n) - 3 = f(n - 2)$ edges. Now (6.4) implies that the theorem follows by induction.

Thus, we may suppose that $G_1 = y$ is a vertex of degree 1, adjacent to x, and $G_2 \cup x = G \setminus y$ is a C_4-saturated graph. Then, by (6.2), G_2 consists of two levels A and B: the neighbours of x (denoted by A) and the vertices not adjacent to x but adjacent to some vertex of A.

As any edge (ya), $a \in A$, produces a C_4, the vertex set A induces a 1-regular graph in G, that is, $|A|$ is even and $A \cup \{x\}$ contains exactly $\frac{3}{2}|A|$ edges of G.

Since G is C_4-free, every $b \in B$ is adjacent to exactly one $a \in A$. Moreover, if there are two vertices $b_1, b_2 \in B$ of degree 1, then their neighbours are adjacent in A. Consequently, there are at most two vertices in B with degree 1. Therefore, by $|A| + |B| = n - 2$,

$$f(n) \geqq \tfrac{3}{2}|A| + |B| + \left[\frac{|B| - 1}{2}\right] + 1 \geqq \left[\frac{3|A| + 3|B| + 1}{2}\right] = \left[\frac{3n - 5}{2}\right].$$

By (6.4)–(6.6), it is easy to check that equality holds only for the graphs (a) and (b) in Fig. 1. (Observe that in case of equality G has at least two vertices of degree 1, the other vertices in B have to induce a 1-regular graph and each of them must have degree 2.)

II. Thus, it is enough to show that C_5 is the *unique* 2-*connected* and C_4-*saturated* graph of at most $f(n)$ edges. From now on, *suppose that G is* 2-*connected*. We distinguish between three cases, according to the behaviour of vertices of degree 2.

A) There are two adjacent vertices x, x' of degree 2 (see Fig. 2).

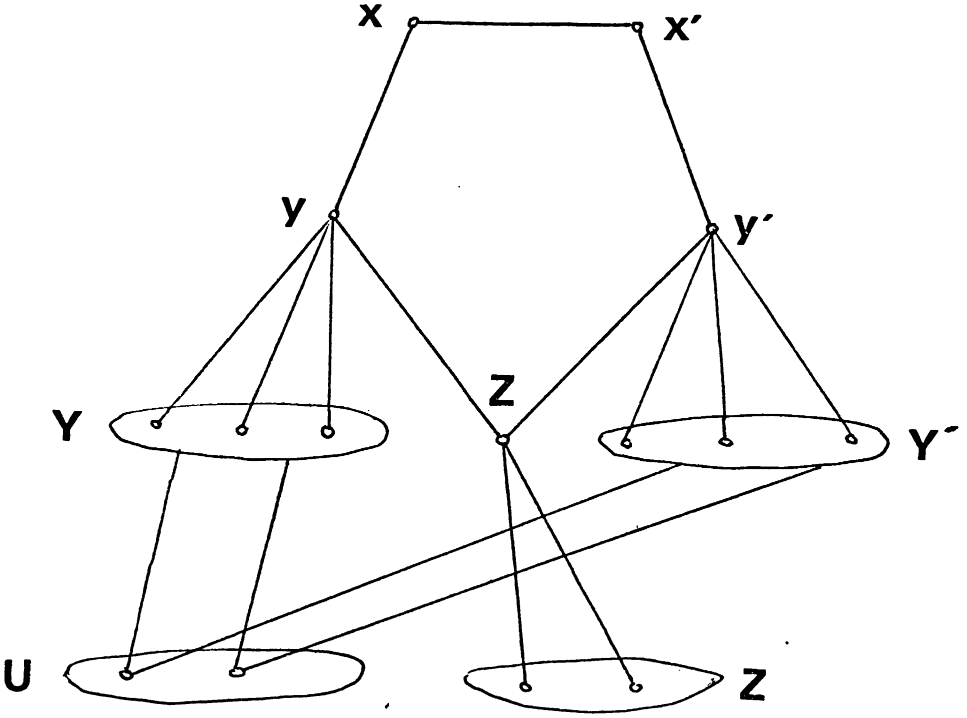

Fig. 2.

If y and y' are the neighbours of x and x', resp., then they are not adjacent and they have exactly one common neighbour z. Denote by Y and Y' the sets of their other neighbours. The remaining vertices are adjacent to z (call their set Z) or they have neighbours in Y *and* Y' because the distance of any vertex from x and x' is at most 3.

If $v \in Y$, (vx) gives a C_4, therefore v is adjacent to some vertex in $Y \cup \{z\}$. A similar property holds for Y'. Moreover, since z is not a cut-vertex, there are at least $|Z|$ edges incident to Z in the subgraph induced by $U \cup Z$. Thus,

$$f(n) \geqq 5 + |Y| + |Y'| + \left[\frac{|Y| + 1}{2}\right] + \left[\frac{|Y'| + 1}{2}\right] + 2(|U| + |Z|) \geqq$$

$$\geqq \tfrac{3}{2}(|Y| + |Y'| + |U| + |Z|) + 5 + \tfrac{1}{2}(|U| + |Z|) \geqq \tfrac{3}{2}(n - 5) + 5 = f(n)$$

with equality only if $U = Z = \emptyset$, $|Y|$ and $|Y'|$ even, and so (by 2-connectivity) $Y = Y' = \emptyset$ and $G = C_5$ as stated.

B) There is a vertex x of degree 2, with neighbours y and y' such that y is not adjacent to y'.

Denote by Y and Y' the set of neighbours of y and y', resp., in $G \setminus x$. (Now $Y \cap Y' = \emptyset$.) If $v \in Y \cup Y'$ then (vx) gives a C_4, therefore $Y \cup Y'$ contains at least $[\frac{1}{2}(|Y| + |Y'| + 1)]$ edges. The other vertices, forming a set called Z are adjacent to $Y \cup Y'$ (by (6.2)) and have degree $\geqq 2$. Thus,

$$f(n) \geqq 2 + |Y| + |Y'| + \left[\frac{|Y| + |Y'| + 1}{2}\right] + |Z| + \left[\frac{|Z| + 1}{2}\right] \geqq$$

$$\geqq \tfrac{3}{2}(|Y| + |Y'| + |Z|) + 2 = \tfrac{3}{2}(n - 3) + 2 = f(n)$$

with equality only if Z, as well as $Y \cup Y'$, induces a 1-regular subgraph in G. But in this case there are two *adjacent* vertices of degree 2 in Z (if $Z \neq \emptyset$) or in $Y \cup Y'$ (if $Z = \emptyset$) and we are back to case A).

C) Every vertex of degree 2 is contained by a triangle.

Let G' be the subgraph induced by the vertices of degree $\geqq 3$ in G. Call an edge of G' *red* if it forms a triangle with some vertex of degree 2 in G, and call the other edges of G' *blue*.

Let $X_1, \ldots, X_k$ be the partition of the vertices of G' in which the X_i's induce the connected components in the graph of red edges. (That is, if a vertex v' of G' is contained by no red edge then v' itself is a one-element class in the partition. Such a vertex has degree $\geqq 3$ in G', too.) With the help of the X_i's we define a partition of $V(G)$ as follows:

$$V_i = X_i \cup \{v \in V(G) \setminus V(G'): \text{the red edge belonging to } v \text{ lies in } X_i\}\,.$$

Clearly, every red edge of X_i is incident to exactly one triangle (meeting $V_i \setminus X_i$). Since $f(n) < 3n/2$, there is a V_i such that, in G, the average degree of the vertices belonging to V_i is *less* than 3. If there are t red edges in X_i $(t \geqq 1)$, then $|V_i| = |X_i| + t \leqq 2t + 1$ and V_i contains $\geqq 3t$ edges (the red edges define edge-disjoint triangles of G). Thus, $|X_i| = t + 1$, and the red edges form a spanning tree T of X_i. Moreover, $V_i \neq V(G)$, because $f(|V_i|) = f(2t + 1) = 3t - 1 < 3t$.

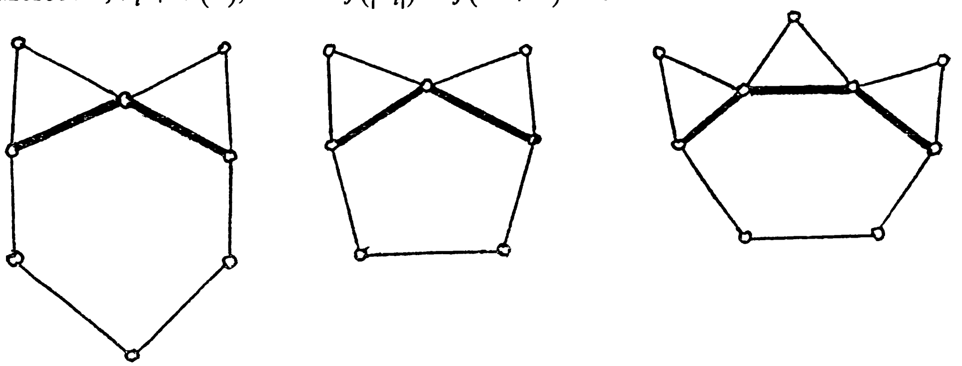

Fig. 3.

By 2-connectivity, there are at least two edges between V_i and $V(G) \setminus V_i$, so that X_i cannot contain any blue edges. Each endpoint $v \in T$ has degree $\geqq 3$ in G, hence there is a blue edge e_v from v to $V(G) \setminus V_i$. Thus, V_i contains at most $(6t + 2 - j)/2 \leqq$ $\leqq 3t$ edges, where j is the number of endpoints of T. Since $j \geqq 2$, equality must hold, i.e., T is a path with endpoints v_1 and v_2.

Denote by $y_k \in V(G) \setminus V_i$ the endpoint of the edge e_k incident to v_k, $k = 1, 2$. Clearly, $y_1 \neq y_2$ (G is 2-connected) and the path T consists of at most three vertices (by (6.2)). Therefore, if $|X_i| > 2$, G should be one of the graphs of Fig. 3 (bold lines indicate T), each having more than $f(n)$ edges. Thus, T is an edge.

Let Y_k denote the set of vertices being outside T and adjacent to y_k, $k = 1, 2$. Since $V_i \cup Y_1 \cup Y_2 \cup \{y_1, y_2\} = V(G)$, and each vertex of $Y_1 \cup Y_2$ has degree $\geqq 2$,

$$|E(G)| = f(n) \geqq 5 + |Y_1| + |Y_2| + |Y_1 \cup Y_2|/2 \geqq 5 + 3(n - 5)/2 \geqq f(n),$$

moreover, equality can hold only if $Y_1 \cap Y_2 = \emptyset$, and $Y_1 \cup Y_2$ induces subgraph of pairwise disjoint edges. Then there are two adjacent vertices (both in $Y_1 \cup Y_2$) of degree two and we are back to case A) again.

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