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# A CLASS OF BANACH LATTICES AND POSITIVE OPERATORS

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By an operator we mean a bounded linear transformation. Let $B$ be a real Banach lattice. A set of all positive operators mapping $B$ into $B$ is denoted by $\mathcal{L}_+(B)$ i.e. $T \in \mathcal{L}_+(B)$ if and only if $Tx \geqslant 0$ for all $x \geqslant 0$. We say that a Banach lattice $B$ has the property $W$ if the isometric domain

$$M(T) = \{ x \in B : \|Tx\| = \|T\| \|x\| \}$$

is a linear subspace of $B$ for all $T \in \mathcal{L}_+(B)$.

In [1] it was shown that $L^p$-spaces, $1 < p < \infty$, have the property $W$. The proof of this result is based on properties of doubly stochastic operators established by Ryff [4],[5]. In the class of Orlicz spaces $L^{\phi}(R)$ (with $\phi : R_+ \to R_+$ strictly convex and $\phi(0) = 0$), equipped with the Minkowski norm only $L^p$-spaces have the property $W$ (see [2]). In view of the above facts it would be interesting to know whether there exist spaces which are not $L^p$-spaces and which have the property $W$.

In this note we give an example of a two dimensional Orlicz space with the property $W$, which is not an $l_2^p$-space. Next we consider other properties of the two-dimensional Banach lattice with the property $W$.

<u>Theorem 1.</u> Let $B$ be a Banach lattice with the property $W$. Then $B$ is strictly convex.

Proof. To get a contradiction suppose that $B$ is not strictly convex. Then there exist distinct positive vectors $u_1$, $u_2$ such that $\| a u_1 + (1-a) u_2 \| = 1$ for all $a \in [0,1]$. Let $f \in B^*$ be such that $\|f\| = f(u_1 + u_2)/2 = 1$. Then $f(u_1) = f(u_2) = 1$. Obviously $f_+(u_1) = f_+(u_2) = \|f_+\| = 1$. Now consider the operator $T$ defined by $Tx = x_0 f_+(x)$, where $x_0 \in B$ is a fixed vector, $x_0 \geqslant 0$, $\|x_0\| = 1$. We have $u_1, u_2 \in M(T)$ and $u_1 - u_2 \notin M(T)$, so $M(T)$ is not a linear space. This contradiction proves our Theorem.

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This paper is in final form and no version of it will be submitted for publication elsewhere.

## The two-dimensional case.

Example. Let $B_0$ denote $R^2$, equipped with the norm

$$\|(x,y)\| = \sqrt{x^2 + |xy| + y^2}$$

$(x,y) \in R^2$ . Obviously $B_0$ is not an $l^p$-space. Note that $B_0$ is an Orlicz space with the Minkowski norm

$$\|(x,y)\|_\phi = \inf\left\{ \alpha : \phi(|x/\alpha|) + \phi(|y/\alpha|) \leqslant 1 \right\}$$

where

$$\phi(t) = \begin{cases} \frac{3+\sqrt{3}}{8}\left[2+t - \sqrt{4-3t^2}\right] & \text{for } 0 \leqslant t \leqslant \frac{\sqrt{3}}{3} \\ \frac{3+\sqrt{3}}{4} t + \frac{1- 3}{4} & \text{for } t \geqslant \frac{\sqrt{3}}{3} \end{cases}$$

It should be pointed out that each two-dimensional Banach lattice with the norm satisfying $\|(x,y)\| = \|(y,x)\|$ is an Orlicz space, with the Minkowski norm. This description does not extend to 3-dimensional spaces (see [3]).

Let $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{L}_+(B_0)$ , that is $a,b,c,d \geqslant 0$ . We claim that $M(T)$ is a linear subspace of $B_0$ . We may and do assume that $\|T\|=1$. If $M(T)$ has exactly one linearly independent vector, then $M(T)$ is obviously a linear subspace. Thus we need to show that if there are two linearly independent vectors in $M(T)$, say $(x_1,y_1)$ , $(x_2,y_2)$ , then $T$ is an isometry. We have $\|T((x,y))\|^2 \leqslant \|(x,y)\|^2$ . Thus

$$A x^2 + B|xy| + C y^2 \leqslant x^2 + |xy| + y^2$$

where $A=a^2+ac+c^2$ , $B=2ab+ad+bc+2cd$ , $C=b^2+bd+d^2$ , and the equality holds for $(x_1,y_1)$ , $(x_2,y_2)$ . It is not hard to see that this implies $A=B=C=1$ . Therefore $a^2b^2 + c^2d^2 + (a^2+c^2)bd + (b^2+d^2)ac + 3abcd = (B^2 - AC)/3 = 0$ . Since $a,b,c,d \geqslant 0$ and $A=C=1$ we obtain $a=d=1$ , $b=c=0$ or $a=d=0$ , $b=c=1$ , i.e. $T$ is an isometry . Therefore $B_0$ has the property W .

Remark. Let B have the property W and dim B=2. Let $T \in \mathcal{L}_+(B)$ be such that $T^{-1} \in \mathcal{L}_+(B)$ . Then either $T/\|T\|$ is an isometry or else there exists exactly one $x_0$ such that $x_0 \geqslant 0$ , $\|x_0\|=1$ and

$$\|Tx_0\| = \inf\left\{\|Tx\| : x \in B , \|x\|=1\right\}$$

Indeed, suppose that $T$ is not an isometry . Then $T^{-1}$ is not an isometry and dim $M(T^{-1})=1$. Let $0 \neq y_0 \in M(T^{-1})$ . The vector

$x_0 = T^{-1}(y_0) / \| T^{-1}(y_0)\|$ satisfies the above equality .

<u>Theorem 2</u>. Let $(R^2, \|\cdot\|)$ have the property W and let $\|(1,0)\| = \|(0,1)\|$. Then $\|(x,y)\| = \|(y,x)\|$ for all $x,y \in R$ .

Proof. Consider the operator $T_a = \begin{bmatrix} 0 & 2-a \\ a & 0 \end{bmatrix}$ . We claim that $T_a$ is an isometry for some $a \in [0,2]$ . To get a contradiction suppose that $\dim M(T_a) = 1$ for all $a \in [0,2]$. Put

$$e_\alpha = (\cos\alpha, \sin\alpha) / \|(\cos\alpha, \sin\alpha)\|$$

$\alpha \in [0, \pi/2]$ . We can define a function $f: [0,2] \to [0, \pi/2]$ such that $e_{f(a)} \in M(T_a)$ . By the Remark for each $a \in (0,2)$ we can find a unique $g(a) \in [0, \pi/2]$ such that $\|T_a e_{g(a)}\| = \inf\{\|T_a x\| : \|x\| = 1\}$ , and we put $g(0) = 0$ , $g(2) = \pi/2$.
It is not hard to see that the functions f and g are continuous. Moreover $f(0) = \pi/2$ and $f(2) = 0$ . By the Darboux property of the continuous function $f-g$ on $[0,2]$ there exists $a_0$ such that $f(a_0) = g(a_0)$ . We have

$$\| T e_{g(a_0)}\| = \inf\{\|T_{a_0}x\| : \|x\| = 1\} \leqslant \sup\{\|T_{a_0}x\| : \|x\| = 1\} = \|Te_{f(a_0)}\|$$

Thus $T_{a_0} / \|T_{a_0}\|$ is an isometry. Hence $\|T_{a_0}((1,0))\| = \|T_{a_0}((0,1))\|$

and $a_0 / \|T_{a_0}\| = (2-a_0) / \|T_{a_0}\| = 1$ , so $\|T_{a_0}\| = a_0 = 1$ .

Therefore $\|(x,y)\| = \|T_{a_0}((x,y))\| = \|(y,x)\|$ .

<u>Proposition</u>. Suppose $(R^2, \|\cdot\|)$ has the property W. Then positive isometries are exactly the operators of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Proof. In view of Theorem 2 the operators having the above form are isometries.

Now assume that $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , where $a,b,c,d \geqslant 0$ . is an isometry. Then $\|T((1,-1))\| = \|(|a-b|, |c-d|)\| \leqslant \|(a+b, c+d)\| = \|T((1,1))\| = \|T((1,-1))\|$ . Thus $|a-b| = a+b$ and $|c-d| = c+d$ , so $ab = cd = 0$ , which completes the proof.

<u>Theorem 3</u>. Let B be a two-dimensional space with the property W and suppose $B^*$ is strictly convex. Then $B^*$ has the property W.

Proof. Let $T \in \mathcal{L}(B^*)$ and $\|T\| = 1$ . We need to show that if there exist two linearly independent vectors, say $v_1, v_2$, in $M(T)$ then $T$ is an isometry. Since $B$ and $B^*$ are strictly convex, there exists a one-to-one correspondence $B^* \ni u^* \longrightarrow u \in B$ such that $\langle u,u^* \rangle = \|u\| \|u^*\|$ and $\|u\| = \|u^*\|$. Thus we have $\|v_i^*\|^2 = \|Tv_i^*\|^2 = \langle Tv_i^*, (Tv_i^*)^* \rangle = \langle v_i^*, T^*(Tv_i^*)^* \rangle$ and $(Tv_i^*)^* \in M(T^*)$, $i=1,2$ ; also $(Tv_1^*)^* \neq (Tv_2^*)^*$. Since $B$ has the property W and $(Tv_1^*)^*$, $(Tv_2^*)^*$ are linearly independent, the operator $T^* \in L(B^*)$ is an isometry. Therefore, by Proposition, $T$ is also an isometry, which completes the proof.

Problems. Characterize the Banach lattices with the property W. In particular describe the norms $\|\cdot\|$ on $R^2$ such that $(R^2, \|\cdot\|)$ has the property W.

Can the strict convexity of $B^*$ be omitted in the assumption of Theorem 3 ?

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