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# Almost regular operators and integral operators on rearrangement invariant p-spaces of functions, $0 < p <= 1$

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# ALMOST REGULAR OPERATORS AND INTEGRAL OPERATORS ON REARRANGEMENT INVARIANT p-SPACES OF FUNCTIONS, $0<p\leqslant 1$

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This paper is divided in two sections. In the first one we introduce the almost regular operators and some related classes of operators on a function p-space, $0<p\leqslant 1$. A characterization theorem for the almost regular operators on a rearrangement invariant p-space X of functions, such that $1<p_X\leqslant q_X<\infty$, is given. This theorem extends Theorem 2.1 [11] stated and proved only for $X=L_q$, $1<q<\infty$. The interest of almost regular operators is mainly due to the fact that both classes of regular and of integral operators are included in the former one. We extend in the second part of the paper Corollary 4.2 -[11] (see also [6]) in the setting of p-Banach function spaces. Finally we get another (simpler) proof of Theorem 2.4-[2] in the particular case of a rearrangement invariant Banach space X of functions.

## 1. Almost regular operators on rearrangement invariant p-spaces of functions, $0<p\leqslant 1$.

In this paper we deal with operators defined on a rearrangement invariant p-space X of functions, $0<p\leqslant 1$.

First we recall the necessary definitions. (See [10]).

In what follows without contrary mention all the spaces will be real spaces. Let p a real number such that $0<p\leqslant 1$.

We consider a p-Banach lattice X of functions on $I=[0,1]$ which fulfills the following conditions.

1) The functions of X are p-integrable (with respect to Lebesgue measure $\mu$).

2) If $f\in X$ and $g\in L_0(I)$ (the space of all Lebesgue measurable functions on I) such that $|g|\leqslant|f|$ $\mu$-a.e., then it follows that $g\in X$ and $\|g\|_X\leqslant\|f\|_X$.

3) The characteristic functions $\chi_A$ belong to X for all $A\subseteq I$ such that $\mu(A)<\infty$.

4) The p-norm $\|f\|_X$ of X is p-convex, i.e. the $\mu$-measurable

This paper is in final form and no version of it will be submitted for publication elsewhere.

function $(\sum_{i=1}^{n} |f_i|^p)^{1/p}$ belongs to X for $f_1,\dots,f_n \in X$ and moreover

$$(*) \qquad \| (\sum_{i=1}^{n} |f_i|^p)^{1/p} \|_X \leq (\sum_{i=1}^{n} \|f_i\|_X^p)^{1/p}.$$

5) (The Riesz-Fischer condition). If $f_1,\dots,f_n,\dots$ are elements of X and if $\sum_{i=1}^{\infty} \|f_i\|_X^p < \infty$, then the $\mu$-measurable function $(\sum_{i=1}^{\infty} |f_i(t)|^p)^{1/p}$ belongs to X.

Such a space X is called a p-<u>Köthe space of functions</u>, $0 < p \leq 1$.

The condition 4) is the most important one. It is automatically fulfilled for p = 1. Also the Riesz-Fischer condition is a consequence of the inequality (*) for p = 1. This possible implication for $0 < p \leq 1$ is unknown to the another.

Let X be a p-Köthe space of functions on I. We denote by $X_{(p)}$ the set $\{x : I \longrightarrow \mathbb{R}$ ; such that the function $t \longrightarrow x(t)^{1/p} := |x(t)|^{1/p} \operatorname{sgn} x(t)$ belongs to $X\}$. Endowed with the pointwise order and the norm $\|x\|_{(p)} := \| |x|^{1/p} \|_X^p$, $X_{(p)}$ becomes a Köthe space of functions on I, i.e. a 1-Köthe space of functions on I .

Now let X be a Köthe space of functions on I. We denote by $X^{(p)}$ the set $\{x : I \longrightarrow \mathbb{R}$; such that the function $x^p$ belongs to $X\}$.

We consider for $x \in X^{(p)}$ the p-norm

$$\|x\|^{(p)} := \| |x|^p \|_X^{1/p} .$$

Then $X^{(p)}$ becomes a p-Köthe space of functions on I.

Now we can consider the Köthe dual of $X_{(p)}$, $[X_{(p)}]' := \{g : I \longrightarrow \mathbb{R}$; such that $\int_0^1 |f(t)g(t)|\,dt < \infty$ for all $f \in X_{(p)}\}$.

We introduce on $[X_{(p)}]'$ the norm

$$\|g\| := \sup_{\|f\|_p \leq 1} \int_0^1 |f(t)g(t)|\,dt.$$

Hence $[X_{(p)}]'$ becomes a Köthe space of functions on I.

Then X is a vector sublattice of $X'' := \{[X_{(p)}]''\}^{(p)}$ but generally it is not a p-Banach subspace of it.

A p-Köthe space X of functions on I is called a <u>rearrangement invariant</u> p-<u>space of functions</u> (briefly r.i.p-space) if the following conditions hold.

1) For every $f \in X$ and every measure preserving automorphism $\tau : I \longrightarrow I$ the function $f \circ \tau$ belongs to X and moreover $\|f \circ \tau\|_X = \|f\|_X$.

2) X is a p-Banach subspace of X" and X is either maximal i.e. $X = X''$; or minimal i.e. the subspace of all simple p-integrable functions is dense in X.

3) We have the canonical inclusions

$$L_\infty(0,1) \subset X \subset L_p\ (0,1)$$

such that the norms of these maps are less than 1. (We denote by $\|T\|$ the expression $\sup\{\|T x\|;\ \|x\|_X \leq 1\}$, where $T : X \longrightarrow Y$ is a linear and bounded operator acting between the p-Banach spaces X and Y).

More details about r.i. p-spaces may be found in [10].

We recall now the definition of Boyd indices of a r.i.p-space X.

For $0 < s < \infty$ we define the operator $D_s$ as follows. For every measurable function f on $[0,1]$, put

$$(D_s f)(t) = \begin{cases} f(t/s) & t \leq \min\ (1,s) \\ 0 & s < t \leq 1. \end{cases}$$

Now we can define the so-called Boyd indices $p_X$, $q_X$

$$p_X = \lim_{s\to\infty} \frac{\log s}{\log\|D_s\|_X} = \sup_{s>1} \frac{\log s}{\log\|D_s\|_X}$$

$$q_X = \lim_{s\to 0^+} \frac{\log s}{\log\|D_s\|_X} = \sup_{0<s<1} \frac{\log s}{\log\|D_s\|_X}$$

It is known [10] that $p \leq p_X \leq q_X \leq \infty$.

For two topological vector spaces X and Y we denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from X into Y and by $\mathcal{L}(X)$ the space $\mathcal{L}(X,X)$.

In what follows X will be a r.i. p-space, $0 < p \leq 1$, such that $1 < p_X \leq q_X < \infty$.

Let $E \subset I$ be a measurable subset. Then by $\chi_E T$ we denote the operator $f \longrightarrow \chi_E T(f)$ defined for $f \in X$.

For $T \in \mathcal{L}(X)$ we put (if it exists) $|T|f = \sup_{|g| \leq f} |Tg|$, where $0 \leq f \in X$ the supremum being calculated in $L_0 := L_0(I)$.

If $(E_n)_{n=1}^\infty$ is an increasing sequence of measurable subsets of I such that $\bigcup_n E_n = I$, then we say that $(E_n)_{n=1}^\infty$ is an exhaustive sequence.

Now let $T \in \mathcal{L}(X)$. T is called an almost regular operator if there exists an exhaustive sequence $(E_n)_{n=1}^\infty$ such that $|\chi_{E_n} T| \in \mathcal{L}(X)$.

In this definition we assume implicitely that the pointwise supremum $\sup\limits_{|g|\leq f} |Tg|$ exists as a measurable function on I for $\leq f \in X$.

Consequently, if $T \in \mathcal{L}(X)$ is an almost regular operator then $j \circ T \in \mathcal{L}(X, L_0)$ is a regular operator (that is $|j \circ T| \in \mathcal{L}(X, L_0)$), where $j : X \longrightarrow L_0$ is the canonical injection.

Moreover $|j \circ T|$ is an order continuous operator, that is if $f_n \downarrow 0$ in X then $|j \circ T| f_n \downarrow 0$ in $L_0$. Since X is an order continuous p-Banach lattice (that is if $f_n \downarrow 0$ then $\|f_n\|_X \xrightarrow[n]{} 0$) having nontrivial Boyd indices, then for a sequence $f_n \downarrow 0$ in X it follows that $\|f_n\|_X \xrightarrow[n]{} 0$, which in turn implies that $|j \circ T| f_n \xrightarrow[n]{} 0$ in $L_0$. But $|j \circ T| f_n \downarrow g \in L_0$, hence $g = 0$.

Then it follows that $j \circ T$ is a difference of two positive and order continuous operators from X into $L_0$.

So we have proved the following result.

Lemma 1. *Let X be a r.i. p-space of functions on I such that $1 < p_X \leq q_X < \infty$ and let $T \in \mathcal{L}(X)$ be an almost regular operator. Then $j \circ T = U_1 - U_2$, where $U_1, U_2$ are positive and order continuous operators from X into $L_0$.*

The converse of Lemma 1 is also true for $X = L_q$, $1 < q < \infty$ and this is nothing else than Nikishin's Theorem. (See Thm.4 [8]).

For a general r.i. p-space X a weaker result holds.

Proposition 2. *Let X as in Lemma 1 and let $T \in \mathcal{L}(X)$ such that $j \circ T = U_1 - U_2$, where $U_1, U_2$ are positive operators in $\mathcal{L}(X, L_0)$. Then for every $\varepsilon > 0$ there exists a measurable subset $E \subset I$ such that $\mu(E) \geq 1 - \varepsilon$ and $|\chi_E T \chi_E| \in \mathcal{L}(X)$.*

(We recall that $U\chi_E$ means the operator defined by $f \longrightarrow U(\chi_E f)$ for $f \in X$. Moreover by Thms. 1.8 and 1.7 - [4] every positive operator on X is continuous).

Proof. Obviously it is sufficient to prove the result for positive operators only.

Let r,q be positive numbers such that $1 < q < p_X \leq q_X < r < \infty$.

Then it is known that $L_r$ is canonically embedded into X and X is canonically embedded into $L_q$. (See [10]). Consequently $j \circ T$ acts as a positive continuous operator from $L_r$ into $L_0$. (Of corse here is an often used abuse of notation). By Nikishin's Theorem (Thm.4 - [8]) we get a measurable subset $F \subset I$ such that $\mu(F) \geq 1 - \varepsilon/2$ and

$$\chi_F T \in \mathcal{L}(L_r). \tag{1}$$

Let's remark that by above inclusions the Köthe dual X' is nontrivial (i.e. $X' := \{g : I \longrightarrow \mathbb{R}\ ;\ \int_0^1 |g(t)f(t)|\,dt < \infty$ for every

$f\in X\} \neq \{0\})$ and let's denote by T' the formal Köthe adjoint of T.

Now let $G\subset I$ be a measurable subset such that $\mu(G)\geqslant 1-\varepsilon/2$ and $(\chi_G T)(1)\in L_\infty$. Then $\chi_G T\in\mathcal{L}(L_\infty)$ (again we deal with an abuse of notation) and by the proof of Lemma 1 it follows that $j\circ T$ is an order-continuous operator from X into $L_0$, which in turn implies that $\chi_G T$ is an order-continuous operator on $L_\infty$ too.

Then $T'\chi_G = (\chi_G T)'\in\mathcal{L}(L_1)$ and $j'\circ(T'\chi_G)\in\mathcal{L}(L_{q'}, L_0)$, where $j' : X' \longrightarrow L_0$ is the canonical inclusion and $1/q' + 1/q = 1$.

Again by Nikishin's Theorem we get a measurable subset $H\subseteq I$ such that $\mu(H)\geqslant 1-\varepsilon/2$ and such that

(2) $$\chi_H T'\chi_H\in\mathcal{L}(L_{q'}).$$

Since $L_{q'}$ is a reflexive space then

(3) $$\chi_H T\chi_H\in\mathcal{L}(L_q).$$

By (1) and (3) and applying Theorem 7 - [10] (which extends Boyd's Interpolation Theorem at the p-Banach function spaces setting) we get a measurable subset $E\subset I$ such that $\mu(E)\geqslant 1-\varepsilon$ and such that $\chi_E T\chi_E\in\mathcal{L}(X)$. ∎

I don't know if Nikishin's Theorem can be extended for a general X.

So in the case $X = L_q$, $1<q<\infty$, $T\in\mathcal{L}(X)$ is an almost regular operator if and only if $|j\circ T|\in\mathcal{L}(X, L_0)$.

In view of Lemma 1 it is natural to introduce a now notion.

Let $T\in\mathcal{L}(X)$ such that $j\circ T\in\mathcal{L}(X,L_0)$ be a regular operator. Then we call T a $L_0$-<u>regular operator on</u> X.

Hence the almost regular and $L_0$-regular operators on $L_q$, $1<q<\infty$, coincide. The coincidence for a general X is still open.

It is natural to ask about the relations between various classes of operators on X.

For instance it is obvious that the regular operators are allways almost regular operators, consequently they are $L_0$- regular operators.

But the converse is not true. We recall that an <u>integral operator</u> $T\in\mathcal{L}(X)$ is an operator given by the formula

$$T\,x(s) = \int_0^1 K(s,t)x(t)dt$$

for every $x\in X$, where K(s,t) is a real valued measurable function defined on $I\times I$.

Then there exists an integral and non-regular operator T on $L_2$ (see [4]). Namely T is given by the kernel

$$K(s,t) = \sum_{n=1}^{\infty} \frac{\chi_{E_n}(s)}{\sqrt{\mu(E_n)}} \varphi_n(t)$$

where $(E_n)_{n=1}^{\infty}$ is a sequence of pairwise disjoint subsets of $I$ such that $\sum_{n=1}^{\infty} \sqrt{\mu(E_n)} < \infty$, and $(\varphi_n)_{n=1}^{\infty}$ is a complete orthonormal system in $L_2$.

Another example of such integral operator is to be found in [5].

On the other hand every integral operator T on $L_2$ is $L_0$-regular (see for instance Lemma 1.6 - [4]). By the proceding remarks T is also an almost regular operator on $L_2$.

Hence the class of regular operators is strictly included in the class of almost regular or in the class of $L_0$-regular operators.

Moreover the class of integral operators is included in the class of $L_0$-regular operators. This inclusion is also strict since the identity operator $I \in \mathcal{L}(L_2)$ is obviously a $L_0$-regular operator but not an integral operator. (See Theorem 8.5 - [3]).

The class of $L_0$-regular (almost regular) is not too large as the following example shows.

<u>Example 3. There exists an operator $T \in \mathcal{L}(L_2)$ which is not a $L_0$-regular (hence is not an almost regular) operator.</u>

<u>Proof</u>. Let first $(c_n)_{n\in \mathbb{Z}} \in \ell_2(\mathbb{Z})$ a sequence such that $\sum_{n\in\mathbb{Z}} |c_n||\cos 2\pi nt|$ is divergent for almost every $t \in [0,1]$.

Such a sequence exists by a theorem of A.N.Kolmogorov (see [9], p.64). Let now $f \in L_2(\mathbb{R})$ defined by

$$f(t) = |c_n| \text{ for } t \in (n,n+1], \quad n \in \mathbb{Z}.$$

Then we consider $U : L_2(\mathbb{R}) \longrightarrow L_2(I)$ given by the formula

$$[U(g)](t) = \sum_{n\in\mathbb{Z}} \left(\int_n^{n+1} g(x)dx\right)\cos 2\pi tn$$

where $g \in L_2(\mathbb{R})$ and $t \in I$. Then

$$(*) \qquad |J \circ U|(f) = \sup_{|g|\leq f} \left|\sum_{n\in\mathbb{Z}} \left(\int_n^{n+1} g(x)dx\right)\cos 2\pi nt\right| \geq$$

$$\geq \sup_{|g|\leq f} \sum_{n\in\mathbb{Z}} \left|\int_n^{n+1} g(x)dx\right||\cos 2\pi nt| = \sum_{n\in\mathbb{Z}} \left(\int_n^{n+1} f(x)dx\right)|\cos 2\pi nt| =$$

$$= \sum_{n\in\mathbb{Z}} |c_n| \cdot |\cos 2\pi nt| = \infty$$

for a.e. $t \in I$.

Let now $\varphi: \mathbb{R} \longrightarrow I$ a measure-theoretic isomorphism, for instance $\varphi(x) = e^{-e^x}$ for $x \in \mathbb{R}$, (see [3]- § 6 for the definition), and let $V_\varphi : L_2(I) \longrightarrow L_2(\mathbb{R})$ be the order isometry given by Theorem 6.1 - [3].

By the relation (*) the operator $T \in \mathcal{L}(L_2)$ given by $T = UV_\varphi$ is not an $L_0$-regular operator. ∎

Another example of this kind is given by Korotkov in [4]. Korotkov uses the spectral theory to prove the assertion of Example 3.

Now we shall give a characterization of almost regular operators on r.i. p-spaces of functions X. This result extends Theorem 2.1 - [11]. First we denote by $\rho_r^n \in \mathcal{L}(L_0)$ the operator defined by

$$\rho_r^n(f) = f \cdot g_n^{-1/r}$$

for $f \in L_0$, $n \in \mathbb{N}$ and $1 < r < \infty$, where $g_n$ is a density (i.e. $g_n(t) > 0$ everywhere on $[0,1]$ and moreover $\|g_n\|_1 = 1$).

Theorem 4. Let X be a r.i. p-space such that $1 < p_X \leq q_X < \infty$ for $0 < p \leq 1$. The following assertions are equivalent.

a) T is an almost regular operator on X.

b) For all r,q such that $1 < q < p_X \leq q_X < r < \infty$, there exists a sequence of densities $(g_n)_{n=1}^{\infty}$ and an exhaustive sequence $(E_n)_{n=1}^{\infty}$ of subsets of I such that

$$\rho_q^n(\chi_{E_n} T)(\rho_r^n)^{-1} \in \mathcal{L}(L_s(g_n d\mu)) \cap \mathcal{L}(L_s(g_n^{q/r}\, d\mu))$$

for all $1 \leq s \leq \infty$ and all $n \in \mathbb{N}$.

Proof. a) b) The proof is very likely with the proof of Theorem 2.1 - [11].

Let $(E_n)_{n=1}^{\infty}$ be an exhaustive sequence of subsets of I such that $|\chi_{E_n} T| = \chi_{E_n} |T| \in \mathcal{L}(X)$ for all $n \in \mathbb{N}$.

We denote by $T_n := \| |\chi_{E_n} T| \|^{-1} \cdot |\chi_{E_n} T|$. Then it is known ( see [10] - Proposition 5) that we have continuous inclusions

$$L_r(\mu) \subset X \subset L_q(\mu)$$

such that their norms should to be less than a constant $C(r,q)$ depending only on r and q.

Let q' and r' the conjugate numbers of q and r and let's define

$$S_n : (L_1)_+ \longrightarrow (L_1)_+ \qquad \text{by}$$

$$S_n(h) = \frac{1}{2}\, [T_n(h^{1/r})]^q + \frac{1}{2}\, [T_n'(h^{1/q'})]^{r'} \quad \text{for } h \in (L_1)_+$$

where $T_n' : X' \longrightarrow X'$ is the adjoint of $T_n$, which maps the Köthe dual $X' = X^*$ into itself.

Then

$$\|S_n h\|_1 = \int_0^1 S_n h \, d\mu = \frac{1}{2} \cdot \int_0^1 \left[T_n(h^{1/r})\right]^q d\mu + \frac{1}{2}\int_0^1 \left[T_n'(h^{1/q'})\right]^{r'} d\mu \leq$$

$$\leq \frac{1}{2} C(q,r)(\|T_n(h^{1/r})\|_X^q + \|T_n'(h^{1/q'})\|_{X'}^{r'}) \leq$$

$$\leq \frac{1}{2} C(q,r)\left[\| h^{1/r} \|_X^q + \|h^{1/q'}\|_{X'}^{r'}\right].$$

Moreover if $0 \leq h \leq g$ then $0 \leq S_n(h) \leq S_n(g)$.

Let now $0 < f_0 \in L_1(\mu)$ such that $\|f_0\|_1 = 1$ and let $a > 1$ such that $(2a)^{1/q} \leq 2$ and $(2a)^{1/r'} \leq 2$.

Denoting by $f_{k+1} = f_0 + \frac{1}{a} S_n(f_k)$ for $k \in \mathbb{N}$ we get that

$$f_0 \leq f_1 \leq \dots \text{ and } \|f_k\|_1 \leq \|f_0\|_1 + \frac{1}{2a} C(r,q)\left[ \|f_{k-1}^{1/r}\|_X^q + \right.$$

$$\left. + \|f_{k-1}^{1/q'}\|_{X'}^{r'}\right] \leq \|f_0\|_1 + \frac{1}{2a} C_1(r,q)\left[\|f_{k-1}\|_1^{q/r} + \|f_{k-1}\|_1^{r'/q'}\right] \leq$$

$$\leq 1 + \frac{1}{2a} C_1(r,q)\left[ \|f_k\|_1^{q/r} + \|f_k\|_1^{r'/q'}\right] \quad \text{for all } k \in \mathbb{N}.$$

If $\sup_k \|f_k\|_1 = +\infty$, then dividing the last inequality by $\|f_k\|_1$, we get

$$1 \leq \frac{1}{\|f_k\|_1} + \frac{1}{2a} C_1(r,q)\left[\frac{1}{\|f_k\|_1^{1-q/r}} + \frac{1}{\|f_k\|_1^{1-r'/q'}}\right].$$

Passing to a subsequence $(f_{k_\ell})_\ell$ such that $\|f_{k_\ell}\|_1 \xrightarrow[\ell]{} \infty$, we obtain $1 \leq a$ and this is a contradiction. Thus $\sup_k \|f_k\|_1 = M(r,q) < \infty$.

By Fatou's Lemma it follows that $f_k$ is norm-convergent to $f^n := \sup_k f_k \in L_1(\mu)$. Consequently

$$f^n = f_0 + \frac{1}{2} S_n(f^n)$$

and

$$S_n(f^n) \leq a f^n.$$

Put

$$g_n := \frac{f^n}{\|f^n\|_1}$$

Then,

$$T_n(g_n^{1/r}) \leq (2S_n(g_n))^{1/q} \leq 2g_n^{1/2}, \quad T_n'(g_n^{1/q'}) \leq (2S_n(g_n))^{1/r'} \leq 2g_n^{1/r'}.$$

If $|f| \leq 1$ then

$$|\rho_q^n(\chi_{E_n}T)(\rho_r^n)^{-1}(f)| \leq \rho_q^n|\chi_{E_n}T|(\rho_r^n)^{-1}(1) = \||\chi_{E_n}T|\| g_n^{-1/q} T_n(g_n^{1/r}) \leq$$

$$\leq 2\||\chi_{E_n} T|\| \cdot 1$$

and

$$|\rho^n_{r'}(\chi_{E_n}T)'(\rho^n_{q'})^{-1}(f)| \leq \rho^n_{r'}|T'_n|\cdot(\rho^n_{q'})^{-1}(1) =$$

$$= \| |\chi_{E_n}T| \| g_n^{-1/r'}\cdot T'_n\,(g_n^{1/q'}) \leq 2\| |\chi_{E_n}T| \|\cdot 1.$$

Consequently $\rho^n_q(\chi_{E_n}T)(\rho^n_r)^{-1}$ and $\rho^n_{r'}(\chi_{E_n}T)'(\rho^n_{q'})^{-1}$ map $L_\infty(g_n d\mu)$ into $L_\infty(g_n d\mu)$ and $L_\infty(g_n^{q/r}\,d\mu)$ into itself. (Obviously $g_n^{q/r} \in L_1(\mu)$ and thus $g_n^{q/r}\|g_n^{q/r}\|^{-1}$ is a density for $n \in \mathbb{N}$).

Since $[(\rho^n_q)(\chi_{E_n}T)(\rho^n_r)^{-1}]' = \rho^n_{r'}(\chi_{E_n}T)'(\rho^n_{q'})^{-1}$ then $\rho^n_q(\chi_{E_n}T)(\rho^n_r)^{-1} \in \mathcal{L}(L_s(g_n d\mu)) \cap \mathcal{L}(L_s(g_n^{q/r}d\mu))$ where s is equal either with 1 or with $+\infty$.

By Riesz-Thorin interpolation theorem it follows that $\rho^n_q(\chi_{E_n}T)(\rho^n_r)^{-1} \in \mathcal{L}(L_s(g_n d\mu) \cap \mathcal{L}(L_s(g_n^{q/r}d\mu))$ for all $s$, $1 \leq s \leq \infty$.

b) $\Rightarrow$ a) Since $|\rho^n_q(\chi_{E_n}T)(\rho^n_r)^{-1}| \in \mathcal{L}(L_s(g_n d\mu)) \cap \mathcal{L}(L_s(g_n^{q/r}d\mu))$ for $s=1$ or $s = +\infty$ ($\mathcal{L}(L_1(d\nu))$ and $\mathcal{L}(L_\infty(d\nu))$ are Banach lattices) then by Riesz-Thorin interpolation theorem it follows that

$$|\rho^n_q(\chi_{E_n}T)(\rho^n_r)^{-1}| \in \mathcal{L}(L_s(g_n d\mu)) \cap \mathcal{L}(L_s(g_n^{q/r}d\mu))$$

for all $1 \leq s \leq \infty$. Particularly

$$|\rho^n_q(\chi_{E_n}T)(\rho^n_r)^{-1}| \in \mathcal{L}(L_r(g_n d\mu)) \tag{1}$$

Then

$$(\rho^n_r)^{-1}|\rho^n_q(\chi_{E_n}T)(\rho^n_r)^{-1}|\rho^n_r \in \mathcal{L}(L_r(\mu)). \tag{2}$$

Indeed, for $0 \leq f \in L_r(\mu)$ we have $\int f^r \cdot g_n^{-1} g_n\, d\mu = \int f^r\, d\mu < \infty$ thus $f g_n^{-1/r} \in L_r(g_n d\mu)$. It follows that

$$(\rho^n_r)^{-1}|\rho^n_q(\chi_{E_n}T)(\rho^n_r)^{-1}|\rho^n_r(f) = (\rho^n_r)^{-1}\cdot \sup_{|h|\leq f} |\rho^n_q\, T(h)|\chi_{E_n} =$$

$$= g_n^{1/r-1/q}\cdot \sup_{|h|\leq f} |Th|\chi_{E_n}.$$

By (1) we have $\chi_{E_n}\cdot g_n^{-1/q}\cdot \sup_{|h|\leq f} |Th| \in L_r(g_n\, d\mu)$, that is

$$(\rho^n_r)^{-1}|\rho^n_q(\chi_{E_n}T)(\rho^n_r)^{-1}|\rho^n_r(f) = g_n^{1/r-1/q}\cdot \sup_{|h|\leq f} |Th|\chi_{E_n} \in L_r(\mu)$$

for all $0 \leq f \in L_r(\mu)$.

On the other hand we have

$$|\rho^n_q(\chi_{E_n}T)(\rho^n_r)^{-1}| \in \mathcal{L}(L_q(g_n^{q/r}\, d\mu)). \tag{3}$$

Now we shall show that

$$(\rho_r^n)^{-1}|\rho_q^n(\chi_{E_n}T)(\rho_r^n)^{-1}|\rho_r^n \in \mathcal{L}(L_q(\mu)). \qquad (4)$$

It is clear that $fg_n^{-1} \in L_q(g_n^{q/r}\, d\mu)$ for $0 \leq f \in L_q(\mu)$, thus $|\rho_q^n(\chi_{E_n}T)(\rho_r^n)^{-1}| \cdot (fg_n^{-1/r}) \in L_q(g_n^{q/r}\, d\mu)$. Then it follows that $g_n^{-1/q} \cdot \sup_{|h|\leq f} |Th|\chi_{E_n} \in L_q(g_n^{q/r}\, d\mu)$, that is $g_n^{1/r-1/q}\chi_{E_n} \cdot \sup_{|h|\leq f} |Th| \in L_q(\mu)$.

Using Theorem 7 - [10], by (2) and (4), it follows that

$$(\rho_r^n)^{-1}|\rho_q^n(\chi_{E_n}T)(\rho_r^n)^{-1}|\rho_r^n \in \mathcal{L}(X).$$

Hence $g_n^{1/r-1/q}\chi_{E_n}|T|f \in X$ for any $0 \leq f \in X$. Denote by $E_{n,m}$ the set $\{t \in E_n \ ;\ g_n(t) \leq m\}$. Then $\bigcup_{m=1}^{\infty} E_{n,m} = E_n$ and

$$m^{1/r-1/q}\chi_{E_{n,m}}|T|f \leq g_n^{1/r-1/q}\chi_{E_{n,m}}|T|(f) \in X$$

for all $0 \leq f \in X$.

Thus $\chi_{E_{n,m}}|T|(f) \in X$ for all $n,m \in \mathbb{N}$ and for all $0 \leq f \in X$.

If $F_k = \bigcup_{n+m\leq k+1} E_{n,m}$ for $k \in \mathbb{N}$ then $(F_k)_{k=1}^{\infty}$ is an exhaustive sequence of subsets of I and

$$\chi_{F_k}|T|(f) \leq \sum_{n+m\leq k+1} \chi_{E_{n,m}}|T|(f) \in X$$

for all $0 \leq f \in X$ and all $k \in \mathbb{N}$.

It follows that $|\chi_{F_k} T| = \chi_{F_k}|T| \in \mathcal{L}(X)$ for all $k \in \mathbb{N}$, that is it follows that T is an almost regular operator on X. ■

We can strengthen the implication a)$\Longrightarrow$b) for the regular operators as follows.

<u>Corollary 5</u>. <u>Let</u> X <u>be a r.i.</u> p-<u>space of functions on</u> $I = [0,1]$ <u>such that</u> $1 < p_X \leq q_X < \infty$.

<u>If</u> T <u>is a regular operator on</u> X <u>then we have the following assertion: for every</u> r,q <u>such that</u> $1 < q < p_X \leq q_X < r < \infty$ <u>there exists a density</u> g <u>such that</u>

$$\rho_q T(\rho_r)^{-1} \in \mathcal{L}(L_s(g\,d\mu)) \cap \mathcal{L}(L_s(g^{q/r}\,d\mu))$$

<u>for all</u> s <u>such that</u> $1 \leq s \leq \infty$.

<u>Proof</u>. Simply replace $T_n$ by $T_o = \| |T| \|^{-1} \cdot |T|$ in the proof of a)$\Longrightarrow$b) of Theorem 4. Then we get a density g with required properties instead of a sequence $(g_n)_{n=1}^{\infty}$. ■

2. Integral operators on rearrangement invariant p-spaces of functions on $[0,1]$, for $0<p\leq 1$.

In this section we deal with integral operators on a r.i.p-space X, such that $1<p_X\leq q_X<\infty$.

It is to be mentioned that Korotkov has shown that integral operators on $L_2$ cannot constitute a left ideal. (See Example 4.12- [4]), but the question if they constitute a right ideal remains still open.

In what follows we are interested to give some compactness criteria for integral operators on rearrangement invariant p-spaces of functions on I, for $0<p\leq 1$, extending the previous known result in the Banach function spaces setting. (See [6]- p.156, where a similar compactness criterion for Banach function spaces is stated without proof).

First we give without proof an easy extension of Theorem 1.3. 10 - [5]. The sole difference in the proofs appears only in the use of Theorem 7 - [10] instead of Riesz-Thorin interpolation theorem.

Theorem 6. Let X be a r.i. p-space of functions on I and let r,q real numbers such that $1<q<p_X\leq q_X<r<\infty$. If $T\in\mathcal{L}(L_r)\cap\mathcal{L}(L_q)$ and if T is moreover a compact operator on $L_r$ then $T\in\mathcal{L}(X)$ and T is a compact operator on X.

The following theorem extends Theorem 4.1 - [11] in the p-Banach functions spaces setting. See also Theorem 3.2 - [4].

Theorem 7. Let X be a r.i. p-space of functions on I such that $1<p_X\leq q_X<\infty$ and let $T\in\mathcal{L}(X)$ an integral operator.

Then there exists an exhaustive sequence $(E_n)_{n=1}^{\infty}$ such that $\chi_{E_n} T\chi_{E_n}$ is a compact operator on X for every $n\in\mathbb{N}$.

Proof. Let's assume first that T is a regular operator on X. By Corollary 5 it follows that, for $1<q<p_X\leq q_X<r<\infty$, there exists a density g such that

$$\rho_q|T|(\rho_r)^{-1}=|\rho_q\ T(\rho_r)^{-1}|\in\mathcal{L}(L_s(g d\mu))\cap\mathcal{L}(L_s\ (g^{q/r}\ d\mu))\text{ for }1\leq s\leq\infty.$$

Since T is an integral operator then by Lemma 3.1 - [11] it follows that there exist the exhaustive sequence $(F_{n1})_{n=1}^{\infty}$, $(F_{n2})_{n=1}^{\infty}$ such that $|\rho_q T(\rho_r)^{-1}|\chi_{F_{n1}}\in\mathcal{L}(L_1(g d\mu))$, $|\rho_q T(\rho_r)^{-1}|\chi_{F_{n2}}\in\mathcal{L}(L_1\ (g^{q/r}\ d\mu))$ and moreover these operators are compact for all $n\in\mathbb{N}$.

Using Theorem 2.5.10 - [5] it follows easy that there exists an exhaustive sequence $(F_n)_{n=1}^{\infty}$ such that $|\rho_q\ T(\rho_r)^{-1}|\chi_{F_n}$ is a compact operator on the both spaces $L_1(g d\mu)$ and $L_1(g^{q/r}\ d\mu)$.

Since $|\rho_q\ T(\rho_r)^{-1}|\chi_{F_n}\in\mathcal{L}(L_s(g d\mu))\cap\mathcal{L}(L_s(g^{q/r}d\mu))$ for all $1\leq s\leq\infty$

and let $n\in\mathbb{N}$. Theorem 1.3.10 - [5] shows us that $|\rho_q T\chi_{F_n}(\rho_r)^{-1}|$ is a compact operator on the spaces $L_r(gd\mu)$ and $L_q(g^{q/r}\,d\mu)$ for all $n\in\mathbb{N}$. Further reasoning as in the proof of implication b)$\Rightarrow$a) of Theorem 4 it follows that $g^{1/r-1/q}\,T\chi_{F_n}$ is a compact operator on the spaces $L_r$ and $L_q$ for all $n\in\mathbb{N}$. Using again Theorem 2.5.10 - [5] we get an exhaustive sequence $(E_n)_{n=1}^{\infty}$ such that $\chi_{E_n}T\chi_{E_n}$ is a compact operator on $L_r$ and $L_q$ for all $n\in\mathbb{N}$.

By Theorem 6 we get now that $\chi_{E_n}T\chi_{E_n}$ is a compact operator on X for every $n\in\mathbb{N}$.

Let now $T\in\mathcal{L}(X)$ be an integral operator. Since by Theorem 3.1.- [11] T is a $L_0$-regular operator, then by Proposition 2 there exists an exhaustive sequence $(E_n)_{n=1}^{\infty}$ of subsets of I such that $|\chi_{E_n}T\chi_{E_n}|\in\mathcal{L}(X)$. By the first part of the proof there exists an exhaustive sequence $(F_n)_{n=1}^{\infty}$ such that $\chi_{F_n}T\chi_{F_n}$ are compact operators on both spaces $L_r$ and $L_q$ for all $n\in\mathbb{N}$. By Theorem 6 it follows that $\chi_{F_n}T\chi_{F_n}$ are compact operators on X for every $n\in\mathbb{N}$. ∎

We give now the compactness criterion extending Corollary 4.2. -[11]. See also [6] - p.156.

Theorem 8. Let X be a r.i. p-space on functions on I, such that $1<p_X\le q_X<\infty$ and let $T\in\mathcal{L}(X)$ be an integral operator.

Then T is compact if and only if $\lim_n\|\chi_{E_n}T\|_{\mathcal{L}(X)}=0$ for every decreasing sequence of subsets $(E_n)_{n=1}^{\infty}$ such that $\bigcap_{n=1}^{\infty}E_n=\emptyset$.

Proof. Let T be a compact operator on X. Since $1<p_X\le q_X<\infty$ the Haar system is an unconditional basis in X. (See Theorem 13 -[10]).

By Mazur-Phillips' Theorem, also true for p-Banach spaces ( see -[7]) it follows that $\lim_{\mu(E_n)\to 0}\|\chi_{E_n}T\|_{\mathcal{L}(X)}=0$ for every sequence $(E_n)_{n=1}^{\infty}$ of dyadic intervals. Hence $\lim_{\mu(E_n)\to 0}\|\chi_{E_n}T\|_{\mathcal{L}(X)}=0$ for every decreasing sequence $(E_n)_{n=1}^{\infty}$ such that $\bigcap_{n=1}^{\infty}E_n=\emptyset$.

If $\lim_{E_n\downarrow\emptyset}\|\chi_{E_n}T\|_{\mathcal{L}(X)}=0$ (here $E_n\downarrow\emptyset$ means that $E_1\supseteq E_2\supseteq\dots$ and $\bigcap_n^{\infty}E_n=\emptyset$), then by Theorem 7 there exists an exhaustive sequence $(F_n)_{n=1}^{\infty}$ such that $\chi_{F_n}T\chi_{F_n}$ is a compact operator for every $n\in\mathbb{N}$. Since $\|T-\chi_{F_n}T\chi_{F_n}\|^p\le\|T-\chi_{F_n}T\|^p+\|\chi_{F_n}T\chi_{F_n}-\chi_{F_n}T\|^p$ then

$$\lim_{n\to\infty} \| T - \chi_{F_n} T \chi_{F_n} \|^p \leq 2 \overline{\lim_{F_n \downarrow \emptyset}} \| \chi_{F_n} T \| = 0.$$

Hence T is a compact operator. ∎

As an application of Theorem 8 we give a simpler proof of Theorem 2.4 - [2], however in the particular case of a r.i. Banach space of functions X with $1 < p_X \leq q_X < \infty$.

We consider that our proof deserves the publication because the original proof depends on deep results of [1].

Theorem 9. Let X be a r.i. Banach space of functions on I such that $1 < p_X \leq q_X < \infty$.

(i) Let $T \in \mathcal{L}(X)$ be an integral operator. T is compact if and only if $\lim_n \| T'(f_n) \| = 0$ for each norm bounded and disjoint sequence $(f_n)_{n=1}^{\infty}$ of elements of X'.

(ii) Let $T \in \mathcal{L}(X)$ be such that T' is an integral operator. Then T is compact if and only if $\lim_n \| T(f_n) \| = 0$ for each norm-bounded and disjoint sequence of elements of X.

Proof. (i) If $(f_n)_n$ is as in the statement of Theorem 9, then, denoting $E_n = \operatorname{supp} f_n$, by the proof of Theorem 8 it follows that $\lim_n \| T'\chi_{E_n} \|_{\mathcal{L}(X')} = 0$. Hence $\lim_n \| T'(f_n) \|_{X'} = 0$.

We shall prove that $\sum_{n=1}^{\infty} T'\chi_{E_n}$ converges for every sequence of disjoint subsets $(E_n)_{n=1}^{\infty}$ of I. Assume the contrary and then it would exists $\delta > 0$ and a subsequence $(m_j)_{j=1}^{\infty}$ of natural numbers such that

$$\Big\| \sum_{n=m_j+1}^{m_{j+1}} T'\chi_{E_n} \Big\|_{\mathcal{L}(X')} \geq \delta \text{ for every } j \in \mathbb{N}, \text{ that is } \| T'\chi_{F_j} \|_{\mathcal{L}(X')} \geq \delta$$

for $j \in \mathbb{N}$, where $F_j = \bigcup_{n=m_j+1}^{m_{j+1}} E_n$. But this last inequality contradicts the relation $\lim_n \| T'(f_n) \|_{X'} = 0$ for every norm-bounded and disjoint sequence $(f_n)_{n=1}^{\infty}$ in X'.

Consequently $\sum_{n=1}^{\infty} T'\chi_{F_n}$ converges. Let now a sequence $F_n \downarrow \emptyset$. We denote by $E_k = F_k \setminus F_{k+1}$, $k \in \mathbb{N}$. Then $\| T'\chi_{F_k} \| = \| \sum_{j=k}^{\infty} T'\chi_{E_j} \|$ and for every $\varepsilon > 0$, there exists $k(\varepsilon)$ such that $k \geq k(\varepsilon)$ implies that $\| T'\chi_{F_k} \|_{\mathcal{L}(X')} \leq \varepsilon$. Hence $\lim_j \| T'\chi_{F_j} \| = 0$ and moreover $\lim_j \| \chi_{F_j} T \|_{\mathcal{L}(X)} = 0$. By Theorem 8 we get that T is a compact operator.

(ii) Since T' is a compact operator on X' for a norm-bounded and disjoint sequence $(f_n)_n$ of elements of X we have by (i) that

$\lim_n \|T''(f_n)\|_{X''} = 0$, and moreover $\lim_n \|T(f_n)\|_X = 0$. By the proof of (i) it follows that $\lim_n \|T\chi_{E_n}\|_{\mathcal{L}(X)} = 0$ for every sequence $E_n \downarrow \emptyset$. Thus $\lim_n \|T''\chi_{E_n}\|_{\mathcal{L}(X'')} = 0$ and by Theorem 8 it follows that T', and hence also T, is a compact operator. ∎

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