ON THE FRÉCHET DIFFERENTIABILITY OF DISTANCE FUNCTIONS

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1. It is well known that in some Banach spaces (which are called Asplund spaces) any continuous convex function is Fréchet differentiable at all points except those which belong to some first category set. In [2] and [3] more stronger estimates of the smallness of the set A_f of points of Fréchet nondifferentiability of continuous convex function f in spaces with a separable dual are given. In [2] it is proved that A_f is σ-porous and in [3] this result is improved by showing that A_f is even angle small.

A subset M of a real Banach space X is said to be α-angle porous (where $\alpha > 0$) if for any $x \in M$ and every $r > 0$ one may find z, $\|z-x\| < r$, and $g \in X^*$ such that

$$M \cap \{y \in X \; ; \; (y-z,g) > \alpha \|g\| \, \|y-z\|\} = \emptyset \quad .$$

If, for every α positive, M can be written as a countable union of α-angle porous sets, we say that M is angle small.

Let X be a Banach space and $M \subset X$ an arbitrary nonempty closed subset of X. Denote by d_M the distance function and by P_M the metric projection determined by the set M. Suppose that X^* is separable and X has an uniformly Fréchet differentiable norm. Then, according to [5], the distance function d_M is Fréchet differentiable at all points of $X-M$ except those which belong to a σ-porous set. If X is a separable Hilbert space, then it is known [4] that the distance function d_M is locally δ-convex in $X-M$ (a function is said to be δ-convex if it is a difference of two continuous convex functions). Therefore we easily obtain that the set of points from $X-M$ at which d_M is not Fréchet differentiable is even angle small.

The purpose of the present article is to prove that this assertion is also true in the case of a Banach space X which has a separable dual and an uniformly Fréchet differentiable norm.

Using a result of [1] we obtain immediately that in Banach spaces which has an uniformly Fréchet differentiable norm, a separable dual and a Fréchet differentiable dual norm on X^* any metric projection P_M is single-valued and continuous at all points of X-M except those which belong to an angle small set.

In the following we use methods from [3] and [5] .

2. In the following X is a real Banach space.

DEFINITION 1. [5] . Let F be a real function defined in X . We say that $g \in X^*$ is an almost superdifferential of F at $x \in X$ if $\limsup_{h \to 0} \left(F(x+h) - F(x) - (h,g) \right) \|h\|^{-1} \leq 0$.

DEFINITION 2. Let F be a real function defined on an open set $G \subset X$. We say that F is uniformly almost superdifferentiable on G if for any $x \in G$ we can choose an almost superdifferential $g(x)$ of F at x such that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\left(F(x+h) - F(x) - (h,g(x)) \right) \|h\|^{-1} \leq \varepsilon$ whenever $0 < \|h\| \leq \delta$ and $x+h \in G$.

DEFINITION 3. Let $g : X \to X^*$ be a singlevalued mapping defined on an open set $G \subset X$. We say that g is LAN (locally almost nonincreasing) on G if for any $x \in G$ and $\varepsilon > 0$ there exists a neighbourhood U of x such that for any $y, z \in U$

$$(y-z, g(y) - g(z)) \leq \varepsilon \|y-z\|$$

DEFINITION 4. Let $\{f_\alpha ; \alpha \in A\}$ be a system of real functions on an open set $G \subset X$. We say that $\{f_\alpha ; \alpha \in A\}$ is uniformly Fréchet differentiable on G if all f_α are Fréchet differentiable on G and if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f_\alpha(y+h) - f_\alpha(y) - (h, f'_\alpha(y))| \leq \varepsilon \|h\|$$

whenever $y \in G$, $\alpha \in A$, $\|h\| < \delta$

LEMMA 1. Let $\{f_\alpha ; \alpha \in A\}$ be a system of K-Lipschitz functions on an open set $G \subset X$ which is uniformly Fréchet differentiable on G . Suppose that the function

$$F(x) := \inf \{ f_\alpha(x) ; \alpha \in A \} \quad \text{is finite on } G .$$

Then F is K-Lipschitz and uniformly almost superdifferentiable on G .

PROOF. It is possible to repeat word by word the proof of Lemma 1 from [5] , since under present hypothesis the corresponding δ does not depend on x .

LEMMA 2. Let F be a real function which is uniformly almost superdifferentiable on an open set $G \subset X$ and let $g(x)$ be as in

Definition 2. Then

(i) the mapping $g : x \to g(x)$ is LAN on G and

(ii) if g is continuous at $a \in G$, then F is Fréchet differentiable at a.

PROOF. Let $x \in G$ and $\varepsilon > 0$ be fixed. Let $\delta > 0$ corresponds to ε as in Definition 2. Then for any $y, z \in G$ from the $\delta/2$ - neigbourhood of x we have $\|y-z\| \leq \delta$ and consequently

$$F(y) - F(z) \leq \varepsilon\|y-z\| + (y-z, g(z)) \quad \text{and}$$
$$F(z) - F(y) \leq \varepsilon\|y-z\| + (z-y, g(y)) .$$

Therefore $(z-y,\ g(z)-g(y)) \leq 2\varepsilon \|y-z\|$, which proves (i). Now suppose that g is continuous at a and let $\varepsilon > 0$ be given. Choose a $\delta > 0$ corresponding to ε by Definition 2 and find $\omega > 0$ such that $\|g(x) - g(a)\| < \varepsilon$ whenever $\|x-a\| < \omega$. If $\|x-a\| < \min(\delta, \omega)$, then

(1) $$F(x) - F(a) \leq \varepsilon\|x-a\| + (x-a, g(a)) \quad \text{and}$$
$$F(a) - F(x) \leq \varepsilon\|x-a\| + (a-x, g(x)) .$$

The last inequality implies

(2) $$F(x) - F(a) \geq (x-a, g(a)) + (x-a, g(x)-g(a)) - \varepsilon\|x-a\| =$$
$$\geq (x-a, g(a)) - 2\varepsilon\|x-a\| .$$

The inequalities (1) and (2) imply that $g(a)$ is the Fréchet derivative of F at a.

LEMMA 3. Let X^* be separable and let $g: X \to X^*$ be LAN on an open set $G \subset X$. Then the set A of all points of discontinuity of g is an angle small set.

PROOF. Put $$A_n = \{x \in G\ ;\ \limsup_{y \to x} \|g(y) - g(x)\| > 1/n\}$$

and choose an arbitrary $\alpha > 0$. Find a sequence (y_k) dense in X^* and define $A_{n,k} = \{x \in A_n\ ;\ \|g(x) - y_k\| < \alpha/4n\}$. Obviously $A = \bigcup A_{n,k}$ and

(3) $$\|g(x) - g(y)\| < \alpha/2n \quad \text{whenever} \quad x, y \in A_{n,k} .$$

Since g is LAN on G and X is separable we can find for any n a countable open covering $(H_p^n)_{p=1}^{\infty}$ of G such that

(4) $$(y-z, g(y)-g(z)) < (\alpha/2n)\ \|y-z\| \quad \text{for} \quad y, z \in H_p^n .$$

Putting $A_{n,k,p} = A_{n,k} \cap H_p^n$ we have $A = \bigcup A_{n,k,p}$ and consequently it is sufficient to prove that any set $A_{n,k,p}$ is α-angle porous. Let n, k, p, $x \in A_{n,k,p}$ and $r > 0$ be fixed. Find $z \in H_p^n$ such that $\|z-x\| < r$ and $\|g(z)-g(x)\| > 1/n$. To finish the proof it is sufficient to show that the set

$$B := A_{n,k,p} \cap \{y \in X\ ;\ (y-z,\ g(x) - g(z)) > \alpha\|y-z\|\,\|g(x)-g(z)\|\}$$

is empty. Suppose on the contrary that there exists $y \in B$. On account of (3) and (4) we obtain

$(\alpha/2n)\,\|y-z\| \geq \|y-z\|\,\|g(x)-g(y)\| \geq (y-z, g(x)-g(y)) =$
$= (y-z, g(x)-g(z)) + (y-z, g(z)-g(y)) > \|y-z\|\,\|g(x)-g(z)\| -$
$- (\alpha/2n)\,\|y-z\| > (\alpha/n)\,\|y-z\| - (\alpha/2n)\,\|y-z\|$
which is a contradiction.

PROPOSITION 1. Let X^* be separable and let F be a real function which is uniformly almost superdifferentiable on an open set $G \subset X$. Then F is Fréchet differentiable at all points of G except those which belong to an angle small set.

PROOF. Proposition is an immediate consequence of Lemma 2 and Lemma 3.

PROPOSITION 2. Let X^* be separable and let $\{f_\alpha ; \alpha \in A\}$ and F be as in Lemma 1. Then F is Fréchet differentiable at all points of G except those which belong to an angle small set.

PROOF. Proposition is an immediate consequence of Lemma 1 and Proposition 1.

THEOREM. Suppose that X has a separable dual and an uniformly Fréchet differentiable norm. Let M be an arbitrary nonempty closed subset of X. Then the distance function d_M is Fréchet differentiable at all points of $X-M$ except of those which belong to an angle small set.

PROOF. Let $U \subset X-M$ be an open bounded set for which $\mathrm{dist}\,(U,M) > 0$. It is easy to see that there exists a bounded set $A \subset M$ such that $d_M(x) = d_A(x)$ for any $x \in U$. Denoting $f_\alpha(x) = \|x-\alpha\|$ we have $d_M(x) = \inf\{f_\alpha(x) ; \alpha \in A\}$. Since the norm of X is uniformly Fréchet differentiable (on the unit sphere) it is easy to see that the system $\{f_\alpha ; \alpha \in A\}$ is uniformly Fréchet differentiable on U. Using Lemma 1 (for $K=1$) and Proposition 2 we obtain that d_M is Fréchet differentiable at all points of U except those which belong to an angle small set. Since a countable union of angle small sets is obviously angle small and G is separable, it is easy to finish the proof.

Using Corollary 2.7. from [1] we obtain immediately the following consequence.

COROLLARY. Suppose that X has an uniformly Fréchet differentiable norm, X^* is separable and the norm of X^* is Fréchet differentiable. Let M be an arbitrary nonempty closed subset of X. Then the metric projection P_M is singlevalued and continuous at all points of $X-M$ except those which belong to an angle small set.

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