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Epimorphims and cowellpoweredness of epireflective subcategories of Top


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Abstract. A functor $F_A : \text{Top} \rightarrow \text{Top}$ induced by a given epireflective subcategory $A$ of the category $\text{Top}$ of topological spaces is used to characterize epimorphisms in some familiar epireflective subcategories of $\text{Top}$ and to solve for these subcategories, the problem of the cowell-poweredness. Furthermore an ordinal number $E_O(X)$, for each $X \in \text{Top}$, is introduced and it is computed in several examples. As an application it is shown that there is no epireflective subcategory of $\text{Top}$ which is properly contained in the subcategory $\text{Top}_2$ of all Hausdorff spaces and whose extremal epireflective hull is $\text{Top}_2$.

1. In 1975 Salbany ([14]) introduced a closure operation $[\cdot]_A : 2^X \rightarrow 2^X$ defined on subsets of a topological space $X$ by a class $A$ of topological spaces. In 1980 Giuli ([6]) used that closure operation to study epi-reflections in epireflective subcategories of $\text{Top}$. He pointed out that epimorphisms in an epireflective subcategory $A$ of $\text{Top}$ are precisely the continuous maps which are dense with respect to $[\cdot]_A$. Recently Dikranjan and Giuli ([4]) characterized $[\cdot]_A$ for some familiar epireflective subcategories $A$ of $\text{Top}$. They showed that, as in the classical case of Hausdorff spaces, the closure operation $[\cdot]_A$ characterizes the spaces $X$ of $A$ in terms of the $A$-closure of the diagonal $\Delta_X$ for $A = \text{Top}_0, \text{FT}_2, \text{Top}_{24}, \text{P}(0\text{-dim})$, (see 1.1 below).

In this paper we will use the previous closure operation to define, for each epireflective subcategory $A$ of $\text{Top}$, a functor $F_A : \text{Top} \rightarrow \text{Top}$. Then some sufficient conditions for the cowellpoweredness of $A$ are given and they are used to answer the question of the cowellpoweredness of some epireflective subcategories of $\text{Top}$. Furthermore an ordinal number $E_O(X)$ (called epimorphic order of $X$ with $A$)

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* Talk given by the second named author. The paper is in its final form and will not be published anywhere else.
respect to $A$ is introduced for each $X\in\text{Top}$ and in several examples it is computed. Iterations of the functor $F_A$ and the relation with the $A$-reflection functor are also studied.

We will use the previous closure operation in a forthcoming paper for a new approach to the study of $A$-minimal and $A$-closed spaces ([5]).

1.1. The following subcategories of $\text{Top}$ are symbolized as follows:

- $\text{Top}_i = \text{the subcategory of topological spaces satisfying the $T_i$-axiom (i=0,1,2)}$
- $\text{FT}_2 = \text{The subcategory of functionally Hausdorff spaces, i.e., spaces $X$ such that for any two different points $x_1, x_2$ there exists a continuous map $f:X\to \mathbb{R}$ with $f(x_1) \neq f(x_2)$.}$
- $\text{Top}_3 = \text{The subcategory of regular Hausdorff spaces.}$
- $\text{P(Top}_3) = \text{The subcategory consisting of spaces whose topology is finer than a regular Hausdorff topology.}$
- $\text{Top}_{2\gamma} = \text{The subcategory of Urysohn spaces, i.e., spaces such that for any two different points there exist disjoint closed nbds.}$
- $\text{Top}_{3\gamma} = \text{The subcategory of completely regular Hausdorff spaces.}$
- $\text{0-dim} = \text{The subcategory of 0-dimensional spaces, i.e., Hausdorff spaces with a base of clopen sets.}$
- $\text{P(0-dim)} = \text{The subcategory of spaces whose topology is finer than a 0-dimensional topology, i.e., spaces in which every point is the intersection of the clopen sets containing it ([12]).}$

We recall that a full and isomorphism-closed subcategory $A$ of $\text{Top}$ is said to be epireflective (respectively bireflective, extremally epireflective) in $\text{Top}$ if for each topological space $X$ there exist $r_A(X)$ belonging to $A$ and an epimorphism (respectively bimorphism, extremal epimorphism) $r_A:X\to r_A(X)$ such that, for each $A\in A$ and continuous map $f:X\to A$ there exists a (unique) continuous map $f':r_A(X)\to A$ satisfying the condition $r_Af'=f$. $r_A$ is called the $A$-reflection of $X$.

$A$ is epireflective in $\text{Top}$ iff it is closed under the formation of products and subspaces ( = extremal subobjects). It is extremally epireflective iff it is epireflective and contains finer topologies. It is bireflective iff it is epireflective and contains (all) indiscrete spaces.

Every class $B$ of topological spaces admits an epireflective hull $E(B)$ (i.e., a smallest epireflective subcategory containing $A$), an
extremal epireflective hull $P(B)$ and a bireflective hull $I(B)$.

All categories listed in 1.1. are epireflective in $\text{Top}$, $\text{Top}_i$, for $i = 0,1,2,2\frac{1}{2}$, and $\text{FT}_2$ are extremally epireflective in $\text{Top}$. For all categories $A$ listed in 1.1. the subcategory $\text{A} = \{ X \in \text{Top} : r_0(X) \in A \}$ (where $r_0$ is the $\text{Top}_0$-reflection) is bireflective in $\text{Top}_o$. For $\text{Top}_{3\frac{1}{2}}$, $\text{FT}_2$ and $\text{Top}_{3\frac{1}{2}}$ (subcategory of completely regular spaces) are respectively the epireflective hull, the extremal epireflective hull and the bireflective hull of the real line $\mathbb{R}$ in $\text{Top}_o$.

For general results on epireflective subcategories of $\text{Top}$ see [7,8]. The categorical terminology is that of [10].

In what follows $A$ will denote an epireflective subcategory of $\text{Top}$. For each pair of continuous maps $(f,g:X \to Y)$, $\text{Eq}(f,g)$ will denote the equalizer in $\text{Top}$ of $f$ and $g$ (i.e., $\text{Eq}(f,g) = \{ x \in X : f(x) = g(x) \}$).

1.2. - Definitions. (a) A subset $F$ of a topological space $X$ is said to be closed with respect to $A$ (in short $A$-closed) in $X$ if there exist $A \subseteq A$ and continuous maps $f,g:X \to A$ such that $\text{Eq}(f,g) = F$.

b) We will define $A$-closure of a subset $M$ of $X$ as follows:

$$[M]^X_A = \{ F \subseteq X : M \subseteq F \text{ and } F \text{ is } A \text{-closed} \}$$

When no confusion is possible we write $[M]_A$ or simply $[M]$ instead of $[M]^X_A$.

c) If $x \notin M$ and $f,g:X \to A, A \subseteq A$, are continuous maps such that $M \subseteq \text{Eq}(f,g)$ and $f(x) \neq g(x)$ then, $(f,g)$ is said to be an $A$-separating pair for $(x,M)$.

By definition $x \notin [M]_A$ iff there exists an $A$-separating pair for $(x,M)$. The family of all $A$-closed sets of a topological space $X$ trivially contains $X$ and, by the productivity of $A$, it is closed under the formation of intersections (i.e., it is a Moore family). Thus the $A$-closure is a closure operation in the sense of Birkhoff ([2]). Furthermore $[\emptyset]_A = \emptyset$ for all epireflective subcategories $A$ different from the trivial subcategory $Sgl$ consisting of topological spaces whose underlying sets have at most one point. Even if $[M]_A \cup [N]_A \subset [M \cup N]_A$ for each $M,N \subseteq X$, the epireflective hull of an infinite strongly rigid space (the continuous self-maps are precisely the constant maps and the identity map ([10])) provides an example of a non-additive closure operation ([3,4]).

2. The following lemma is very useful in the sequel.
2.1. Lemma. (a) For each $X \in \text{Top}$ and $M \subseteq X$, the following holds:

$$[M]^X_A = (r_A^{-1})^* ([r^*_A(M)]^A (X)).$$

Thus $A$-closure is additive (thus a Kuratowski operation) for each $X \in \text{Top}$ iff it is so for each $A \in A$.

(b) For each $X \in P(A)$ and $M \subseteq X$, the following holds:


Thus $P(A)$-closure is a Kuratowski operation iff $A$-closure is.

Proof. (a) By 1.2 (x) of [4], $r_A ([M]^X_A) \subseteq [r^*_A(M)]^A (X)$, so

$$[M]^X_A \subseteq (r_A^{-1})^* ([r^*_A(M)]^A (X)).$$

On the other hand, if $x \notin [M]^X_A$ and $(f, g : X \rightarrow A)$ is an $A$-separating pair for $(x, M)$, then $(f^*, g^* : r_A (X) \rightarrow A)$ where $f^* r_A = f$ and $g^* r_A = g$, is an $A$-separating pair for $(r_A(X), r_A(M))$, so $x \notin (r_A^{-1})^* ([r^*_A(M)]^A (X))$.

(b) For each $X \in P(A)$, $r_A : X \rightarrow r_A (X)$ is the identity on the underlying sets then, it follows from (a) that $[M]^X_{P(A)} = [M]^X_{P(A)}$.

Furthermore $[M]^X_{P(A)} \subseteq [M]^X_{P(A)}$ follows from the inclusion $A \subseteq P(A)$.

To show that $[M]^X_{P(A)} \subseteq [M]^X_{P(A)}$ take $x \notin [M]^X_{P(A)}$ and a $P(A)$-separating pair $(f, g : X \rightarrow Y)$ for $(x, M)$. Then $(f^*, g^* : r_A (X) \rightarrow r_A (Y))$, where $f^* r_A = f$ and $g^* r_A = g$, is an $A$-separating pair for $(x, M)$ in $r_A (X)$, so $x \notin [M]^X_{P(A)}$. For the last equality note that $r_A : X \rightarrow r_A (X)$ is the identity on the underlying sets then (a) gives $[M]^X_A = [M]^X_A$ for every $M \subseteq X$.

For each $(X, \tau) \in \text{Top}$, $\tau_A$ will denote the topology generated in $X$ by the $A$-closure, i.e., the coarsest topology on $X$ for which all $A$-closed sets are closed. $F_A : \text{Top} \rightarrow \text{Top}$ will denote the functor which assigns to $(X, \tau) \in \text{Top}$ the space $(X, \tau_A)$. For each continuous map $f : (X, \tau) \rightarrow (Y, \tau')$ in $\text{Top}$ the continuity of $f = F_A (f) : (X, \tau_A) \rightarrow (Y, \tau')$ follows from 1.2 (x) of [4].

By 2.1 of [4] for every $(X, \tau) \in \text{Top}$, $\tau_A$ is the initial topology on $X$ induced by the map $X \rightarrow F_A (r_A X)$, where $r_A$ is the $A$-reflection of $X$. This is why $(X, \tau_A)$ is discrete iff $r_A X$ is a singleton. On the other hand, if $A \neq \text{Sgl}$, then for each $(X, \tau) \in \text{Top}$, $(X, \tau_A) \in \text{Top}_1$ iff $r_A : X \rightarrow r_A X$ is injective. In particular if $A$ is extremally epireflec-
Epimorphisms and cowellpoweredness of... 125

tive, then \((X, Z) \in \mathbf{Top}_1 \) iff \((X, Z) \in \mathbf{A}\). Conditions ensuring \((X, Z) \in \mathbf{Top}_2 \) are discussed in 2.8.

Till the end of this section, we study the properties of the functor \(F_A\). Set \(A = \{X \in \mathbf{Top} : \mathbf{F}(X) = X\}\). Clearly \((X, Z) \in \mathbf{A}\) iff \(r_A(X, Z) \in \mathbf{A}\) and \(X\) has the initial topology with respect to \(r_A : X \rightarrow r_A X\).

In the following theorem we give explicitly \(F_A\) for various categories \(A\) including those listed in 1.1. First recall the notion of θ-closure introduced by Velichko ([17]). For \((X, Z) \in \mathbf{Top}\) and \(M \subset X\), \(\mathbf{C} = \{x \in X : \text{for each nbd } V \text{ of } x, \exists s \in V \text{ s.t. } M \subset V\}\). A subset \(M\) of \(X\) is said to be θ-closed (θ-open) if \(M = \mathbf{C}(M)\). The θ-closure is additive but not idempotent in general. The idempotent hull of \(\mathbf{C}\) is \(\bigcap_{\mathbf{Top}_2}^{\mathbf{A}}\) since for each \((X, Z) \in \mathbf{Top}\) and \(M \subset X\), \(\mathbf{C}(M) \subset \bigcap_{\mathbf{Top}_2}^{\mathbf{A}}\) and \(M\) is θ-closed iff \(M\) is \(\mathbf{Top}_2\)-closed (see 2.5(b) of [4]). θ-closure was also studied by Schröder [15].

2.2. Theorem. (a) If \(A\) is bireflective (resp. \(A = \mathbf{Top}_1\)) then \(\mathbf{C} = \mathbf{A}\) is the discrete topology for every \((X, Z) \in \mathbf{Top}\) (resp. \((X, Z) \in \mathbf{A}\)).

(b) If \(A = \mathbf{Top}_1\), \(i = 2, 3, 3^*\), or \(A = \mathbf{O-dim}\), then \(\mathbf{C} = \mathbf{Z}\) for each \((X, Z) \in \mathbf{A}\).

(c) If \(A = \mathbf{P}(B)\) then for each \((X, Z) \in \mathbf{A}\), \(\mathbf{C} = \mathbf{Z}_B\), where \((X, Z)\) is the \(B\)-reflection of \((X, Z)\). Thus the functors \(F_A\) and \(F_B\) coincide.

(d) For \(B = \mathbf{Top}_3, \mathbf{Top}_3^{3^*}\) and \(\mathbf{O-dim}\) and \(A = \mathbf{P}(B)\), \(F_A\) coincides on \(A\) with the \(B\)-reflection.

(e) For \(A = \mathbf{Top}_0\) and \((X, Z) \in \mathbf{Top}_0\), \(\mathbf{C} = \mathbf{A}\) is the topology on \(X\) having, as open base, all locally closed subsets of \((X, Z)\) (finite intersections of open and closed sets in \((X, Z)\)). Thus \(\mathbf{C} = \mathbf{Z}\) and \((X, Z) \in \mathbf{O-dim}\).

(f) For \(A = \mathbf{Top}_3^{3^*}\) and \((X, Z) \in \mathbf{A}\), \(U \in \mathbf{Z}_A\) iff \(U\) is θ-open. In particular \(\mathbf{Z} = \mathbf{Z}_A\) iff \((X, Z) \in \mathbf{Top}_0\).

Proof. (a): By 1.10 (a) of [4] in this case the \(A\)-closure coincides with the identity operator.

(b): By 2.8 (i) of [4] in this case the \(A\)-closure coincides with the ordinary closure.

(c): It follows from 2.1 (b). (d): It follows from (b) and (c).

(e): As pointed out in 2.9 of [4] in this case the \(\mathbf{Top}_0\)-closure coincides with the well-known front-closure ([17],[12]),

\[ \mathbf{C} = \{x \in X : \text{for each nbd } V \text{ of } x, \exists s \in V \text{ s.t. } M \subset V\}. \]

Thus \(U \subset X\) is \(\mathbf{Z}_A\)-open iff for each \(x \in U\) there exists a nbd \(V\) such that \(V \cap (X \setminus U) = \emptyset\), i.e. \(\exists x \in V \subset U\). Clearly any \(V \cap V_i\) is clopen in \((X, Z)\), so \((X, Z) \in \mathbf{O-dim}\).
(f) Obviously $\text{Cl}_\theta$ is additive, thus $\bigcup_{\text{Top}_\theta}$ being its idempotent hull will be a Kuratowski operator (in fact, $\text{Cl}_\theta([M]\cup[N]) = \text{Cl}_\theta([M])\cup\text{Cl}_\theta([N]) \subset ([M]\cup[N]) = [M]\cup[N]$, so $[M]\cup[N]$ is $\theta$-closed, thus $\text{Top}_\theta$ closed). The last assertion is proved in 2.4(ii).

2.3 Remarks (a). The $A$-closure is additive in all subcategories $A$ of $\text{Top}$ listed in 1.1. We do not know any example of non additive $A$-closure operation different from the case $A=$epireflective hull of a class of strongly rigid spaces.

(b) By the explicit form of $\tau_{\text{Top}_\theta}$ it can be seen easily that for $(X,\tau)\in\text{Top}_\theta$, $\tau_{\text{Top}_\theta}$ is discrete iff for each $x\in X$ there exists a $\tau$-nbd $V$ such that $\{x\} = \overline{\{x\}} \cap V$.

The subcategory of such spaces of $\text{Top}_\theta$ will be denoted by $\text{T}_\theta$.

(c) The functor $F_{\text{Top}_\theta}$ preserves embeddings and finite products (more precisely, for each family $\{(X_i,\tau_i)\}_{i\in I}$ in $\text{Top}_\theta$ with $(X,\tau) = \prod_{i\in I}(X_i,\tau_i)$, $\tau = \prod_{i\in I}(\tau_i)$ holds iff all but a finite number of the spaces $X_i$ are singletons).

In general the functor $F_A$ is submultiplicative, i.e., for each family $\{(X_i,\tau_i)\}_{i\in I}$ in $\text{Top}_\theta$, $(\prod_{i\in I}(X_i,\tau_i)) \supseteq \prod_{i\in I}(\tau_i)$, the following examples show that in general $F_A$ does not preserve neither embeddings nor finite products.

2.4. Examples (a) Let $(H,\tau')$ be the space given in 1.3 of [4]. Then $F_{\{0,0\}}$ is discrete in $(H,\tau')$, while $F_{\{0,0\}}$ is not discrete as a subspace of $(H,\tau')$, $\tau'$ again compact.

(b) Let $A=\mathbb{E}(\{X,\tau\}$, where $(X,\tau)$ is an infinite strongly rigid space. Then $\tau_A$ is the cofinite topology on $X$, so $\Delta_X$ is not closed in $(XXX,\tau_A \times \tau_A^*)$. On the other hand $\Delta_X$ is the equalizer of the projections, so $\Delta_X$ is closed in $(X^2, (\tau_A^*)^2)$. Thus $(\tau_A^*)^2 \supseteq \tau_A \times \tau_A$.

2.5 Proposition. If $F_A$ is finitely multiplicative, then for each $(X,\tau) \in \text{Top}_A$, $(X,\tau_A) \in \text{Top}_2$.

Proof. Consider $\Delta_X$ in $(XXX,\tau_A \times \tau_A^*)$; since $\Delta_X$ is always $A$-closed in $(XXX,\tau_A \times \tau_A^*)$ and $(\tau_A^*)^2 = \tau_A \times \tau_A$, this implies that $\Delta_X$ is closed in $(XXX,\tau_A \times \tau_A^*)$, so $(X,\tau_A) \in \text{Top}_2$.

In the following Section we show that there exists $(X,\tau) \in \text{Top}_2$ with $(X,\tau) \notin \text{Top}_2$. Hence $F_{\text{Top}_2}$ is not finitely multiplicative.

Till the end of this section we study conditions which ensure $\tau_A \in \tau$ or $(X,\tau_A)$ discrete.

For $(X,\tau) \in \text{Top}$ denote by $I(X,\tau)$ the set of all isolated points.
2.6 Lemma. For any epireflective subcategory $A$ of $\text{Top}$ and each $(X,\mathfrak{F}) \in A$
$$I(X,\mathfrak{F}) \subseteq I(X,\mathfrak{F}_A).$$
Moreover, (*) holds for each $(X,\mathfrak{F}) \in \text{Top}$ iff $A$ is bireflective or $A=\text{Top}_0$.

Proof. Consider first the case when $A$ is neither bireflective nor $\text{Top}_0$. Then $A \subseteq \text{Top}_1$, so for every $(X,\mathfrak{F}) \in A$, $(X,\mathfrak{F}_A) \in \text{Top}_1$ holds. Therefore each isolated point of $(X,\mathfrak{F})$ is $\mathfrak{F}$-clopen, thus also $\mathfrak{F}_A$-clopen by 1.2 (vi) of [4]. This proves (*). Remark that (*) does not hold for Sierpinski's two-points space $(S,\mathfrak{F})$ (two points 0,1 with $\emptyset$ unique proper open set) since $I(S,\mathfrak{F}) \neq \emptyset$ and $I(S,\mathfrak{F}_A) = \emptyset$ (the space $(S,\mathfrak{F}_A)$ is indiscrete since the reflection of $(S,\mathfrak{F})$ in $A$ is a singleton because of $A \subseteq \text{Top}_1$).

It remains to show that (*) holds for every $(X,\mathfrak{F}) \in \text{Top}$ if $A$ is bireflective or $A=\text{Top}_0$. This is obvious in the first case since $\mathfrak{F}_A$ is always discrete according to 2.2 (a). Assume $A=\text{Top}_0$ and take an arbitrary $(X,\mathfrak{F}) \in \text{Top}$. Then for each $x \in I(X,\mathfrak{F})$ the characteristic (continuous) map $f:X \rightarrow S$ of the open set $\{x\}$ and the constant at 1 form an $A$-separating pair for $(x,X\setminus\{x\})$ so $x \in I(X,\mathfrak{F}_A)$.

In the following proposition we show that the converse inclusion of (*) for any space $(X,\mathfrak{F}) \in A$ implies $A \subseteq \text{Top}_0$.

2.7 Proposition. For each epireflective subcategory $A$ of $\text{Top}$ the following conditions are equivalent:

(a) $A \subseteq \text{Top}_2$;
(b) for each $(X,\mathfrak{F}) \in A$ $\mathfrak{F}_A \subseteq \mathfrak{F}$;
(c) for each $(X,\mathfrak{F}) \in \text{Top}$ $\mathfrak{F}_A \subseteq \mathfrak{F}$;
(d) for each $(X,\mathfrak{F}) \in A$ $I(X,\mathfrak{F}) = I(X,\mathfrak{F}_A)$;
(e) every $(X,\mathfrak{F}) \in A$ is discrete whenever $(X,\mathfrak{F}_A)$ is discrete.

Proof. The equivalence (a) $\iff$ (b) was given in 1.10 (b) from [4]. The equivalence (b) $\iff$ (c) follows from 2.1 (a). Clearly (b) implies (d) and (d) implies (e). To finish the proof we have to show (e)$\Rightarrow$(a).

We can assume without loss of generality that $A$ is extremally epireflective. In fact, if $B=P(A)$ then because of 2.2 (c) the functors $F_A$ and $F_B$ coincide. To show that each $(X,\mathfrak{F}) \in B$ satisfies (e), consider the reflection $(X,\mathfrak{F}')$ of $(X,\mathfrak{F})$ in $A$. Then by 2.2 (c) $\mathfrak{F}_B = \mathfrak{F}'$. Now if $\mathfrak{F}$ is discrete then by (e) $(X,\mathfrak{F}')$ is discrete, thus $(X,\mathfrak{F})$ is discrete too. So we can assume that $A$ is extremally epireflective.
If $A$ is bireflective then $A=\text{Top}$ and (e) is not verified since $\mathcal{E}_{\text{Top}}$ is always discrete. Therefore $A \subseteq \text{Top}_0$. Now $A=\text{Top}_0$ contradicts (e) since there exists a non-discrete space $(X,\mathcal{E}) \notin \text{Top}_0$, then $\mathcal{E}_{\text{Top}_0}$ is discrete.

We have shown that (e) implies $A \subseteq \text{Top}_1$. Assume there exists a space $(X,\mathcal{E}) \notin A$ such that $(X,\mathcal{E}) \notin \text{Top}_2$. Then there exist two distinct points $x$ and $y$ in $X$ such that for any nbd $V$ of $x$ and any nbd $U$ of $y$ in $(X,\mathcal{E})$  

$$V \cap U \neq \emptyset.$$ 

Now set $Y = \{p \cup X \sim \{x,y\}\}$ and consider the following topology $\mathcal{E}$ on $Y$. All points different from $p$ are isolated, for nbds system of $p$ take all intersections (***) added the point $p$. Clearly $\mathcal{E}$ is non discrete because of (**). Consider the maps $f_x$ and $f_y$ of $Y$ into $X$ defined by, 

$$f_x(u) = f_y(u) = u \text{ if } u \neq p \text{ and } f_x(p) = x, f_y(p) = y.$$ 

The continuity of $f_x$ and $f_y$ follows directly from the definition of $\mathcal{E}$. On the other hand both maps are injective, hence $(Y, \mathcal{E}) \notin A$ because $X \notin A$ and $A$ is extremally epireflective. Now the space $(Y, \mathcal{E})$ does not satisfy (e) since $\mathcal{E}_{A}$ is discrete. In fact by 2.6 $I(Y, \mathcal{E}_A) \ni I(Y, \mathcal{E}) = Y \setminus \{p\}$ and $(f_x, f_y)$ is an $A$-separating pair for $(p, Y \setminus \{p\})$, so $\{p\}$ is $\mathcal{E}'$-open.

3. It is well known that $\text{Top}_2$ is a cowellpowered category, i.e., the class of all $\text{Top}_2$-epimorphisms (i.e. dense continuous maps) with domain a fixed Hausdorff space has a representative set ([7]). In 1975 Herrlich [5] first produced an example of a non cowellpowered epireflective subcategory of $\text{Top}$: the epireflective hull of a proper class of strongly rigid spaces such that the continuous maps between them are precisely the identities or the constant maps.

In 1983 Schröder showed that $\text{Top}_{2_{\mathcal{E}}}$ is not cowellpowered ([16]). He produced for each ordinal number $\beta$ a Urysohn space $\mathcal{Y}_\beta$ of cardinality $\aleph_0 \cdot \text{card}(\beta)$ and an embedding $e_\beta : \mathcal{Q} \hookrightarrow \mathcal{Y}_\beta$, where $\mathcal{Q}$ is the space of rational numbers with the usual topology, such that $e_\beta$ is a $\text{Top}_{2_{\mathcal{E}}}$-epimorphism.

In what follows we shall show that all remaining categories listed in 1.1 are cowellpowered. The following proposition given in [4] and [6] will be used.

3.1 Proposition. $f : X \to Y$ is an $A$-epimorphism iff $f(X)$ is $A$-dense in $Y$, i.e., $[f(X)]_A = Y$.

3.2 Lemma. Let $A$ and $B$ be epireflective subcategories of Top and let
F:A → B be a functor satisfying the following conditions:

1. F preserves epimorphisms, i.e., for each A-epimorphism f:X → Y the map F(f):F(X) → F(Y) is a B-epimorphism;
2. F is a concrete functor, i.e., if U:Top → Set is the forgetful functor then UF = U.

Then A is cowellpowered whenever B is cowellpowered.

Proof. Trivial.

3.3 Corollary. Let B be a cowellpowered epireflective subcategory of Top, then so is P(B).

Proof. For A=P(B) and F=r_B—the B-reflection—apply 3.2. Clearly F satisfies (2), on the other hand, by 2.1.(b), f:X → Y is an epimorphism in A iff f=F(f):F(X) → F(Y) is an epimorphism in B. Thus F satisfies also (1).

3.4 Corollary. If A is an epireflective subcategory of Top such that for each (X,τ) ∈ A, τ_A ∈ Top_2, then A is cowellpowered.

Proof. For B=Top_2 and F=F_A we apply 3.2. Obviously (2) holds; on the other hand for each epimorphism f:X → Y in A f(X) is A-dense in Y by virtue of 3.1. Therefore f(X) is dense in F(Y) hence f:F(X) → F(Y) is an epimorphism in B=Top_2 and Top_2 is cowellpowered.

For all subcategories of Top listed in 1.1 except Top_2 with A has Hausdorff so all they are cowellpowered.

3.5 Corollary. If A is an epireflective subcategory of Top such that F_A is finitely multiplicative than A is cowellpowered.

Proof. By virtue of 2.5, A satisfies the condition in 3.4, so A is cowellpowered.

Some familiar extremally epireflective subcategories of Top are the extremal epireflective hull of a proper epireflective subcategory (e.g. FT_s=P(3_T)). Top_2 does not have that property as the following proposition shows.

3.6 Proposition. If A is an extremally epireflective subcategory of Top and for every (X,τ) ∈ A, τ_A = τ, then there does not exist a proper epireflective subcategory B ⊆ A such that P(B) = A.

Proof. Since τ_A = τ for each (X,τ) ∈ A, by virtue of proposition 2.7, A ⊆ Top_2. Assume there exists an epireflective subcategory B of Top such that A=P(B).

By 2.2(c), for each (X,τ) ∈ A with B-reflection r_B(X,τ) = (X,σ), r_A = r_B holds. Since B ⊆ A ⊆ Top_2, σ_B ≤ σ_A, thus we get τ = τ_A = σ_B ≤ σ = τ_B.
On the other hand always \( \mathcal{E} \subseteq \mathcal{E} \) holds, so for each \( (X, \mathcal{E}) \in A \), \( r_B(X, \mathcal{E}) = (X, \mathcal{E}) \).
Therefore \( B = A \).

3.7 Question. Does there exist such a \( B \) as in 3.6 for \( A = \text{Top} \)? By virtue of 3.3 such a \( B \) will not be coweighted.

4. In this section we consider iterations of the functor \( F_A : \text{Top} \to \text{Top} \) defined in section 2. Let \( A \) be epireflective subcategory of \( \text{Top} \); then for each ordinal number \( \alpha \) we define a topology \( \mathcal{E}_A^\alpha \) on \( X \) in the following way: \( \mathcal{E}_A^0 = \mathcal{E} \) and \( \mathcal{E}_A^{\alpha+1} = (\mathcal{E}_A^\alpha)^\mathcal{A} \) for any \( \alpha \); if \( \alpha \) is a limit ordinal \( \mathcal{E}_A^\alpha = \inf \mathcal{E}_A^\beta \). It is easy to check that setting \( F_A(X, \mathcal{E}) = (X, \mathcal{E}_A^\alpha) \) we get a functor \( F_A : \text{Top} \to \text{Top} \). By virtue of 2.7 if \( A \subseteq \text{Top} \) for each \( (X, \mathcal{E}) \in \text{Top} \), the topologies \( \mathcal{E}_A^\alpha \) form a decreasing chain, so there will exist an ordinal number \( \alpha \) such that \( \mathcal{E}_A^{\alpha+1} = \mathcal{E}_A^\alpha \).

4.1 Definition. Let \( A \subseteq \text{Top} \) and \( (X, \mathcal{E}) \in \text{Top} \); the smallest ordinal \( \alpha \), such that \( \mathcal{E}_A^{\alpha+1} = \mathcal{E}_A^\alpha \) will be called epimorphic order of \( (X, \mathcal{E}) \) with respect to \( A \) and will be denoted by \( E_0^A(X, \mathcal{E}) \).

In particular \( E_0^A(X) = 0 \) iff \( X \in A \), otherwise \( E_0^A(X) = 1 + E_0^A(F(X)) \) with easy check.

Epimorphic order can be defined in a similar way also for categories \( A \) such that \( V_*^V \) for each \( (X, \mathcal{E}) \in \text{Top} \).

4.2 Examples. Let \( (X, \mathcal{E}) \) be an infinite strongly rigid space and \( A = E\{(X, \mathcal{E})| \text{ } \} \); then \( \mathcal{E}_A^\alpha \) is the cofinite topology on \( X \), so \( r_A(X, \mathcal{E}) \) is a singleton, therefore \( (X, \mathcal{E}_A) \) is indiscrete, so \( E_0^A(X, \mathcal{E}) = 2 \).

(b) Let \( B \subseteq B \) and \( A = P(B) \), then \( E_0^A(X, \mathcal{E}) = 0 \) iff \( r_A(X, \mathcal{E}) \in B \) and \( X \) has the initial topology with respect to \( X \to r_A(X, \mathcal{E}) \), otherwise \( E_0^A(X, \mathcal{E}) = 1 \).

(c) Let \( (Y_\beta, \mathcal{E}_\beta) \) be the Ubrishon space constructed in [16] for an ordinal \( \beta \) satisfying \( 1 < \beta < \omega + 1 \); then \( E_0^{\text{Top}_2}(Y_\beta, \mathcal{E}_\beta) = 2 \) while \( E_0^{\text{Top}_2}(Y_1, \mathcal{E}_1) = 1 \). Moreover \( F^{\text{Top}_2}(Y_\beta, \mathcal{E}_\beta) \) of these ordinals.

(d) If \( A \) is bireflective and \( X \in \text{Top} \) then \( E_0^A(X) = 0 \) iff \( X \) is discrete, otherwise \( E_0^A(X) = 1 \).

(c) For \( A = \text{Top}_1 \), \( E_0^A(X) = 0 \) iff \( r_{\text{Top}_1}(X) \) is discrete and \( X \) has the initial topology with respect to \( X \to r_{\text{Top}_1}(X) \), otherwise \( E_0^A(X) = 1 \).

(f) For \( A = \text{Top}_0 \) and \( X \in \text{Top} \), \( E_0^A(X) = 0 \) iff \( r_{\text{Top}_0}(X) \) is discrete and \( X \) has the initial topology with respect to \( X \to r_{\text{Top}_0}(X) \);
\( E_0^A(X) = 1 \) iff \( r_{\text{Top}_0}(X) \) is non discrete and belongs to \( T_D \), \( E_0^A(X) = 2 \) iff \( r_{\text{Top}_0}(X) \notin T_D \).
We have no examples of epimorphic order greater than 2.

In order to calculate easier the epimorphic order we have to know better the interrelation between the functors $F_A$ and $r_A$. In what follows we omit the index $A$ for brevity, $A$ is always contained in Top$_2$ and $\prod_{x \in k}$.p.

For each $x \in$ Top consider the diagram

$$
\begin{aligned}
X & \xrightarrow{id} FX \\
\downarrow & \downarrow r \\
rX & \xrightarrow{S_x} rFX
\end{aligned}
$$

By the definition of $r$ there exists a unique continuous map $S_x : rX \rightarrow rFX$ which makes commutative the diagram.

**4.3 Lemma.** The map $S_x : rX \rightarrow rFX$ defined above is continuous when we consider on $rX$ the topology generated by the $A$-closure, i.e. $S_x : FrX \rightarrow rFX$ is continuous.

**Proof.** We have to show that for each closed set $M$ in $rFX$, $S_x^{-1}(M)$ is $A$-closed in $rX$. By the continuity of $r_1$, $r_1^{-1}(M)$ is closed in $FX$. By 2.1 (a)

$$
[r_1^{-1} M]^X = r_1^{-1}([r_1^{-1} M]^X) = r_1^{-1}M
$$

on the other hand $r_1 = S_x r$, so $r(r_1^{-1} M) = S_x^{-1}(M)$, thus $r_1^{-1}M = Fr(r^{-1} M) = FrS_x^{-1} M$.

Applying $r$ we get $S_x^{-1} M = [S_x^{-1} M]$ which proves the continuity of $S_x : FrX \rightarrow rFX$.

**4.4 Proposition.** For each natural number $n$ and each $x \in \omega$, $rF^n rX$ is naturally isomorphic to $rF^n X$.

**Proof.** For any natural $k < n$ the above lemma applied to the space $Y = r^{k}X$ provides a natural continuous map $S_k : FrF^{k}X \rightarrow rF^{k+1}X$ which makes commutative the following diagram

$$
\begin{aligned}
F^{k+1}X & \xrightarrow{S_k} rF^{k+1}X \\
\downarrow r_k & \downarrow r_{k+1} \\
FrF^{k}X & \xrightarrow{S_k} rF^{k+1}X
\end{aligned}
$$

where $r_k$ and $r_{k+1}$ are the corresponding reflections. Applying the functor $F^{n-k-1}$ we get the commutative diagram

$$
\begin{aligned}
rF^n rX & \xrightarrow{S_n} rF^n X \\
\downarrow & \downarrow r_n \\
rF^n rX & = = rF^n X
\end{aligned}
$$
where \( F \) is the A-reflection. By the definition of the reflection there exists a unique continuous map \( S_n: rF^n rX \rightarrow rF^n X \) such that \( Snr = S \).

Let us see that \( S \) is an isomorphism. Again by the properties of the reflection there exists a unique continuous map \( \&: rF^n X \rightarrow rF^n rX \) such that \( \&r = \&_o \).

Consider now the composition \( \# = \& S \circ \& F_{rF} X \) by the definition of \( S \) and \( \& \) we get \( \# = \& S \circ r = \&_o \).

Thus the restriction of \( \& S \) on \( \&_o (F^n X) \) is the identity.

Since \( \&_o \) is an epimorphism this gives \( \& S = \text{id} \) on \( rF^n rX \). In the same way one proves that \( S \circ \& \) is the identity on \( rF^n X \).

4.5 Remark. Consider the semigroup \( \Sigma \) of all functors \( \text{Top} \rightarrow \text{Top} \) generated by \( r \) and \( F \). By the definition of \( r \), \( r = r^2 \) holds. On the other hand 4.4 shows that, for any \( n \), there is an equivalence between \( nF^n rX \) and \( nF^n X \). Let \( \Sigma \) be the quotient of \( \Sigma \) with respect to the equivalence of functors. Then the functors \( F^n \) and \( F^{nF} \) with \( m, n \) and \( t \) non-negative integers \( (F^0 \) is the identity functor) represent all elements of \( \Sigma \). The multiplication is given by

\[
(F^m F^n)^t = F^{m + n} F^t, \quad (F^m F^n)^t = F^m (F^t F^n) = F^{m + n} F^t.
\]

It was mentioned in section 2 that for any \( X \in \text{Top} \) \( F^n X \rightarrow F^n X \) is initial. Proposition 4.4 enables us to show it for any natural \( n \).

4.6 Corollary. For any \( X \in \text{Top} \) and for any positive integer \( n \), \( F^n X \rightarrow F^n rX \) is initial.

Proof: By the definition of \( F^n X \), \( F^n X \rightarrow F^n F^{n-1} X \) is initial. By 4.4, \( F^n F^{n-1} X \) is naturally isomorphic to \( F^{n-1} F^n X \).

Consider the commutative diagram

\[
\begin{array}{ccc}
F^{n-1} X & \xrightarrow{f} & rF^{n-1} X \\
r & & S_{n-1} \\
F^{n-1} X & \xleftarrow{r_1} & rF^{n-1} X \\
\end{array}
\]

where \( S_{n-1} \) is the natural isomorphism given in 4.4, \( r \) and \( r_1 \) are reflections.

Applying the functor \( F \) we get the commutative diagram

\[
\begin{array}{ccc}
F^n X & \xrightarrow{f} & F F^{n-1} X \\
r & & S_{n-1} \\
F^n X & \xleftarrow{r_1} & F F^{n-1} X \\
\end{array}
\]

(*) \( \Sigma \) is finite for all categories listed in 1.1 except may be \( \text{Top}_{2n} \) (See 4.2)
4.7 Remark. (a) The assertion of the above corollary is no valid for \( n=0 \) (see (4.12 (b)).

(b) We do not know whether 4.6 is true for infinite ordinals. A positive answer would imply the validity of the following corollary for arbitrary non-zero ordinals.

4.8 Corollary. Let \( n \) be a positive integer and \( X \in \text{Top} \) with \( rX \in A_0 \). Then \( E_0^A(X)=n \) iff \( E_0^A(rX)=n \).

Proof. By 4.6 \( E_0^A(X) \leq E_0^A(rX) \) since \( F_{n+1}^A rX = F_n^A rX \) would imply \( F_{n+1}^A X = F_n^A X \). Since \( X \rightarrow rX \) is surjective, different topologies on \( rX \) give rise to different initial topologies on \( X \), i.e., \( F_{n+1}^A X = F_n^A X \) would imply \( F_{n+1}^A rX = F_n^A rX \), thus \( E_0^A(rX) \leq E_0^A(X) \).

It may happen \( rX \in A_0 \), i.e., \( E_0^A(rX)=0 \) and \( E_0^A(X)=1 \) if \( X \rightarrow rX \) is not initial. The above corollary permits easier calculation of the epimorphic order.

4.9 Example. Let \( (Y_\rho, \mathcal{C}_\rho) \) denotes the Urishon space constructed for the ordinal \( \rho \) in \([15]\). If \( \rho > \omega +1 \) one can see that \( Z=F_{\text{Top}_2^\omega}(Y_\rho, \mathcal{C}_\rho) \) is not even Hausdorff. However for every \( \rho > \omega +1 \) the Hausdorff reflection of \( Z \) is already Urishon, i.e. \( r_{\text{Top}_2^\omega} Z=r_{\text{Top}_2^\omega} Z \). Moreover there exist a continuous bijection \( F_{\text{Top}_2^\omega}(Y_{\omega+1}, \mathcal{C}_{\omega+1}) \rightarrow rZ \) such that \( rZ \rightarrow F_{\text{Top}_2^\omega}(Y_{\omega+1}, \mathcal{C}_{\omega+1}) \) is continuous and not open.

Since \( E_0^\text{Top}_2^\omega(Y_{\omega+1}, \mathcal{C}_{\omega+1})=2 \) this implies \( E_0^\text{Top}_2^\omega(rZ)=1 \). By corollary 4.8 \( E_0^\text{Top}_2^\omega(Z)=1 \), so by the definitions of epimorphic order

\[ E_0^\text{Top}_2^\omega(Y_\rho, \mathcal{C}_\rho)=2 \] for \( \rho > \omega +1 \).

The above example justifies the following definition.

4.10 Definition. Let \( \beta \) be an ordinal number, denote by \( A^{(\beta)} \) the category of all spaces \( X \in A \) such that \( F_X(X) \in A \) for each \( \gamma \in \beta \).

Set \( A^{(\omega)}=\bigcap A^{(\beta)} \), i.e., \( A^{(\omega)} \) is the category of all spaces \( X \in A \) such that \( F_X(X) \in A \) for each \( \gamma \leq E_0(X) \).

For example in 4.9 \( Y_\rho \in \text{Top}_2^{(1)} \); for \( \rho \) as in 4.2 (b) \( A^{(\omega)}=A \).

4.11 Theorem. Let \( A \) be an epireflective subcategory of \( \text{Top} \), then:

(a) \( A_0 \) is bireflective in \( \text{Top} \).

(b) \( A\cap A_0 \) is bireflective in \( A \), thus \( A\cap A_0 \subseteq A^{(\omega)} \subseteq A \cap A_0 \).

(c) If \( A \) is extremally epireflective then, for each ordinal \( \rho \),
A (ρ) and A (ω) are extremally epireflective; in particular A (ω) = P(A ∩ A).

Proof. (a) For any X ∈ Top define r_A^o(X) = F_A^o(X), where λ = E_0 A.

Now for every Y ∈ A and any map f: X → Y applying F_A^c we get
f = F_A^c(f): F_A^c(X) → F_A^c(Y) = Y. Thus r_A^o is a bireflection of Top in A.

(b) Follows from (a).

(c) Let Y ∈ A (ρ), then F_P(Y) ∈ A. For any subspace X of Y applying to the embedding i:X → Y the functor F_P we get i = F_P(X) → F_P(Y).

Since A is extremally epireflective this implies F_P(X) ∈ A. For any family {X_i} of spaces in A (ρ), F_P(X_i) ∈ A, therefore F_P(Π_i X_i), having a topology finer than that of Π_i F_P(X_i), belongs to A. Therefore A (ρ) is extremally epireflective.

The rest is obvious.

4.12 Remark. Analogous theorem can be proved for categories A which satisfy r_A ∈ r_A for each (X, r) ∈ Top. In such a case A_0 is a coreflective subcategory of A and the coreflection is given by F^*(X) → X where r = E_0 A(X).

(b) In 4.9 Z → r Top_2 (Z) is not initial (this shows that in general FX → rFX is not initial).

(c) Since A_0 is a bireflective subcategory of Top, Top = P(A) holds. On the other hand always A_0 ≠ Top. In fact, assume A ⊆ A_0, then by 2.7 A ⊆ Top_2. Since X ∈ A_0 iff r_X ∈ A_0 and X → r_X is initial, it suffices to find X ∈ Top such that X → r_X is not initial. Now A ⊆ Top_2 provides the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{r_X} & A \\
\downarrow{r_{Top_2}} & & \downarrow{r_X} \\
Top_2 X & \xrightarrow{r_X} & A
\end{array}
\]

this is why a space X such that X → r_X is not initial, will do (take, for example, the space Z from (b)).

The following theorem characterizes the categories A satisfying A (ω) = A.

4.13 Theorem. For an extremally epireflective subcategory A of Top the following conditions are equivalent:

(a) there exists an epireflective subcategory B of Top such that B ⊆ B_0 and A = P(B).

(b) A (ω) = A.
Proof. \((a) \Rightarrow (b)\) is obvious since, for any \((X, \tau) \leq A, (X, \tau) B \leq \text{A}\). On the other hand \((b) \Rightarrow (a)\) follows from 4.11 with \(B = \text{A} \cap \text{A}_0\).

4.14 Remarks. (a) By 4.2 (b) both conditions in 4.13 imply \(\text{E}_0^\text{A}(X) \leq 1\) for any \(X \in \text{Top}\). We do not know whether the converse is also true. Observe that if \(\text{E}_0^\text{A}(X) \leq 1\) for every \(X \in \text{A}\), than by 4.8 \(\text{E}_0^\text{A}(X) \leq 1\) for every \(X \in \text{Top}\).

(b) In general for any extremally epireflective subcategory \(\text{A}\) of \(\text{Top}, \text{A}^\omega = \text{P} (\text{A} \cap \text{A}_0)\) according to 4.11 (c), thus for \(X \in \text{A}\), \(\text{E}_0^\text{A}(X) \leq 1\) iff \(X \notin \text{A}_0\). On the other hand it may happen \(\text{E}_0^\text{A}(X) \geq \text{E}_0^\text{A}(X)(\omega)\) (take for example \(X = \omega\) as in 4.2 (c); then for \(\text{A} = \text{Top}, \text{A} \cap \text{A}_0 = \text{Top}_3\), therefore \(\text{A}^\omega = \text{P} (\text{Top}_3)\) and \(X \in \text{A}\), \(\text{E}_0^\text{A}(X) = 2 \geq \text{E}_0^\text{A}(\omega) (X) = 1\)).

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