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Z_2 - GRADED EXTENSIONS OF THE ORTHOGONAL LIE ALGEBRAS

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We are going to classify all Z_2-graded extensions of so(m,n) Lie algebras, with gradind representation being the spinorial one. The non-trivial extensions for all signatures (m,n) do not exist neither in the category of Lie algebras nor in the category of Lie superalgebras[3]. Defining the spinorial extension to be an object of the category of Z_2-graded ε-Lie algebras we can give complete classification of the extensions of this canonical type.

I. The Lie algebra so(m,n), with structural relations

$$[M_{ab},M_{cd}] = 2(\eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac} + \eta_{ad}M_{bc})$$

can be realized in Clifford algebra C(m,n) of the orthogonal space E(m,n)[6], as the set of bivectors $\Lambda^2 E(m,n)$ with the commutator as Lie bracket. The spinor module S(m,n)[5] of this Lie algebra can be identified: with a minimal one-sided (say left-sided) ideal in even subalgebra $C^+(m,n)$ of C(m,n) algebra, when $(m,n)\neq(4,0)$, (2,2);

or with the direct sum $S(m,n)=S_+(m,n)\oplus S_-(m,n)$ of minimal ideals, on which $C^+(m,n)$ has faithfull (though reducible) representation, when (m,n)=(4,0),(2,2). These two last signatures are distinguished by non-simplicity of corresponding orthogonal Lie algebras, for which faithfull representations are reducible.

The representation of so(m,n) Lie algebra,

$$\Lambda^2 E(m,n) \ni \Sigma \to \tau_\Sigma \in \mathrm{End}(S(m,n))$$

where the map τ_Σ is defined according to

$$S(m,n) \ni \psi \to \tau_\Sigma(\psi) := \Sigma\psi \in S(m,n) \ ,$$

is called spinorial, and it is integrable to the fundamental representation of $\mathrm{Spin}^o(m,n)$ - the connected component of unity of Spin(m,n) group. Detailed discussion of the properties of S(m,n) modules and classification of so(m,n) invariant antiautomorphisms of $C^{+}(m,n)$ algebras can be found in Ref.5.

II. For the notion of spinorial extension to be defined we need the following
Definition 1. (of Z_2-graded ε-Lie algebra)[7]

The structure $(A, \langle\ ,\ \rangle)$, where

$$A = A_{(o)} \oplus A_{(1)}$$ is Z_2-graded vector space and

$$A \times A \ni (X,Y) \to \langle X,Y\rangle \in A$$

is bilinear map, consistent with Z_2-gradation i.e.

$$\langle A_{(o)},A_{(o)}\rangle \subset A_{(o)} \quad , \quad \langle A_{(o)},A_{(1)}\rangle \subset A_{(1)} \quad , \quad \langle A_{(1)},A_{(1)}\rangle \subset A_{(o)} \quad ,$$

and moreover for the elements homogeneous in sense of Z_2-gradation, the following identities are satisfied

$$\langle X,Y\rangle = -\varepsilon(X,Y)\langle Y,X\rangle$$

$$\langle X,\langle Y,Z\rangle\rangle = \langle\langle X,Y\rangle,Z\rangle + \varepsilon(X,Y)\langle Y,\langle X,Z\rangle\rangle$$

where $\varepsilon \equiv 1$ or $\varepsilon(X,Y)=(-1)^{\deg X \deg Y}$, is called Z_2-graded ε-Lie algebra.
Note that the category of Z_2-graded ε-Lie algebras contains two subcategories: Category of Lie algebras, which corresponds to the first choice of the function ε, and the category of Lie superalgebras, which corresponds to second choice of commutation factor.
We are now in a position to formulate the following natural
Definition 2. (of spinorial extension of so(m,n) Lie algebra)
The spinorial extension of the orthogonal Lie algebra so(m,n) will be called a triple $A(m,n)=(S(m,n),Q,1)$, where $S(m,n)=S_{(o)}(m,n) \oplus S_{(1)}(m,n)$ is central Z_2-graded ε-Lie algebra, and

1) $Q : S(m,n) \to S_{(1)}(m,n)$ is an isomorphism of vector spaces (S(m,n)-spinor module of so(m,n))
2) $1 : so(m,n) \to S_{(o)}(m,n)$ is a monomorphism of Lie algebra, such that

$$\forall \Sigma \in so(m,n)\ \forall \psi \in S(m,n) \quad \langle 1(\Sigma),Q(\psi)\rangle = Q(\Sigma\psi)$$

3) $S_{(o)}(m,n) = \langle S_{(1)}(m,n),\ S_{(1)}(m,n)\rangle$

The conditions 1) and 2) mean that $S(m,n)$ is in essence spinorial extension where-as 3) means that this extension is minimal one.

Definition 3. (of the category A(m,n) of spinorial extensions)

The category A(m,n) of spinorial extensions of so(m,n) Lie algebra is the category consisting of spinorial extensions $A(m,n) := (S(m,n),Q,1)$, with morphisms $\Phi: S(m,n) \to \tilde{S}(m,n)$ being the homomorphisms of Z_2-graded ε-Lie algebras, such that the following diagram

$$\begin{array}{ccccc} & \overset{1}{\nearrow} & S_{(0)}(m,n) \ni 1(\Sigma) : & S_{(1)}(m,n) \to & S_{(1)}(m,n) \\ so(m,n) & & \Phi\downarrow & \Phi\downarrow & \Phi\downarrow \\ & \underset{\tilde{1}}{\searrow} & \tilde{S}_{(0)}(m,n) \ni \tilde{1}(\Sigma) : & \tilde{S}_{(1)}(m,n) \to & \tilde{S}_{(1)}(m,n) \end{array}$$

is commutative.

The commutativity of above diagram, i.e.

$$\Phi\langle 1(\Sigma), Q(\psi)\rangle = (\tilde{1}(\Sigma), \Phi Q(\psi))$$

means that the structure of spinorial extension is preserved by morphisms of the category A(m,n). Almost immediately one can prove the following

LEMMA 1.

1° If $(m,n) \neq (2,2),(4,0)$ and the category A(m,n) is not empty, then it contains only simple objects.

2° The category A(2,2) and A(4,0) consists of the objects being the direct sums of two simple ideals.

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The properties of morphisms of A(m,n) and Lemma 1. implies the following

LEMMA 2.

Arbitrary morphism $\Phi \in Mor(m,n)$ is an equivalence.

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Since now A(m,n) will denote the quotient category with identified equivalent elements.

Using uniqueness of EndS(m,n) - valued, so(m,n) invariant bilinear maps from $S(m,n)\otimes S(m,n)$, which are ε-skew symmetric and for which generalized Jacobi identity is satisfied (def 1), we obtain

PROPOSITION 1.

The categories A(m,n) consist of at most two elements: one Lie algebra and one Lie superalgebra, and their content is following:

1° A(2,2) : ${}^2Osp(1|2;R)$,

A(4,0) : ${}^2sp(2)$, ${}^2\alpha uu(1|1;H)$

3° Euclidean serie A(m,0)

$so(2l+1;R)$ $m=0,1,7 \pmod 8$

$\left.\begin{array}{l} su(2l+1) \\ su_{\alpha}u(1|2l;C) \end{array}\right\}$; m=2,6 (mod 8)

$\left.\begin{array}{l} sp(l+1) \\ \alpha uu(1|l;H) \end{array}\right\}$; m=3,4,5 (mod 8)

4° Real series

a) m-n=0 (mod 8)

so(l+1,l) ; m+n=0 (mod 8)

Osp(1|2l;R) ; m+n=4 (mod 8)

b) m-n=1,7 (mod 8)

so(l+1,l) ; m+n=1,7 (mod 8)

Osp(1|2l;R) ; m+n=3,5 (mod 8)

5° Complex analytic serie

m-n=2,6 (mod 8)

Osp(1|2l;$\mathbb{C}$) ; m+n=4 (mod 8)

so(2l+1;$\mathbb{C}$) ; m+n=0 (mod 8)

6° Unitary serie

m-n=2,6 (mod 8)

$\left.\begin{array}{l} su(l+1,l) \\ su_{\alpha}u(1|2l;\mathbb{C}) \end{array}\right\}$; m+n=2,6 (mod 8)

7° Quaternionic series

a) m+n=4 (mod 8)

$\left.\begin{array}{l} so^{*}(2l+2) \\ u_{\alpha}u(1|l;H) \end{array}\right\}$; m+n=0 (mod 8)

$\left.\begin{array}{l} sp(l/2+1,l/2) \\ \alpha uu(1|l/2,l/2;H) \end{array}\right\}$; m+n=4 (mod 8)

b) m+n=3,5 (mod 8)

$\left.\begin{array}{l} so^{*}(2l+2) \\ u_{\alpha}u(1|l;H) \end{array}\right\}$; m+n=1,7 (mod 8)

$$\left.\begin{array}{l} sp(1/2+1,1/2) \\ \alpha uu(1|1/2,1/2;H) \end{array}\right\} \; ; \; m+n=3,5 \pmod 8$$

where $2l=2^s$ with $s=[m+n/_2]-1$.
The notation used above is that of Refs. 3,4.

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To obtain non-trivial extension in the case 1^o it is necessary to enlarge $S_{(1)}(m,n)$ to the direct sum of two irreducible spinor modules, and then resulting simple Z_2-graded ε-Lie algebra is of the class sl(1;2l;F) where F=R or H.
Above classification has probably great significance for theoretical physics, particularly in better understanding of spin-statistics connection and the theorems of Refs. 1,2, the results of which are extrapolated to, recently developing, field theories in orthogonal spaces, different than E(3,1)- Minkowski space-time. The uniqueness of the extension of so(3,1) Lie algebra is due to complex analyticity of spin(3,1) group, and this fact is also crucial one for mentioned theorems. Our results indicate an impossibility of the extrapolation of mentioned type.

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