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THE TOPOLOGY OF THE YANG-MILLS THEORY OVER TORUS

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ABSTRACT

Some results on the topology of the Yang-Mills theory over torus are summarized. The interpretation of t Hooft electric and magnetic fluxes is given.

INTRODUCTION

One of the method of field quantization is the Shrodinger approach (Hamiltonian formulation,Q-space...) very popular at early stages of Constructive quantum field theory (see for example [1]). It is the most effective method for the bound-state problems. [2],[3],[4]. In this approach one starts with the classical configuration space $\mathfrak{M}$ and next consider the Hilbert space of states as $L^2(\mathfrak{M},d\mu)$ with a suitably defined measure μ. The quantization program is then the construction of Hamiltonian and quantum field operators in $L^2(\mathfrak{M},d\mu)$ or rather their regularized (cut-off) versions. Two kinds of regularizations are necessary-volume and ultraviolet ones. Finally one should remove all cut-offs using the renormalisation procedure i.e. redefining the bare parameters.

Only some steps in this direction was made with mathematical rigorous in continuum (i.e. not on lattice) for the Yang-Mills field theory [5],[6],[7],[8]. This is because the proper definition of the (regularized) measure μ is still lacking. However the configuration space $\mathfrak{M}$ for pure gauge theory was studied with the great geometrical details [9],[10],[11]. The formal (i.e. acting on twice differentiable functions on $\mathfrak{M}$) regularized Hamiltonian was constructed [6],[7]. It was also suggested that well-defined, gauge invariant Wilson loops, may be used as field operators at least in the axiomatic framework [12].

This notes are expanded and modified version of lectures at 12th Winter School on Astract Analysis in Srni. We briefly summarized the mathematically rigorous results on Schrodinger representation of

This paper is in final form and no version of it will be submitted for publication elsewhere.

Yang-Mills theory. The original contribution is the discussion of topological aspects (like Θ-vacua) of the theory in this framework. Our main attempt is to relate some of the ideas of [13],[14],[15] with the well-known mathematical constructions in order to have a geometrical background for the (static) quark confinement considerations.

The paper is organized as follows:

Section 1. The geometry of the configuration space

2. The topology of $\mathfrak{M}$.

3. The classification of the principal bundles over some low dimmensional manifolds.

Finally, let us mention that some problems closely related to discuss in this paper, in particular

-Frohlich-Duurhus theorem

-'t Hooft and Wilson loops

-Hamiltonian in the presence of electric fluxes,

are omitted in order to describe rather the topological structure of Yang-Mills field theory.

I

A volume cut-off is introduced in these notes by working with periodic boundary conditions. Thus we take the space-like surface M to be a torus T^3. G stands for a compact, connected Lie group. Our main choice is $G = SU(N)/Z(N)$.

Gauge potentials are, by definition, connections in principal bundle $P(M,G)$ and gauge transformations are automorphisms of this bundle [16],[17].

The true configuration space for the Yang-Mills theory is then the orbit space for the action of authomorphism on connections.

To be more specific, let $\mathcal{A}$ denotes the space of C^∞ connections and $\mathcal{G}$ the group of C^∞ gauges.

For technical purposes we consider rather the subgroup $\mathcal{G}_o$ consisting of gauges fixing a given point of P. Because $\mathcal{G}_o$ is the group of authomorfisms of P, $\mathcal{G}_o$ acts on A. The first fact is the following [9]:

<u>Proposition</u>: The orbit space $\mathfrak{M} = \mathcal{A}/\mathcal{G}_o$ is a (Frechet) manifold. This is because the action of $\mathcal{G}_o$ on $\mathcal{A}$ is free and proper.

$\mathfrak{M}$ posses the additional hierarchy of (apriori stronger than natural C^∞-one) topologies and differentiable structures. They are obtained by considering all objects (i.e. connections and gauges) of Sobolev class H^k $(k > 3)$.

The space $\mathcal{A}^k$ of H^k connections is an affine space modeled on a Hilbert space. The group $\mathcal{G}_o^k$ is a Hilbert Lie group. However it can be proved that the action of $\mathcal{G}_o^{k+1}$ on $\mathcal{A}^k$ is C^∞, proper and free. Thus, we have [10],[11]:

Proposition: The orbit space $\mathfrak{M}^k = \mathcal{A}^k/\mathcal{G}_o^{k+1}$ is a C^∞-Hilbert manifold.

The identification of $\mathfrak{M}^k$ and $\mathfrak{M}$ is based on the following two observations

- In each $\mathcal{G}_o^{k+1}$ orbit in $\mathcal{A}^k$ there exists a C^∞ connection
- If two C^∞-connections are $\mathcal{G}_o^{k+1}$-equivalent, then, they are $\mathcal{G}_o$-equivalent.

One can use this additional Sobolev structures to the ultraviolet regularization [5],[6],[7]. Even though the construction of the configuration space does not depend on the whether or not the bundle $P(M,G)$ is trivial, the nontriviality of $P(M,G)$ is of the physica relevance.

The classification of principal bundles over T^3 can be made "by hand" (see section 3). The result is that they are labeled by elements of $\Pi_1(G) \times \Pi_1(G) \times \Pi_1(G)$. To see the physical signíficence of these numbers consider $G = U(1)$. It is the case of electrodynamics and a connection A is then a magnetic potential, and its curvature $F(A)$ is a magnetic field.

Let us compute magnetic fluxes flowing in x,y,z-direction i.e. integrals $\int_{T^2_x,T^2_y,T^2_2} F(A)$ where $T^2_{x,y,z}$ are two-dimensional toruses in T^3 "orthogonal" to the corresponding directions. The result is exactly the number of $\Pi_1(G) \times \Pi_1(G) \times \Pi_1(G) = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ which clasyfies the bundle from which magnetic potential A was taken [18]. This follows from the fact that bundles $P(T^3,G)$ are determined by their restrictions to T^2_x,T^2_y,T^2_2 (theorem 3) and from properties of Chern classes [18]. Basing on this analogy we interprete according to t Hooft [14] the numbers $(m_x,m_y,m_z) \in \Pi_1(SU(N)/Z(N))\times\ldots =$ $= Z(N) \times \mathbb{Z}(N) \times \mathbb{Z}(N)$ for non abelian Yang-Mills theory as magnetic fluxes running in x,y,z directions. The corresponding configuration spaces $\mathfrak{M}^{m_x,m_y,m_z}$ are called to carry the m_x,m_y,m_z units of magnetic fluxes. Using local trivialisation one can establish the relation beetwen nontriviality of $P(T^3,G)$ and t Hooft "twist" boundary conditions more directly, see [19],[20].

Finally, we remark that the full quotient space $\mathcal{A}^k/\mathcal{G}^{k+1}$ has a structure of stratification onto Hilbert manifolds [11].

Now we will study the topology of the configuration space $\mathfrak{M}$. The reason is that the multiply - connectivity of $\mathfrak{M}$ is the source of the so-called "Θ-vacua" [20],[21]. The argumentation is going as follows [22].

In the Schrodinger approach one is looking for the Hilbert space of states to be the space of wave functions (or rather functionals) i.e. the space of complex functions on the configuration space $\mathfrak{M}$. In the case when $\pi_1(\mathfrak{M})$ is non-trivial one finds them to be multi-valued. Multivaluedness is choosen to be undistinguishable by any observable. This leads to the requirement that multivalued wave functions have to satisfy certain properties. The most easy way to describe them is to use the universal covering space $\hat{\mathfrak{M}}$ of $\mathfrak{M}$. The wave function $\hat{\psi}$ written on $\hat{\mathfrak{M}}$ is singlevalued and satisfies

$$\hat{\psi}(\hat{m}[\omega]) = \Theta([\omega])\hat{\psi}(\hat{m})$$

$\hat{m} \in \hat{\mathfrak{M}}$, $[\omega] \in \pi_1(\mathfrak{M})$, $\hat{m}[\omega]$ is the result of the action of $[\omega]$ on $\hat{m}$. Here Θ is a certain character of $\pi_1(\mathfrak{M})$. $\hat{\psi}$ defines multivalued wave function on $\mathfrak{M}$ by the formula

$$\psi(m) = \bigcup_{\hat{m} \in \pi^{-1}(m)} \hat{\psi}(\hat{m}) \quad \text{where } \pi : \hat{\mathfrak{M}} \to \mathfrak{M} \text{ is a projection.}$$

Characters of $\pi_1(M)$ labels different classes of wave functions and thus lead to the "superselection sectors" of the theory [23]. The consequence is that one must work in each superselection sector separetly. It was suggested that for Q.C.D, Θ is near to identity, otherwise e.g. P and T-symetries should be break [5].

One can return to the singlevalued wave function by suitable redefinition of the momentum operator and consequently, the change of Hamiltonian. This serves the possibility of Θ being measured. To compute the fundamental group of $\mathfrak{M}$ we use the exact sequence for principal fibrations and a fact that $\pi_n(\mathcal{A}) = 0$. [9] Therefore

$$\pi_n(\mathfrak{M}) = \pi_{n-1}(\mathcal{G}_o)$$

and in particular $\pi_1(\mathfrak{M}) = \pi_o(\mathcal{G}_o)$.

Moreover $\pi_o(\mathcal{G}_o) = \pi_o(\mathcal{G})$ = the set of homotopy classes of authomorphisms of $P(M,G)$. (lemma 1)

The computation of $\Pi_o(\mathcal{G})$ is given in Theorem 4

$$\Pi_o(\mathcal{G}) = \Pi_3(G) \times \Pi_1(G) \times \Pi_1(G) \times \Pi_1(G)$$

In our case $G = SU(N)/Z(N)$ so

$$\Pi_1(\mathfrak{M}) = \mathbf{Z} \times \mathbf{Z}(N) \times \mathbf{Z}(N) \times \mathbf{Z}(N).$$

The result is independent of the choice of bundle. The superselection sectors for $SU(N)/Z(N)$ Yang-Mills theory over torus T^3 are then labeled by characters of $Z \times Z(N) \times Z(N) \times Z(N)$ i.e. by elements (Θ, e_x, e_y, e_z) of $S^1 \times Z(N) \times Z(N) \times Z(N)$. Let $\mathcal{G}_\infty$ denotes the component of unity of $\mathcal{G}_o$. Write down a wave function ψ on $\mathcal{A}/\mathcal{G}_\infty = \hat{\mathfrak{M}}$ which is the universal covering of $\mathfrak{M}$.
Then ψ has to satisfies

$$\omega[k_x, k_y, k_z, \nu)\psi = e^{\frac{2\Pi i}{N}(\vec{e}\cdot\vec{K}) + i\Theta\nu}\,\psi$$

where $\omega[k_x, k_y, k_z, \nu] \in \Pi_1(\mathfrak{M}) = \Pi_o(\mathcal{G}) \equiv Z(N) \times Z(N) \times Z(N) \times Z$.
't Hooft, basing on the formulas of the above type gave the physical interpretation for the numbers (Θ, e_x, e_y, e_z). Firstly e_x, e_y, e_z are indicators for the electric fluxes running in x,y,z-directions. The angle Θ indexes the original Θ-vacua of [24],[25] because it origin is the bundle over S^3.

III

This section is devoted to the classification of bundles and authomorphisms on these bundles over T^3 because of it importance in the discussed matter. We find it useful for physicist to give the details of the proofs. We use only the elementary topological notions.

Definition 1

Let $(P_1, \Pi_1, G, M), (P_2, \Pi_2, G, M)$ denote two principal fibre bundles over the manifold M with the structure Lie group G.
The diffeomorphism $\phi: P_1 \to P_2$ will be called an isomorphism of bundles if

1° $\Pi_2\,\phi = \Pi_1$
2° $\bigwedge_{p \in P_1} \bigwedge_{g \in G} \phi(p\,g) = \phi(p)\,g$

If $P_1 = P_2$ the isomorphism ϕ will be called an automorphism or a gauge. The set of all automorphisms ϕ of the principal bundle (P,Π,M,G) forms a group named the gauge group $\mathcal{G}(P,M,G)$. It is necessery to understand gauges also in another way i.e. as sections of certain associated bundle. The typical fibre of this bundle is G. The representation r of G in G is given by the formula:

$$\bigwedge_{g}\bigwedge_{h\in G} r_g(h) = g^{-1}hg$$

The associated (with respect to r) bundle has the following properties:

- sections of this bundle can be identified with gauges
- this bundle is trivial if (P,Π,M,G) is trivial, moreover for given trivialization of (P,Π,G,M) gauges can be represented as maps : $M \to G$

THEOREM 1.
Let (P_1,Π_1,G,S^n), (P_2,Π_2,G,S^n) denote two principal bundles over n-dimensional sphere S^n. Then

$$\Pi_0(\mathcal{G}(P_1,S^n,G)) = \Pi_0(\mathcal{G}(P_2,S^n,G) = \Pi_n(G)$$

where $\Pi_0(\mathcal{G})$ denotes the group of homotopy classes of $\mathcal{G}$
Proof of thm 1.
It is enough to prove that $\Pi_0(\mathcal{G}(P,S^n,G)) = \Pi_0(\mathcal{G}(S^n \times G,S^n,G))$
Let $E \subset S^{n-1}$ denotes the equator of S^n. The equator E devides S^n into two closed subsets N(north) and S(south)

$$N \cup S \cup E = S^n$$

$$N \cap S = E$$

Let us consider two isomorphism of bundles i_N, i_S:

$$i_N : P|_N \to N \times G$$

$$i_S : P|_S \to S \times G$$

Let $\mathcal{G}_{id}(P,S^n,G)$ denotes the group of automorphisms of (P,S^n,G) which are identity on E. The isomorphisms $i_{N,S}$ will be used to construct the isomorphism i,

$i: \mathcal{G}_{id}(P,S^n,G) \to \mathcal{G}_{id}(S^n \times G, S^n, G)$

Let $\phi \in \mathcal{G}_{id}(P,S^n,G)$. If $p \in N \times G$ we put

$i\phi(p) := i_N \ \phi i_N^{-1}(P)$

and if $p \in S \times G$

$i\phi(p) := i_s \phi \ i_s^{-1}(p)$

Let us notice that i is equal to identity on N, so $i\phi$ is a continuous automorphism of $(S^n \times G, S^n, G)$. Now, it is enough to prove that $\Pi_o(\mathcal{G}_{id}(P,S^n,G)) = \Pi_o(\mathcal{G}(P,S^n,G))$. We expect that this fact is obvious for algebraic topologists but in order to make this prove clear and self-contained we will give the details

Proposition 1

If $\phi_1, \phi_2 \in \mathcal{G}_{id}(P,S^n,G)$ are homotopically equivalent in $\mathcal{G}(P,S^n,G)$ then, they are homotopically equivalent in $\mathcal{G}_{id}$ $P,S^n,G)$, that means, that there exists a homotopy ψ_t such that

$\psi_o = \psi_1, \quad \psi_1 = \psi_2$

$\psi_t|_E = id$

Proposition 2

For every $\phi \in$ (P,S^n,G) there exists a homotopy ψ_t, such that

$\psi_o = \phi, \ \phi_1 \in \mathcal{G}_{id}(P,S^n,G)$

The statements of prop.1.2 mean exactly that $\Pi_o(\mathcal{G}_{id}(P,S^n,G) = \Pi_o(\mathcal{G}(P,S^n,G))$. So, to complete the proof of thm.1, it suffices to prove prop.1.2. It needs a lemma

Lemma 1

Let

$\phi_1 : [0,\varepsilon] \times N \to G$

$\phi_2 : \{0\} \times N \to G$ where N is a manifold

and moreover $\phi_1|\{0\} \times N$ is homotopically equivalent to ϕ_2. (we denote this homotopy by η).

Then,there exists a homotopy

$\phi(t,s,x):[0,1]\times[0,\varepsilon]\times N\to G$ such that

$\phi(0,S,X) = \phi_1(S,X)$

$\phi(1,0,X) = \phi_2(0,X)$

$\phi(1,\varepsilon,X) = \phi_1(\varepsilon,X)$

This homotopy is given by the formula

$\phi(t,s,x) = \phi_2(0,X)\eta^{-1}(1 - t\,\frac{\varepsilon-s}{\varepsilon},\ x)\ \phi_1(S,X)$ ∎

Proof of Proposition 1

It is easily seen that using lemma 1 one can reduce the problem to the following. Let ϕ_t be a homotopy joining ϕ_o and ϕ_1 and ϕ_1,ϕ_2 are equal to the identity in the given point x. Then there exists a homotopy ψ_t such that

$\psi_o = \phi_o\ ,\quad \psi_1 = \phi_1$

$\psi_t(x) = id$

This homotopy is given by the formula:

$\psi_t(\cdot) = \phi_t(\cdot)\phi_t^{-1}(x)$ ∎

Proof of proposition 2

It is an immidiate consequence of lemma 1 ∎ ∎

THEOREM 2.

Classes of isomorphic principal bundles over T^3 with compact, connected structure Lie group G are labeled by elements of $\Pi_1(G)\times\Pi_1(G)\times\Pi_1(G)$

Proof.

The proof is based on the following theorem [26] of Steenrod. The principal bundles over $M\times S^1$ are labeled by principal bundles over M and homotopy classes of automorphisms of those bundles. This theorem reduces our problem to the study of bundles over T^2.

Lemma 2.

Principal bundles over T^2 are labeled by the elements of $(\Pi_1(G)$.

Proof of lemma 2

Since all principal bundles with the connected structure group G over S^1 are trivial and in this case automorphisms are simply maps $S^1 \to G$, then the statement follows from Steenrod theorem. ∎

Let T_1, T_2 denote the following subsets of $T^2 = S^1 \times S^1$

$$T_1 = \{p\} \times S^1$$

$$T_2 = S^1 \times \{p\} \quad \text{for certain } p$$

Proposition 3

Let ϕ denotes an automorphism of $(\Pi^{-1}(T_1), \Pi, T_1, G)$ such that $\phi(\{p\}) = \mathrm{id}$. Then, there exists an automorphism $\hat{\phi} \in \mathcal{C}(P, \Pi, T_2, G)$ satysfying:

1° $\hat{\phi}|_{T_1} = \phi$

2° $\hat{\phi}|_{T_2} = \mathrm{id}.$

Proof of prop.3

We construct $\hat{\phi}$ by extension of ϕ to the set $(S^1 - \{p'\}) \times S^1$ where $p' \neq p$ and next by patching it around $\{p'\} \times S^1$ using, if it is necessery, a suitable choosen homotopy (lemma 1) ∎

We can obtain, as a corollary the following fact. Let ϕ_1 be an automorphism of $(\Pi^{-1}(T_1), \Pi, T_1, G)$; $\phi_1(p) = \mathrm{id}$ and let ϕ_2 be an authomorphism of $(\Pi^1(T_2), \Pi, T_2, G)$; $\phi_2(\phi) = \mathrm{id}$.

Then we can construct an authomorphism ϕ of (P, Π, T^2, G) such that

$$\phi|_{T_1} = \phi_1$$

$$\phi|_{T_2} = \phi_2$$

by putting $\phi = \phi_1 \phi_2$.

We have proved that the number of homotopy classes of the authomorphisms of (P, Π, T^2, G) is not less the $\Pi_1(G) \times \Pi_1(G)$. Moreover we have the natural homomorphism h

$$h: \begin{pmatrix} \text{The group of homotopy classes} \\ \text{of authomorphisms of } (P,\Pi,T_1^2,G) \end{pmatrix} \to \begin{pmatrix} \text{The group of homotopy classes} \\ \text{of authomorphisms of} \\ (\Pi^{-1}(T_1),\Pi,T_1,G) \end{pmatrix}$$

$$\times \begin{pmatrix} \text{The group of homotopy classes} \\ \text{of authomorphisms of } (\Pi^{-1}(T_2), \Pi, T_2, G) \end{pmatrix}$$

defined by the restriction.

To complete the proof it is enough to show that h is an isomorphism. In other words if ϕ is homotopically equivalent to the identity on T_1 and T_2 then it is homotopically equivalent to the the identity on T^2.

Let s be a section over $T_1 \cup T_2$, that means

$$s: T_1 \cup T_2 \to P$$
$$\Pi \cdot s = \mathrm{id}|_{T_1 \cup T_2}.$$

We will define an equivalence relation on P by the formula:

$$p \sim p' \Leftrightarrow p, p \in \Pi^{-1}(T_1 \cup T_2) \text{ and}$$
$$p/s(\Pi(p)) = p'/s(\Pi(p'))$$

where $p/s(\Pi(p))$ denotes an element of G such that

$$p = s(\Pi(p))\ p/s(\Pi(p))$$

The quotient space $P/\sim$ is also a G-space and $P/\sim/G$ is the two dimensional sphere S^2. The map $\eta: P \to P/\sim$ given by the above relation commutes with the action of the group G.

We will introduce also an analogical equivalence relation on T^2.

$$m \sim m' \Leftrightarrow m, m' \in T_1 \cup T_2$$

In this way we have the map $\eta': T^2 \to S^2$ because $T^2/\sim = S^2$.

<u>Proposition 4</u>

The following diagram commutes

$$\begin{array}{ccc} P & \xrightarrow{Z} & P/\sim \\ \Pi \downarrow & & \downarrow \Pi' \\ T^2 & \xrightarrow{Z'} & S^2 \end{array}$$

i.e. the pair (η, η') is a morphism of bundles.

∎

The authomorphisms of $(P/\sim, \Pi', S^2, G)$ may be transformed to the automorphisms of (P,Π,T^2,G) by the formulas:

a) $\phi(p) \to \eta^{-1}\phi\eta(p)$ for $\Pi(p) \notin T_1 \cup T_2$
b) $\phi(p) \to p\cdot\phi(\eta(p))$ for $\Pi(p) \in T_1 \cup T_2$

We have used the following meaning of ϕ

$\phi: P/\sim \to G$
$\phi(p\,g) = g^{-1}\phi(p)\,g$ (equivariant G-function on $P/\sim$).

This gives a map $\mathcal{G}((P/\sim,S^2,G)) \to \mathcal{G}((P,T^2,G))$.
The image of this map consists of gauges which are "constant" on $T_1 \cup T_2$. But all homotopically trivial automorphisms on $T_1 \cup T_2$ can be homotopically transform to this form (lemma 1). It means that we have an isomorphism

$$\Pi_0\mathcal{G}(P/\sim,S^2,G) \to \text{Ker } h$$

By virtue of thm 1 we have

$$\Pi_0\mathcal{G}(P/\sim,S^2,G) = \Pi_2(G)$$

For any compact Lie group G, $\Pi_2(G) = 0$ [26] so $\ker h = 0$. ∎
In three dimensional torus T^3 one can distinguish two dimensional toruses T^2_x, T^2_y, T^2_z in natural way. We will also use the abbreviation $T^1_z = T^2_x \cap T^2_y$; $T^1_y = T^2_x \cap T^2_z$; $T^1_x = T^2_y \cap T^2_z$.

THEOREM 3

Each fibre bundle over T^3 is uniquely determined by its restrictions to T^2_x, T^2_y, T^2_z.

Proof

By virtue of Steenrod theorem principal bundles over T^3 are labeled by pairs: (a bundle over T^2_x, a homotopy class of a gauge in T^2_x). According to the theorem 2 the homotopy class of an automorphism is determined by: (a homotopy class of the restriction $\phi|_{T^1_z}$; a homotopy class of a restriction $\phi|_{T^1_y}$). So, finally bundles over T^3 are determined by triplets: (a bundle over T^2_x, a homotopy class of $\phi|_{T^1_x}$, a homotopy class of $\phi|_{T^1_y}$). Let us notice that $T^2_z = T^1_y \times S^1 = (T^1_y \times T^1_x)$. So, if we have a bundle over T^2_x and an automorphism in it with a given homotopy class on T^1_y, we can construct a bundle over T^2_z simply by Steenrod theorem. Analogically ~~ϕ~~ we obtain a bundle over T^2_y. Thus all bundles over T^3 are labeled by (a bundle over T^2_x, a bundle over T^2_y, a bundle over T^2_z). ∎

<u>THEOREM 4</u>

The set of homotopy classes of automorphisms of (P,Π,T^3,G) is equal to $\Pi_1(G) \times \Pi_1(G) \times \Pi_1(G) \times \Pi_3(G)$.

<u>Proof</u>

Let h denotes a homomorphism

$$h: \Pi_o(\mathcal{G}(P,T^3,G) \to \Pi_o(\mathcal{G}(\Pi^{-1}(T^1_x),T^1_x,G)) \times \Pi_o(\mathcal{G}(\Pi^{-1}(T^1_y),T^1_y,G)) \times$$
$$\times \Pi_o(\mathcal{G}(\Pi^{-1}(T^1_z),T^1_z,G))$$

defined by restrictions. Following the idea of corrolary of proof of thm 2. we conclude that h is onto.

Now, we will investigate the kernel of h. Let ϕ be the element of $\mathcal{G}(P,T^3,G)$ which is homotopically trivial on T^1_x, T^1_y, T^1_z. Then, it is homotopically trivial on T^2_x, T^2_y, T^2_z (see thm 2), and we may put

$$\phi|_{T^2_x} = id_{T^2_x} \; ; \quad \phi|_{T^2_y} = id_{T^2_y} \; ; \quad \phi|_{T^2_z} = id_{T^2_z} \; .$$

We denote the set of such automorphisms by $\mathcal{G}_{id}(P,T^3,G)$. Let us notice that the following fact holds (and it is trivial at this step).

<u>Proposition 5</u>

Let ϕ_o and ϕ_1 are automorphisms of (P,Π,T^3,G) such that

- they are homotopically equivalent
- $\phi_o|_V = \phi_1|_V = id_V$

$$V = T^2_x \cup T^2_y \cup T^2_z$$

Then there exists a homotopy ϕ_t such that

$$\phi_o = \phi_o \; ; \quad \phi_1 = \phi_1$$
$$\phi_t|_V = id$$

This Proposition implies, that $\ker h = \Pi_o(\mathcal{G}_{id}(P,T^3,G))$ and, that

$\Pi_o(\mathcal{G}(P,T^3,G)) = \Pi_o(\mathcal{G}_{id}(P,T^3,G)) \times \Pi_o(\mathcal{G}(\Pi^{-1}(T_x^1),T_x^1,G) \times \ldots =$
$= \Pi_o(\mathcal{G}_{id}(P,T^3,G)) \times \Pi_1(G) \times \Pi_1(G) \times \Pi_1(G).$

To complete the proof we need to compute $\Pi_o(G_{id}(P,T^3,G))$. We will construct an isomorphism

$i: \mathcal{G}\,id(P,T^3,G) \to \mathcal{G}\,id(S^3 \times G,S^3,G)$

where $\mathcal{G}\,id\ (S^3 \times G,S^3,G)$ denotes the set of automorphisms of $(S^3 \times G,S^3,G)$ which are identity at a given point $a \in S^3$.
Let s_1 denotes a section over $T^3 - V$

$s_1(T^3 - V) \to P; \quad \Pi s_1 = \Pi$

and let s_2 denotes a section of the bundle $(S_3 \times G,S^3,G)$.
We will define an equivalence relation in T^3 by the following formula

$T^3 \ni m_1 \sim m_2 \in T^3 \quad \Leftrightarrow \quad m_1,m_2 \in V$

Then $T^3/\sim = S^3$ and $[V] = a$. Finally let $\eta: T^3 \to T^3/\sim = S^3$ be the natural projection.
The isomorphism i has the following form:

$\phi \in \mathcal{G}\,id(P,T^3,G), \ p \in S^3 \times G$

$i\phi$ acts on p as follows:

$p\cdot(i\phi) = s_2(\Pi(p)) \cdot \phi s_1\eta^{-1}(P)/s_1\eta^{-1}(\Pi(p) \cdot P/s_2(\Pi(p)).$

But we have $\Pi_o(\mathcal{G}\,id(S^3 \times G,S^3,G)) =$ (prop 1) $\Pi_o(\mathcal{G}(S^3 \times G,S^3,G)) =$
$= \Pi_3(G).$

∎

The table presents the classification of principal fibre bundles with connected, Compact structure group over some low dimensional manifold

Base space	Number of bundles
S^1	only trivial
S^2	$\Pi_1(G)$
T^2	$\Pi_1(G)$

S^3	only trivial
$S^2 \times S^1$	$\Pi_1(G)$
T^3	$\Pi_1(G) \times \Pi_1(G) \times \Pi_1(G)$
S^4	$\Pi_3(G)$
T^4	$\Pi_3(G) \times \underbrace{\Pi_1(G) \times \dots \times \Pi_1(G)}_{\text{6-times}}$

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