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SELF-DUAL MAGNETIC MONOPOLES AND GENERALIZATIONS OF HOLOMORPHIC FUNCTIONS

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Among the functions of two variables the holomorphic ones have a special importance both in mathematics and in physics. It is less evident, how to generalize the Cauchy-Riemann equations to functions of more variables, not because such generalizations are difficult to invent, but because there are so many possibilities. However, we shall see that for four and probably for six variables the most obvious generalizations together form a tight and unique structure and that a reduction of the four-dimensional case to three variables yields nice results, too. At first we shall consider the local forms of the equations, later the consequences of global constraints.

In 2n-dimensional oriented Riemannian manifolds M one may introduce local complex structures in the tangent spaces TM_x of points $x \in M$. If these structures can be integrated to a global complex structure J, holomorphic functions $f: M \to C$ can be defined by

$$J\,df = i\,df. \tag{1}$$

However, J is not unique for $n>1$. Thus one is led to introduce the fibre bundle E over M which has as fibre over x the complex structures in TM_x. This bundle has $n(n+1)/2$ natural complex coordinates. Instead of functions on M one may now consider meromorphic functions in E. At first this seems to introduce unwanted new degrees of freedom, but this is not really the case, as the fibres are compact and support only restricted classes of meromorphic functions.

For $n=1$ the unique J is given by the Hodge $*$ operation, such that one obtains the Cauchy-Riemann equations

$$*df = i\,df. \tag{2}$$

This suggests the alternative generalization

$$* d\omega = i^n d\omega \tag{3}$$

where ω now is an $(n-1)$-form. Eq. (3) is called the self-duality equation for $d\omega$. We shall pay little attention to the distinction between self-duality and anti-self-duality, as $*$ changes sign for reversed orientation.

We see that natural holomorphic maps from M to C do not exist, but that either M or C has to be enlarged to a space of more dimensions. This also applies to the generalizations discussed below.

Eq. (2) implies

$$\Delta f = 0 \tag{4}$$

where

$$\Delta = d*d* - *d*d \tag{5}$$

is the Laplace-Beltrami operator. Thus another generalization of eq. (2) is obtained from the search for a linear equation the solutions of which belong to the kernel of Δ . This is basically the way the Dirac equation was discovered. At least locally one may associate a spin bundle S to the tangent bundle TM and write for sections ψ of S

$$q^\mu \partial_\mu \psi = 0. \tag{6}$$

The fibres of S are $2^{(n-1)}$-dimensional and the q_μ are matrices acting on the fibres and satisfying

$$q_\mu q_\nu^+ + q_\nu q_\mu^+ = 2g_{\mu\nu}, \tag{7}$$

where g is the metric. For conformally flat spaces eq. (6) implies the Laplace eq. (4) for the components of ψ .

Actually eq. (6) is the Weyl equation, which also was used by Fueter as a generalization of the Cauchy-Riemann equations. Indeed, for $n=1$ the q_μ are numbers and may be normalized to $(1,i)$. The Dirac equation is written with

$$\gamma_r = \begin{pmatrix} 0 & q_r \\ q_r^+ & 0 \end{pmatrix} \tag{8}$$

and also may include a mass term.

If the metric is not conformally flat, the Weyl equation does not imply the Laplace equation. Instead one may use the factorization

$$\Delta = (d* + *d)(d* - *d) \tag{9}$$

leading to the Kähler equation

$$(d* - *d)\,\omega = 0. \tag{10}$$

Here the differential form ω has components of various degrees, which may, however, be restricted to be even or odd, writing

$$(-)^K \omega = \pm\omega, \tag{11}$$

where K is the operator which gives the degree of homogeneous forms. Once such a restriction has been imposed, one has

$$i^K * (d* - *d) = \pm (-)^n i\,(d* - *d)\, i^K * \tag{12}$$

and one may impose in addition

$$i^K * \omega = \varepsilon\,\omega \tag{13}$$

where the constant ε has to satisfy

$$\varepsilon^2 = \pm(-)^n. \tag{14}$$

With these restrictions, ω has 4^{n-1} components. Finally for even n these components may be required to be real.

Restricting ω to a homogeneous form of degree n, eq. (13) means that it is self-dual and eq. (10) that it is closed. Thus eq. (3) may be regarded as a special case of the Kähler equation.

On conformally flat spaces, the Kähler eq. (10) reduces to 2^{n+1} Weyl equations, as one sees easily by transforming conformally to flat space and using the translationally invariant forms as a basis. The restrictions of eq. (11) and (13) reduce this number to

2^{n-1}. If in addition one uses real ω, one sees that for n=2 the Kähler equation becomes no more redundant than the Weyl equation.

From now on the manifold M will be assumed to be conformally flat. We have found three different ways to generalize the Cauchy-Riemann equations: Analytic function theory on the bundle E, the self-duality equation and the Weyl equation. For n=2 all three approaches are closely related, which is the basis of Penrose's twistor method[1].

In general the fibres of E are of the form SO(2n)/U(n). For n=2,3 these spaces are projective, namely CP^1 and CP^3 respectively, as one sees from the isomorphisms

$$SO(4) = (SU(2) \times SU(2))/Z_2 \,, \tag{15}$$

$$SO(6) = SU(4)/Z_2 \,. \tag{16}$$

The underlying linear spaces C^2 and C^4 may be identified with the dual of the fibre of the spin bundle S, as it also happens trivially for n=1, where the fibre is a C^1. The dimensions are correct, as

$$\dim_C (SO(2n)/U(n)) + 1 = 2^{n-1} \qquad \text{for } n=1,2,3. \tag{17}$$

For $u \in C^2$ or C^4, resp., analytic functions on E satisfy the equations

$$\frac{\partial f}{\partial \bar{u}} = 0 \tag{18}$$

and

$$(Du) f = 0 \,, \tag{19}$$

with

$$D = \gamma^\mu \partial_\mu \,, \tag{20}$$

when f is written as a function of $x \in M$ and u. For n=1,2 the components of eq. (19) are independent, due to

$$n = 2^{n-1} \qquad \text{for } n=1,2, \tag{21}$$

but for n=3 among the four complex components only three are linearly independent. The latter case has been studied much less than the by now standard twistor formalism for n=2, and we shall not conside: it further.

There are no globally holomorphic non-constant functions on CP^1 or CP^3, so one has to investigate functions with simple poles as the next simplest case. The poles introduce additional degrees of freedom, which one eliminates by forming equivalence classes, using the Čech cohomology $H^1(E(-1))$. Its elements can be written in the form

$$g = g_1 - g_2 , \tag{22}$$

where each g_i has only one simple pole. Now according to eq. (19) Du annihilates g, but it also maps the g_i into analytic functions of CP^1, which have to be constant, as they can be extended to a common globally analytic function. Thus

$$\psi = (Du)g_1 = (Du)g_2 \tag{23}$$

only depends on x. Because of eq. (7) one has

$$u^T \varepsilon D^+ D u = 0 \tag{24}$$

and

$$u^T \varepsilon D^+ \psi = 0 \tag{25}$$

for all u. Thus ψ satisfies the adjoint Weyl equation

$$D^+ \psi = 0 . \tag{26}$$

Conversely, this equation is the integrability condition of eq. (24).

We have seen that for n=2 indeed the various generalizations of holomorphic functions are closely related. But this case has another important feature, which was discovered first by physicists, though there is now also compelling reason for its study inside pure mathematics: The statements made so far generalize easily to the case where M is replaced by a principle fibre bundle, locally MxG, and derivatives are replaced by covariant derivatives given by a connection on this bundle. The connection can be written locally as a 1-form A taking values in the Lie algebra of G. Acting on sections of some associated bundle given by a representation ρ of G, the covariant derivative can be written in the form

$$d_A f = df + \rho(A) \wedge f \tag{27}$$

One has

$$(d_A)^2 = \rho(F) \wedge \tag{28}$$

where the 2-form F is the curvature of the connection.

The twistor approach works as before, as long as the components of eq. (19) remain compatible, when derivatives are replaced by covariant derivatives. The compatibility condition is

$$F = *F \tag{29}$$

i.e. the self-duality equation for the curvature.

Conversely we shall show that small deformations of the self-duality equation yield the Weyl equation in Kähler form. First we have to exclude variations of the potential of the form

$$\delta A = d_A F, \tag{30}$$

as these only yield gauge transformations, i.e. reparametrisations of the bundle. One can achieve orthogonality of δA to all local gauge transformations by requiring

$$d_A * \delta A = 0. \tag{31}$$

Moreover one has

$$\delta F = d_A \delta A. \tag{32}$$

Thus small deformations of the self-duality equation yield solutions of the Kähler equation of type

$$\omega = \delta A - * \delta A \tag{33}$$

i.e. just those solutions of odd degree which fulfil eq. (13) with $\varepsilon = -i$.

So far all considerations have been local on M. With suitable global restrictions one can do much more. In particular the differential operators introduced above become elliptic operators with calculable index. The solutions spaces of the generalizations of the Cauchy-Riemann equations become finite dimensional. As the self-duality equation for the curvature is no longer linear, its solution spaces have an interesting topology, which can be related to the topology of M. For $M=S^4$ all solutions of eqs. (26) and (29) are known, at least up to algebraic manipulations[2].

Another interesting case[3] is $M=R^3 x R^1$, with an SU(2) connection which is required to be invariant under translations in R^1. Moreover one requires the curvature to be square integrable over R^3. If one writes the connection in the form

$$A = \sum_{i=1}^{3} A_i \, dx^i + \varphi \, dx^4, \tag{34}$$

the self-duality equation for the curvature is

$$F(R^3) = * d_A \varphi. \tag{35}$$

This is now an example of a differential equation in an odd dimensional space, which nevertheless is closely related to the Cauchy-Riemann equations, as we shall see.

One can show that the connection reduces asymptotically to a U(1) connection. Thus asymptotically the curvature becomes an exact 2-form, and φ satisfies the Laplace equation. More precisely one can write

$$\varphi = \hat{\varphi} \, \varphi_{as} + O(\exp(-cr)), \tag{36}$$

where $\hat{\varphi}$ is of unit norm and asymptotically is a covariant constant,

$$d_A \hat{\varphi} = O(\exp(-cr)), \tag{37}$$

whereas φ_{as} is a scalar function satisfying

$$\Delta \varphi_{as} = 0 \tag{38}$$

Asymptotically

$$\varphi_{as} = c - \frac{k}{2r} + O(r^{-2}) \tag{39}$$

with integer k, such that φ_{as} is the potential of a magnetic monopole configuration of total charge k. The charge is magnetic rather than electric, because the curvature is in spatial planes, not in space-time planes as for electric charges. Actually, time has not been introduced at all, and the metric in R^3xR^1 has been taken to be the standard positive one.

φ_{as} can be continued to the whole of R^3 with the exception of a finite number of closed algebraic curves and isolated points. If continued around those curves, it becomes multivalued. From the curves and points one may reconstruct the whole solution, though this has not yet been worked out in detail.

Instead, all solutions can be constructed using the Weyl equation (26). The solutions of this equation admit a Fourier analysis, such that one may write

$$\psi(x,z) \sim \exp(ix^4 z). \tag{40}$$

Let $\psi(x,z)$ for given z be an orthonormalized vector of solutions of eq. (26), spanning the space of all solutions which are square integrable over R^3. Furthermore adjust the z dependence, such that

$$\int \psi^+ \frac{\partial}{\partial z} \psi \, d^3x = ix^4. \tag{41}$$

Then one can prove easily that the matrices

$$T^i(z) = \int \psi^+ x^i \psi \, d^3x \tag{42}$$

fulfil the equations

$$\frac{dT_i(z)}{dz} = i\varepsilon_{ijk} T^j(z) T^k(z). \tag{43}$$

The matrices $T^i(z)$ are k-dimensional for $|z|<c$ and vanish outside this interval, as one sees by calculating the index of the Weyl operator for the fundamental representation of SU(2).

The eq. (43) is itself the self-duality equation for the curvature of a U(k) connection

$$T = T^i(z)\, dp_i \tag{44}$$

in a space with coordinates (p^i, z), which is invariant under translations of the p^i. Eq. (43) is integrable in terms of Riemannian θ-functions. The potential A can be obtained back from T using the

Weyl equation in (p^i,z) space with connection T.

All this seems to be rather far removed from the usual theory of holomorphic functions. But using eq. (43) Donaldson has shown that for fixed c the space of self-dual SU(2) monopoles of charge k has the same topology as the space of all holomorphic maps of degree k of CP^1 onto itself which fix one point[4]. These maps are rational and can be written as quotients of two polynomials. The denominator is given by

$$Q(\zeta) = \det(u_i T^i(z) - \zeta), \tag{45}$$

where u is a fixed isotropic vector. By eq. (43) this expression is independent of z.

In Donaldson's construction u has to be fixed, but if it is varied the projective space with coordinates (u,ζ) can be identified with the space of oriented lines in R^3. Eq. (45) determines a curve in this space, and eq. (43) translates into a linear flow in the Jacobian of this curve. Moreover, the family of lines in R^3 given by this curve has an envelope, consisting of closed algebraic curves in R^3 and isolated points - just those curves and points on which φ_{as} is singular.

It is not yet clear, how Donaldson's results generalize to other gauge groups, but at least for the description of the moduli space of more complicated self-dual monopoles, there are plausible conjectures. Let φ_o be the value of the Higgs field at some point of the sphere at infinity. At other points of that sphere one obtains the value by conjugating φ_o with some element of the gauge group G. All possible values correspond to the coset space $G/G(\varphi_o)$, where $G(\varphi_o)$ is the subgroup of G which commutes with φ_o. Such coset spaces have a natural complex structure. Now new results by Atiyah[5] indicate that the moduli space of self-dual monopoles with this asymptotic behaviour corresponds to the holomorphic maps from CP^1 into $G/G(\varphi_o)$. For G=SU(2) one has

$$G/G(\varphi_o) = SU(2)/U(1) = CP^1, \tag{46}$$

which yields Donaldson's result.

Non-linear partial differential equations have not received very much attention by mathematicians, as there are few general results and there seemed to be no point in studying special equations to great depth. Due to many nice results concerning the self-duality equation for gauge field strengths in four dimensions this attitude seems to be changing. It is certainly significant that this equation turned up in mathematical physics.

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