HEREDITARY SUBDIRECTLY IRREDUCIBLE GRAPHS

Jiří Vinárek

0. Introduction

The concept of subdirect irreducibility was introduced for algebras by G. Birkhoff in [1]. It can be defined also for classes of graphs as was done by A. Pultr in [3]: Let $\underline{C}$ be a class of (some) graphs. Then a graph $A \in \underline{C}$ (i.e. a $\underline{C}$-graph A) is said to be subdirectly irreducible (SI) if, whenever an isomorphic copy A' of A is contained as an induced subgraph in a product $\times_{i \in I} B_i$ with $B_i \in \underline{C}$ and $p_j(A') = B_j$ for all the projections, there is a j such that the restriction of p_j to A' is an isomorphism onto B_j.

Having a list of SI $\underline{C}$-graphs, one can construct any $\underline{C}$-graph from subdirectly irreducibles using only operations "product" and "restriction to an induced subgraph". Characterization theorem for SI is given in [4] . This theorem, however, does not solve neither the problem when the list of subdirectly irreducibles is closed to induced subgraphs, nor that one when subdirectly irreducibles are in some sense "homogeneous" (as e.g. in the case of all antireflexive symmetric graphs where SI are just complete graphs). The first problem was solved for antireflexive graphs in [6] , the second one is discussed in this paper.

Notation. Let $\underline{D}$ be a collection of graphs. Then $SP(\underline{D})$ denotes (similarly as in [2]) a class of all the graphs which can be embedded as induced subgraphs into products of graphs from $\underline{D}$.

1. Hereditary subdirect irreducibility

1.1. Definition. A class $\underline{C}$ of graphs closed to categorical products $(\times_{i \in I} (X_i , R_i) = (\times_{i \in I} X_i , R)$ where $((x_i)_I , (y_i)_I) \in R \Leftrightarrow (x_i , y_i) \in R_i$ for any $i \in I)$ and to induced subgraphs is said to be hereditary with respect to subdirect

This paper is in final form and no version of it will be submitted for publication elsewhere.

irreducibility (HSI) if any induced subgraph of a SI $\underline{C}$-graph is again SI.

1.2. Examples 1. HSI classes of graphs are e.g. graphs, antireflexive graphs, symmetric graphs, posets etc.

2. HSI classes of graphs are not e.g. : bipartite graphs, n-chromatic graphs.

1.3. Proposition. Let $\underline{C} \neq$ SET be a class of antireflexive symmetric graphs closed to categorical products and induced subgraphs. Then $\underline{C}$ is HSI iff either $\underline{C}$ is the class of all antireflexive symmetric graphs, or there exists $n \geq 2$ such that $\underline{C} = SP(\{K_n\})$ (where K_n is a complete antireflexive symmetric graph with n vertices).

Proof is given in [5].

1.4. Proposition. Let $\underline{P} \neq$ SET be a class of antireflexive posets, closed to categorical products and suborderings. Then $\underline{P}$ is HSI if either $\underline{P}$ is the class of all antireflexive posets, or there exists $n \geq 2$ such that $\underline{P} = SP(\{L_n\})$ where L_n is a linear ordering on n points.

Proof. Since $\underline{P} \neq$ SET and $\underline{P}$ is closed to suborderings, P contains L_2. Consider two cases :

a) $\underline{P}$ is a class of all antireflexive posets. Then according to [4], A is SI in $\underline{P}$ iff A is a linear ordering. Hence, $\underline{P}$ is HSI.

b) There exists $n \geq 2$ such that L_n is the maximal linear ordering in $\underline{P}$. If $\underline{P}$ is HSI then no $\underline{P}$-graph with more than n vertices can be SI (otherwise an induced discrete graph with 2 vertices would be SI which is a contradiction). Hence, any SI $\underline{P}$-graph is a linear ordering with at most n vertices. Therefore, any $\underline{P}$-graph X is an induced subgraph of L_n^m for some m and $\underline{C} = SP(\{L_n\})$. Q.E.D.

1.5. Remark. Classes of graphs from 1.3 and 1.4 satisfy a stronger property than HSI. Subdirectly irreducibles are not only hereditary but also - in some sense - homogeneous.

This observation can be generalized, using category theory, as follows :

2. Homogeneous hereditary subdirect irreducibility

2.1. Definition. Let $\underline{C}$ be a productive hereditary system of objects. An object $A \in \underline{C}$ is called homogeneous if any two its subobjects of the same cardinality are isomorphic.

$\underline{C}$ is called homogeneously hereditary with respect to subdirect irreducibility (HHSI) if $\underline{C}$ is HSI, any SI object of $\underline{C}$ is homogeneous and any two SI objects of the same cardinality are isomorphic.

2.2. Theorem. A productive hereditary class $\underline{C}$ of graphs is HHSI iff $\underline{C}$ is one of the following classes:

(i) SET (the class of all sets = discrete graphs)

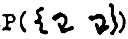
(ii) $SP(\{K_n\})$

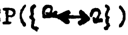
(iii) SYMGRAPH (the class of all antireflexive symmetric graphs)

(iv) $SP(\{L_n\})$

(v) POSET (the class of all antireflexive posets)

(vi) $SP(\{C_3\})$ where $C_3 = (3, \{(0,1),(1,2),(2,0)\})$ (a cycle of length 3)

(vii) SP() (the class of all reflexive discrete graphs)

(viii) SP() (the class of all reflexive complete graphs)

(ix) SP() (the class of all reflexive posets)

(x) SP()

We are going to prove Theorem 2.2 by a series of lemmas :

2.3. Lemma. Let $\underline{C}$ be a HHSI class of graphs. If a two-point discrete graph D_2 is SI in $\underline{C}$ then $\underline{C}$ = SET.

Proof follows evidently from HHSI-property.

2.4. Lemma. Let $\underline{C}$ be a HHSI class of graphs. If K_2 is SI in $\underline{C}$ then either $\underline{C}$ = SYMGRAPH , or $\underline{C} = SP(\{K_n\})$ for some n.

Proof. Let G be a SI $\underline{C}$-graph. HHSI-property implies that any induced subgraph of G with 2 vertices is isomorphic to K_2. Hence, G is isomorphic to some K_k.

Consider two cases :

a) For any nonnegative integer k there exists m such that $m \geq k$ and $K_m \in \underline{C}$. From HSI of $\underline{C}$, it follows that any complete antireflexive graph is SI. Hence, $\underline{C}$ = SYMGRAPH.

b) There exists $n = \max\{k \; ; \; K_k \in \underline{C}\}$. Then $K_m \in \underline{C}$ iff $m \leq n$, and $\underline{C} = SP(\{K_n\})$. Q.E.D.

2.5. Lemma. Let $\underline{C}$ be a HHSI class of graphs. If L_2 is SI and $C_3 \notin \underline{C}$ then either $\underline{C}$ = POSET, or $\underline{C} = SP(\{L_n\})$ for some n.

Proof. Let G be a SI $\underline{C}$-graph. HHSI-property implies that any induced subgraph of G with 2 vertices is isomorphic to L_2. Since $C_3 \notin \underline{C}$, any induced subgraph of G with 3 vertices is isomorphic to L_3. One can check that G is isomorphic to some L_k.

Consider two cases :

a) For any k there exists m such that $m \geq k$ and $L_m \in \underline{C}$. Then any linear ordering is SI and $\underline{C}$ = POSET.

b) There exists $n = \max\{k \; ; \; L_k \in \underline{C}\}$. Then $L_m \in \underline{C}$ iff $m \leq n$, and $\underline{C} = SP(\{L_n\})$. Q.E.D.

<u>2.6. Lemma.</u> Let $\underline{C}$ be a HHSI class of graphs. If C_3 is SI in $\underline{C}$ then $\underline{C} = SP(\{C_3\})$.

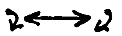
<u>Proof</u>. HHSI-property implies that $L_3 \notin \underline{C}$ and that any SI $\underline{C}$-graph is an antireflexive tournament. One can check easily that any antireflexive tournament with at least 3 vertices contains L_3 as an induced subgraph. Hence, there are no subdirectly irreducibles with at least 4 vertices and $\underline{C} = SP(\{C_3\})$. Q.E.D.

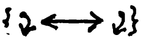
<u>2.7. Lemma.</u> If a HHSI class $\underline{C}$ contains non-trivial graphs with loops then any SI $\underline{C}$-graph is reflexive.

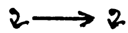
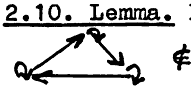
<u>Proof</u> follows directly from HHSI-property of $\underline{C}$.

<u>2.8. Lemma.</u> If  is SI in HHSI $\underline{C}$ then $\underline{C}$ = SP({  }).

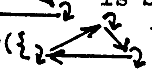
<u>Proof</u>. According to Lemma 2.7, any SI $\underline{C}$-graph is reflexive. Since $\underline{C}$ is HHSI, any SI $\underline{C}$-graph is reflexive discrete. Hence, any $\underline{C}$-graph is reflexive discrete and $\underline{C}$ = SP({  }). Q.E.D.

<u>2.9. Lemma.</u> If  is SI in $\underline{C}$ then $\underline{C}$ = SP({  }).

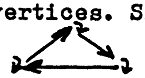
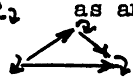
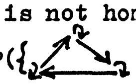
<u>Proof</u>. Since $\underline{C}$ is HHSI, any SI $\underline{C}$-graph is reflexive complete. Hence, any $\underline{C}$-graph is reflexive complete and $\underline{C}$ = SP({  }). Q.E.D.

<u>2.10. Lemma.</u> If  is SI in HHSI $\underline{C}$ and  $\notin$ $\underline{C}$ then $\underline{C}$ = SP({  }).

<u>Proof</u>. Since $\underline{C}$ is productive and hereditary, $\underline{C} \supseteq$ SP({  }). Since  $\notin \underline{C}$, any $\underline{C}$-graph is transitive and $\underline{C}$ = SP({  }). Q.E.D.

<u>2.11. Lemma.</u> If  is SI in HHSI $\underline{C}$ then $\underline{C}$ = SP({  }).

<u>Proof</u>.

Suppose that $\underline{C}$ contains a subdirectly irreducible tournament T with more than 3 vertices. Since T is SI, T is no linear ordering. Hence, T contains  as an induced subgraph. But one can check that T contains  as an induced subgraph as well. Thus, T is not homogeneous, which is a contradiction. Therefore, $\underline{C}$ = SP({  }). Q.E.D.

<u>2.12. Lemma.</u> If $\underline{C}$ is one of the classes listed in 2.2(i) - (x) then $\underline{C}$ is HHSI.

<u>Proof</u> is obvious.

<u>2.13. Proof of Theorem 2.2.</u> follows from Lemmas 2.3 - 2.12.

<u>2.14. Problem.</u> Characterize HSI and HHSI for concrete categories.

REFERENCES

[1] BIRKHOFF G. "Lattice Theory", AMS Colloquium Publ. Vol. 25, Providence RI, 1967.

[2] NEŠETŘIL J. - PULTR A., "On classes of relations and graphs determined by subobjects and factorobjects", Discr. Math. 22 (1978), 287-300.

[3] PULTR A., "On productive classes of graphs determined by prohibiting given subgraphs", Colloq. Math. Soc. J. Bolyai 18 (1976), 805 - 819.

[4] PULTR A. - VINÁREK J. , "Productive classes and subdirect irreducibility, in particular for graphs", Discr. Math. 20 (1977), 159 - 176.

[5] VINÁREK J., "Remarks on dimensions of graphs", Abstracta Eighth Winter School on Abstract Analysis, Praha 1980, 180-193.

[6] VINÁREK J., "Hereditary subdirect irreducibility in graphs", Rep. ZW 193/83, Math.Centrum Amsterdam 1983.

MATEMATICKO-FYZIKÁLNÍ FAKULTA
UNIVERZITY KARLOVY
SOKOLOVSKÁ 83
186 00 PRAHA 8
CZECHOSLOVAKIA