STAR-PRODUCTS ON SYMPLECTIC MANIFOLDS

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Let (M,F) be a symplectic manifold. The manifold M is supposed to be smooth, connected, second countable and of dimension $2m > 2$; unless further indications, objects like functions, forms,... are assumed to be smooth.

We denote by N the space of smooth functions on M. It is an associative algebra for the usual product $\mathcal{M} : (u,v) \to uv$ and a Lie algebra for the Poisson bracket $P : (u,v) \to \{u,v\}$. The space $E(N,\lambda)$ is the vector space of formal series

$$\sum_{k=0}^{\infty} \lambda^k u_k \quad (u_k \in N).$$

A *formal* p-linear map from $E(N,\lambda)^p$ into itself is a map

$$T_\lambda = \sum_{k=0}^{\infty} \lambda^k T_k$$

where each T_k is p-linear and where

$$T_\lambda(u_\lambda,\ldots) = \sum_{k=0}^{\infty} \lambda^k \sum_{i+j+\ldots=k} T_i(u_j,\ldots).$$

A *formal deformation* of $(N,\mathcal{M})$ is a formal bilinear map $\mathcal{M}_\lambda$ from $E(N,\lambda)^2$ into itself such that $\mathcal{M}_0 = \mathcal{M}$ and that $(E(N,\lambda),\mathcal{M}_\lambda)$ is an associative algebra. A formal deformation of (N,P) is defined in a similar way.

Formal deformations of the Lie algebra (N,P) or of $(N,\mathcal{M})$ appeared in the context of quantum mechanics. In the Hamiltonian formulation of classical mechanics, the phase space of observables is N. Passing to quantum mechanics usually changes the nature of observables : they become operators on suitable functional spaces, the Lie algebra structure being given by the commutator.

The first formal deformation of (N,M) appeared as the inverse Weyl transform of the product of operators (Moyal [13]). It was rediscovered by Vey [14]. It has been proposed by Flato, Lichnerowicz, Sternheimer et al. to develop a new framework for quantum mechanics by keeping the space of observables N unchanged, but replacing its structures M and P by a formal deformation $M + \lambda P + \ldots$ An account of the results obtained in this direction from the point of view of quantum mechanics can be found in [8]. Regarding the mathematical study of such deformations, their derivations,..., see [10] and the references therein. More recently, that theory has also been used for the study of representations of Lie groups [1,2,3].

It turns out to be convenient and reasonable for the purposes that we have mentioned to restrict somewhat the notion of formal deformation introduced above.

Definition. A *star-product* is a formal deformation

$$M_\lambda = \sum_{k=0}^{\infty} \lambda^k C_k$$

of (N,M) such that

(i) $C_o = M$, $C_1 = P$,

(ii) $C_k(u,v) = (-1)^k C_k(v,u)$, $\forall\, u,v \in N$,

(iii) each $C_k (k > 1)$ is local and vanishing on the constants.

A *deformed bracket* of N is a formal deformation

$$\mathcal{L}_\lambda = \sum_{k=0}^{\infty} \lambda^k C_k$$

of (N,P) such that each C_k is local and vanishing on the constants.

From the beginning of the theory, the problem of the existence of star-products or deformed brackets has given rise to a number of papers, seeking both for existence criteria and for constructive examples.

The Moyal-Vey star-product is easily described. Suppose that M is an open subset of $\mathbb{R}^{2m}$. Denote by Λ the contravariant tensor obtained by lifting the indices of F by means of the duality defined of F.

The k-th iterated of the Poisson bracket is then defined by

$$P^k(u,v) = \Sigma\, \Lambda^{i_1 j_1} \ldots \Lambda^{i_k j_k} \partial_{i_1 \ldots i_k} u\, \partial_{j_1 \ldots j_k} v.$$

The Moyal-Vey star-product is given by

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} P^k.$$

In order to obtain similar operators on an arbitrary manifold, Vey suggested to replace the partial derivatives by covariant derivatives. Unfortunately, the lack of commutativity of these derivatives leads to very difficult computations. An explicit computation of the five first terms of a star-product can be found in [9]. It leaves few hope for an explicit constructive general process along the same lines.

The natural framework for the study of deformations or formal deformations (from a general algebraic point of view) is provided by the Nijenhuis-Richardson bracket : denote by L a vector space and by $A^p(L,L)$ the space of $(p-1)$-linear alternative maps of L into itself and by $A(L,L)$ the direct sum of these spaces. The Nijenhuis-Richardson bracket, denoted $[\,,\,]$, equippes $A(L,L)$ with a graded Lie algebra structure.

Moreover,

- (L,C) *is a Lie algebra if and only if* $[C,C] = 0$;
- *if* $[C,C] = 0$, $\partial_C : C' \to [C,C']$ *is a coboundary operator on* $A(L,L)$.

Multiplying $\partial_C C'$ by $(-1)^d$, where d is the degree of C', we obtain the Chevalley coboundary related to the adjoint representation of (L,C).

Say that $C_\lambda = \Sigma\, \lambda^i C_i$ is a formal deformation *of order* k if $[C_\lambda, C_\lambda] = 0$ up to the order k ($[C_\lambda, C_\lambda]$ is a formal operator and we mean that the coefficients of $\lambda^o, \ldots, \lambda^k$ are vanishing). Then,

- *if* C_λ *is a formal deformation of order* k *of* C_o ,

$$J_{k+1} = \sum_{\substack{a+b=k+1 \\ a,b\leq k}} [C_a, C_b]$$

is a cocycle for ∂_{C_0} *; if*

$$2\,\partial_{C_0} C_{k+1} = J_{k+1} ,$$

then C_λ *is a formal deformation of order* $k + 1$.

This outlines a step-wise process for constructing a formal deformation and shows that cohomological obstructions may appear, belonging to the third Chevalley cohomology space $H^3(L, C_0)$.

The case of formal deformations of an associative product can be treated in a completely similar way by an appropriate adaptation of the Nijenhuis-Richardson bracket [5,6].

Applied to star-products or deformed brackets, these considerations point out two cohomology spaces : $H^3_{loc,nc}(N,\partial)$, the third Chevalley cohomology space of (N,P), the cochains being assumed local and nc (null on the constants), and $H^3_{loc,nc}(N,\delta)$ the third Hochschild cohomology space of (N,M), again with local nc cochains. These two spaces are known. The space $H^3_{loc,nc}(N,\delta)$ is isomorphic to the space of antisymmetric contravariants 3-tensors on M. In particular, it never vanishes for $2m > 2$; but it vanishes for $2m = 2$, thus, in this case, the existence of star-products is trivial. The space $H^3_{loc,nc}(N,\partial$ is isomorphic to $\mathbb{R} \oplus H^3_d(M)$ (third de Rham cohomology space of M). The factor $\mathbb{R}$ corresponds to the multiples of a cocycle exhibited by Vey, S^3_Γ, which is never a coboundary.

The first existence theorem is due to Vey [14] who proved the existence of formal brackets provided $H^3_d(M) = 0$. Later, Neroslavsky and Vlassov [12] proved the same result for star-products, by a much simpler proof. Lichnerowicz [11] describes a way of obtaining star-products on a certain class of homogeneous spaces, showing in particular that the vanishing of $H^3_d(M)$ is not a necessary condition. Gutt [9] obtains, for symplectic manifolds with a Lie group action, a sufficient condition requiring the existence of an invariant connection and the vanishing of the *invariant* de Rham cohomology space. Cahen and Gutt [3] show the existence of a star-product on the cotangent bundle of a parallelizable manifold (actually their proof extends easily to $T^\star M$ for an arbitrary M). The basic idea of this paper is that by imposing appropriate conditions of the terms C_k of the star-product, one can avoid the cohomological obstructions in the step-wise construction. These conditions turn out to be homogeneity of order $-k$ of C_k with respect to the vector field of homotheties of the fibers. Starting from this fact, the author and Lecomte [4-7] finally proved that no obstructions at all are encountered in constructing star-products or deformed brackets.

Our main results are summarized in the following theorems. Let us say that a deformed bracket C_λ is *1-differentiable* if each C_k is a differential operator of order 1 in each argument.

Recall that C_λ is a deformed bracket of order k if $[C_\lambda, C_\lambda] = 0$ up to the order k. The terms C_i $(i > k)$ of C_λ are irrelevant in that definition. Saying that C_λ extends to a deformed bracket means that we can choose the C_i $(i > k)$ in order to have $[C_\lambda, C_\lambda] = 0$.

Theorem. [7] *Let* (M,F) *be an arbitrary symplectic manifold.*

(i) *Every 1-differentiable deformed bracket of order* k *extends to a 1-differentiable deformed bracket.*

(ii) *Every deformed bracket of order* k *extends to a deformed bracket.*

(iii) *Every star-product of order* $2k$ *extends to a star-product.*

The proofs are based on cohomological properties of the Nijenhuis-Richardson bracket and of the local infinitesimal conformal transformations of the symplectic form.

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