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ON CERTAIN QUANTITES IN FREDHOLM - OPERATOR THEORY AND MIL'MAN'S ISOMETRY SPECTRUM

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§ 1: INTRODUCTION

In this note we look at the following two quantities in the theory of Fredholm operators, which were introduced by M. Schechter [13] and B. Gramsch [11]:

$$\Gamma(T) := \inf_{M\subset X} \| T_{|M} \|$$

$$\Delta(T) := \sup_{M\subset X} \inf_{N\subset M} \| T_{|N} \|$$

Here T is a continuous linear operator from a Banach space X to a Banach space Y (i.e. $T \in L(X,Y)$) and M,N are closed infinite dimensional subspaces of X. In this note for convenience we shall only write subspace if we speak of a closed infinite dimensional subspace.

These quantities provide characterizations of two classes of operators, namely the class of Φ_+-operators (Semi-Fredholm operators with finite - dimensional kernel) and the class of strictly singular operators or Kato-operators (cf. for ex. [12]) because: [13]

$\Delta(T) = 0 \Leftrightarrow T$ strictly singular

$\Gamma(T) > 0 \Leftrightarrow T \in \Phi_+$

The main result of Schechter's paper is the following generalization of the wellknown Krein-Gohberg- and Kato perturbation theorems for (semi-) Fredholm operators: $T,S : X \to Y$ then

$$\Delta(S) < \Gamma(T) \Rightarrow T + S \in \Phi_+ \text{ , ind } (T+S) = \text{ind } (T)$$

Finally we mention that there are dual notions and results for Φ_- - operators and Pelczynski's strictly cosingular operators [cf.8; 14; 15] which however will not be considered here.

§ 2: REPRESENTATION THEOREMS FOR Δ, Γ.

At first glance it seems that Γ and Δ are only of very theoretical interest because (with the exception of some very special cases) there is no hope to calculate $\Gamma(T)$ and $\Delta(T)$ for an operator T from their definition even when T is given in a concrete representation.

But nevertheless with the help of some Banach space techniques, in many cases a much nicer representation of Γ and Δ is possible if we restrict ourselves to

(a) special classes of operators
or (b) special classes of Banach spaces
(namely those with a "good" subspace stucture as we will see later)

As an illustration we state the following result of L.W. Weis and the author, which shows, that for the determination of Γ and Δ it suffices to calculate the norms of restrictions of the operator to special subspaces, if the subspace structure of the considered Banach space is well known.

2.1. PROPOSITION:

Let $X = l^p$ $(1 \leq p < \infty)$ or c_0 and $T \in L(X)$. Then

$$\Delta(T) = \lim_{n\to\infty} \| Q_n \; T \; Q_n \|$$

$$\Gamma(T) = \lim_{n\to\infty} \gamma(Q_n \; T \; Q_n)$$

where Q_n denotes the canonical projection of X onto the span of the unit vector basis starting from index n+1 and γ the minimum modulus of an operator.

Idea of proof: It is possible to choose inductively a sequence $\tilde{x}_n$ of nearly disjoint (normalized) vectors in X such that, roughly speaking,

$$T\tilde{x}_n \underset{(1+\varepsilon)}{\approx} Q_n \; T \; Q_n \tilde{x}_n$$

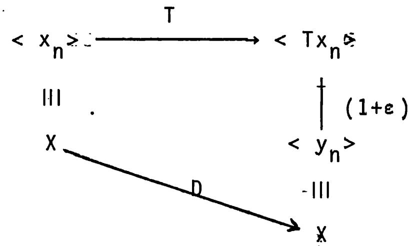
$$\text{and } \| Q_n \; T \; Q_n \; \tilde{x}_n \| \approx \| Q_n \; T \; Q_n \|$$

By truncation and normaization we get normalized disjoint sequences x_n and y_n such that

$$Tx_n \underset{1+\varepsilon}{\approx} \| Q_n T Q_n \| \; y_n$$

(in reality $Tx_{n_k} \approx \| Q_{n_k} T Q_{m_k} \| y_{n_k}$ for some suitable sequences n_k, m_k but this is only of technical importance)

So we can construct subspaces $<x_n>$ and $<y_n>$ of X isometric to X [cf. 6] such that $T|_{<x_n>}$ behaves like a diagonal operator D with diagonal $\| Q_n T Q_n \|$. This situation is represented in the following diagram:



So $\Delta(D) \leq (1+\varepsilon)\, \Delta(T)$. But the calculation of $\Delta(D)$, D being a diagonal operator on X, is very easy. $\Delta(D)$ equals just the limit of the diagonal sequence i.e. $\lim \| Q_n T Q_n \|$ in our case. Trivially [cf. 13] $\Delta(T) \leq \| Q_n T Q_n \| \; \forall n \in \mathbb{N}$ and the above consideration yields the desired result.

The proof of Γ- result is similar. □

As an application of the result just mentioned and as an illustration of the viewpoints § 1,a,b we give the following simple example:

Let $H_2(\mathbb{T})$ be the Hardy-space [cf. 2] and $H : H_2(\mathbb{T}) \to H_2(\mathbb{T})$ a Hankel-operator and $T : H_2(\mathbb{T}) \to H_2(\mathbb{T})$ a Töplitz-operator. Both operators can be represented as operators in l^2 by infinite matrices:

$$H = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \quad ; \; T = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_{-1} & a_0 & a_1 & a_2 & \cdots \\ a_{-2} & a_{-1} & a_0 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

The proposition 1.1 says that $\Delta(H)$ and $\Delta(T)$ are simply the limit of the norms of those operators defined by the submatrices which arise when we cut off the first n rows and columns. (So we get for example

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_{-1} & a_0 & a_1 & a_2 & \cdots \\ a_{-2} & a_{-1} & a_0 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

and $\|Q_n T Q_n\| = \|T\| \; \forall n$ which means that $\Delta(T) = \|T\|$.)

What is essential in this example is that using proposition 1.1 in calculating Δ and Γ we only have to consider subspaces which do not destroy the structure of the operator because Q_nTQ_n remains a Toeplitz operator and Q_nHQ_n remains a Hankel operator.

As a further result of the possibilities in representing Γ, Δ on certain concrete spaces, we mention the following generalization of a result of Pelczynski [9] which says that on L^1-spaces strictly singular operators are always weakly compact and vice versa. This theorem is due to L.W. Weis [unpublished] :

<u>2.2 THEOREM</u> [Weis]

Let (X,μ); (Y,ν) compact measure spaces with regular Borel measures and $T : L^1(X,\mu) \to L^1(Y,\nu)$
Then

$$\Delta(T) = \overline{\lim_{\nu(A)\to 0}} \|\chi_A T\|$$

§ 3 THE ISOMETRY SPECTRUM AND Δ, Γ

The main interest of this note however lies in the connection of Δ and Γ and a notion introduced by V.D. Mil'man in [7]. This is the so called Isometry Spectrum of an operator which will be defined as follows:

Let X,Y be Banach spaces and $T \in L(X,Y)$. Then we call

$$I(T) := \{\alpha\in\mathbb{R}_+ : \forall\varepsilon>0 \;\; \exists M\subset X, \dim M = \infty \text{ such that}$$
$$\big|\, \|Tx\| - \alpha \,\big| < \varepsilon \;\; \forall x\in M, \;\; \|x\| = 1 \}$$

the Isometry Spectrum of T.

So $I(T)$ contains all $\alpha \geq 0$ for which there exists an infinite-dimensional closed subspace M of X where T behaves like the α-product of an isometry.

Trivially there are the following relations to the quantities Δ , Γ :

$$\Delta(T) = 0 \Leftrightarrow T \text{ strictly singular} \Leftrightarrow I(T) = \{o\}$$

and $\quad I(T) \subset [\Gamma(T); \Delta(T)]$.

But if we restrict ourselves, following the ideas above, to Banach spaces with "good" subspace structure, we can even say more:

Let us call C the class of all l^p- saturated Banach spaces in the following sense:

$$X \in C \Leftrightarrow \forall M \subset X, \dim M = \infty \ \forall \varepsilon > 0$$
$$\exists p \in [1,\infty) \quad \exists N \subset M, \dim N = \infty$$
$$\text{such that } N \underset{1+\varepsilon}{\cong} l^p$$

The class C is big enough. This can be seen from the fact that it contains the class of all stable Banach spaces defined by Krivine and Maurey in [5]and therefore especially l^p-, L^p-, Lorentz and some Orlicz-spaces [cf. 10] .

If we consider only the class C we are able to state the following

3.1 PROPOSITION:

Let X, Y be in C.
Then $\Delta(T) = \max I(T)$
$\Gamma(T) = \min I(T)$

i.e. $\Delta(T)$, $\Gamma(T)$ are contained in $I(T)$. Especially follows:
$M \subset X$, $\dim M = \infty \Rightarrow \Delta(T|_M) \in I(T)$

IDEA OF PROOF: We have to show that $\Gamma(T)$ and $\Delta(T)$ are elements of $I(T)$. This is trivial if $\Gamma(T) = 0$ or $\Delta(T) = 0$ since in both cases there are subspaces where T can't be an isomorphism and so $0 \in I(T)$. If $\Delta(T)$ or $\Gamma(T) \neq 0$ then T is Φ_+ or strictly singular according to the characterization in § 1. So there are subspaces M where T is an isomorphism onto TM and which can be chosen in such way that $\| T|_M \| \underset{\varepsilon}{\approx} \Delta(T)$ resp. $\Delta(T)$. But since $X, Y \in C$ we can choose $M \underset{1+\varepsilon}{\cong} l^p$ (take a subspace). So we deal with endomorphisms on l^p. Here we have some additional properties which allow us to find l^p- subspaces where $|\, \|Tx\| - \Delta(,\Gamma)| < \varepsilon \ \forall \|x\| = 1$ □

If we look at proposition 3.1 and the remarks at the beginning of § 3, the following question arises:

When is $I(T) = [\Gamma(T), \Delta(T)]$?

In general $I(T)$ is not equal to $[\Gamma(T), \Delta(T)]$ even in the C-case because we can show that the Isometry Spectrum of an endomorphism in X can split into two disjoint sets if X can be decomposed into the sum of two totally incomparable spaces, as $l^p \oplus l^q$ $p \neq q$ for example.

3.2. PROPOSITION:

Let X, Y be totally incomparable Banach spaces and P,Q denote the projections of $X \oplus Y$ to X and Y. Let i,j denote the inclusions of X, Y in $X \oplus Y$ then

$$I(T) = I(PTi) \cup I(QTj)$$

But even if such a decomposition is not possible, we have not been able to prove an affirmative result for $X \in C$ or X stable. In fact we need much more structure than l^p-saturation. So the proofs of the following positive results are based to a large extent on the structure of the special C-spaces considered.

3.3 THEOREM:

Let $X = c_0, l^p, L^p[0,1]$ $(1 \leq p < \infty)$ and $T \in L(X)$. Then

$$I(T) = [\Gamma(T), \Delta(T)]$$

IDEA OF PROOF: Let us take the l^p-case. It is well-known that l^p is not only in C but l^p is saturated by one and only one $l^{(\cdot)}$-space, namely l^p.

By results of Mityagin [3,8] and Berkson [1] we know that these (complemented) l^p-subspaces can be combined in a connected component of the space of all subspaces induced with a suitable topology. This is the opening- or Schäffer topology [cf. 1]. If we denote (SX,d) the space of all subspaces of a Banach space X with Schäffer-topology d, we can show that the function

$$\Delta_T : (SX,d) \to \mathbb{R}$$
$$M \to \Delta_T(M) := \Delta(T|_M)$$

is continuous. So the image of the above mentioned connected l^p-component (say M) is connected in $\mathbb{R}$ i.e. $\Delta_T(M)$ is an interval.

Since by proposition 3.1 $\Delta_T(M) \in I(T)$ it is easy to see that $\Delta_T(M)$ fills up all of $I(T)$. So $I(T)$ is an interval, namely $[\Gamma(T), \Delta(T)]$.

The proof of the L^p-result contains essentially the same ideas. Here we have two connected subspace components in (SX,d) if $p > 2$ (those l^2 and l^p) according to the structure theorems of Kadets-Pelczynski [4]. So we have at most two disjoint intervals which form $I(T)$.

But it can be shown that they are never disjoint and that therefore $I(T)$ must be an interval.

For $p < 2$ the methods are similar. □

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