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ON THE A.E. CONVERGENCE OF $T^n f/a_n$ IN L_1- SPACE

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1. Introduction

Let (X,Σ,m) be a σ-finite measure space and let T be a linear operator of $L_1(X,\Sigma,m)$. A neccessary condition of the pointwise convergence a.e. of ergodic means $1/n \sum_{i=0}^{n-1} T^i f$, $f \in L_1$, is

(1) $T^n f/n \longrightarrow 0$, a.e.

The condition (1) is fulfiled in many special cases, e.g. for T beeing a positive contraction of both L_1 and L_∞, but of course it is not satisfied in general. The condition (1) does not hold even for positive contractions of L_1 (see [3]).

Let $\{a_n\}$ be an increasing seguence of positive real numbers. We shall investigate the a.e. convergence to zero of $\{T^n f/a_n\}$ for all $f \in L_1$ with respect to the properties of the sequence $\{a_n\}$.

2. The spectral radius of T

The n-th iterate of a linear operator T may have an exponencial streaming determined by the spectral radius λ_T.

Definition. Let T be a linear operator on Banach space B. Then the spectral radius λ_T is defined as

$$\lambda_T = \limsup_n \sqrt[n]{\|T^n\|} .$$

Lemma 1. $\lambda_T = \lim_n \sqrt[n]{\|T^n\|} = \inf_n \sqrt[n]{\|T^n\|}$.

Proof. As $\|T^{n+m}\| \leq \|T^n\| \|T^m\|$, the sequence $\{\log \|T^n\|\}$ forms a subadditive sequence. Thus, there exist

$$\lim_n (\log \|T^n\|)/n = \inf_n (\log \|T^n\|)/n = \lim_n \log \sqrt[n]{\|T^n\|} ,$$

see e.g. [6] .

To eliminate the exponencial trend of T^n in what follows we suppose $\lambda_T = 1$. If $\lambda_T \neq 1$, it is sufficient to investigate the

"This paper is in final form and no version of it will be submitted for publication elsewhere".

linear operator $T^* = T/\lambda_T$.

3. Finite space (X,Σ)

Let (X,Σ) be a finite measurable space, (X,Σ, m) a measure space. A linear operator T acting on $L_1(X,\Sigma,m)$ may be viewed as a matrix (T_{ij}), $\|T\|_1 = M\|T_{ij}\|$, where M depends only on m, $\|.\|_1$ is a norm in L_1-space, $\|.\|$ is a matrix norm. For the sake of simplicity, we identify $T = (T_{ij})$. Let A be Jordan matrix of T, i.e. $T = UAU^{-1}$, $T^n = UA^nU^{-1}$. It is easy to see that $\lambda_T = \lambda_A$. Let $\lambda_T = 1$. Then the matrix A is a block-diagonal matrix with eigen-values λ_i , $\max|\lambda_i| = \lambda_A = 1$. For our pourpose it is sufficient to work with A of the form

$$A = \underbrace{\begin{bmatrix} \lambda 0 \dots 0 \\ 1\lambda 0 \dots 0 \\ 01\lambda 0 \dots 0 \\ \vdots \\ 0 \dots 01\lambda \end{bmatrix}}_{m} = \lambda I_m + B_m \; , |\lambda| = 1 \; , \; I_m \text{ is an unit matrix,} \quad B_m^m = 0_m \; .$$

Then for $n \geq m$, $A^n = \sum_{k=0}^{m-1} \binom{n}{k} B_m^k \lambda^{n-k}$, so that $\lim_n \|A^n\|/n^{m-1} = 1/\ m-1\ !$. All these facts imply the following theorem.

<u>Theorem 1.</u> Let T be a linear operator on finite-dimensional L_1-space, $\lambda_T = 1$. Then

i) for some nonnegative integer k there exists positive finite limit $\lim_n \|T^n\|/n^k$

ii) for any sequence $\{a_n\}$ with the property

(2) $a_n/n^k \longrightarrow \infty$

and for any $f \in L_1$ it holds $T^n f/a_n \longrightarrow 0$, a.e, L_1

iii) the condition (2) is best possible to assure the a.e. convergence of $T^n f/a_n$ to zero.

4. General case

A direct extension of Theorem 1 / parts ii) and iii) / to the general case of underlying measure space is not possible, as shown in the next example.

<u>Example 1.</u> We construct an operator T satisfying $\|T^n\| = 2$, n = 1, 2,..., .e. i) of Theorem 1, for k = 0, an incrasing sequence $\{a_n\}$ of posi ve reals satisfying the condition (2) / even for k = 1 / and a f ction $f \in L_1$, such that $T^n f/a_n \longrightarrow 0$, a.e., does not hold. Th s example is a modification of an example in [5, p. 262].

Le S be an ergodic invertible measure preserving transformation of $\langle 0, 1)$ / with Lebesgue measure / and define also $Sf(x) = f(Sx)$. Take $0 \leq f \in L_1\langle 0, 1)$ such that $f.\log^+ f \notin L_1$. By [7] $\sup_n S^nf/n \notin L_1$. Then there exists an increasing sequence $\{n_i\}$ of integers such that $E(\max_{n \leq n_i} S^nf/n) \geq i^2$. Let $b_n = i$ for $n_{i-1} < n \leq n_i$, $n_0 = 0$. Denote $a_n = n.b_n$. As $E(\max_{n \leq n_i} S^nf/a_n) \geq i$, we have $\sup_n S^nf/a_n \notin L_1$. It is obvious that $a_n/n \to \infty$. For the sake of completness, we continue in presenting Example 1, although the rest is essentially the same as in [5, p. 262] .

We define $X = \langle 0, 2)$ with the Lebesgue sets and measure. By Theorem 4.3. of [5] there is a sub-σ-algebra $\mathcal{L}$ such that $E(S^nf/a_n / \mathcal{L})$ does not converge a.e. Let E denote the conditional expectation operator with respect to $\mathcal{L}$. Define T on $L_1\langle 0, 2)$ by

$$Tg(x) = \begin{cases} g(Sx) & 0 \leq x < 1 \\ ES(1_{\langle 0,1)}g)(x-1) & 1 \leq x < 2 \end{cases} .$$

Clearly T is linear / and positive /,

$$T^ng(x) = \begin{cases} g(S^nx) & 0 \leq x < 1 \\ ES^n(1_{\langle 0,1)}g)(x-1) & 1 \leq x < 2 \end{cases} .$$

We have $\|T^n\|_1 = 2$, $n = 1, 2, \ldots$, $\|T\|_\infty = 1$, $T1 = 1$. Putting f' on $\langle 0, 2)$ as f on $\langle 0, 1)$ and 0 on $\langle 1, 2)$ we have for $1 \leq x < 2$ $T^nf'(x)/a_n = (ES^nf/a_n)(x-1)$, which does not converge on $\langle 1, 2)$.

<u>Remark 1.</u> Similarly we can modify the example of a contraction of L_1 without a.e. convergence of Cèsaro means due to Chacon [3] . By changing the choice of c_n and K_n in [3] we can construct a contraction T of L_1 , $f \in L_1$ and a sequence $\{a_n\}$ satisfying the condition (2) with k = 1 such that

$\liminf_n T^nf/a_n = 0$ a.e.

$\limsup_n T^nf/a_n = \infty$ a.e.

For mean bounded operators, i.e. $\sup_n \|M_n\|_1 = M < \infty$, where $M_n = (I+T+\ldots+T^{n-1})/n$, the problem of a.e. convergence of T^nf/a_n to zero is solved completely by the next theorem.

<u>Theorem 2.</u> Let T be a mean bounded linear operator on $L_1(X, \Sigma, m)$. Then

i) for any increasing sequence $\{a_n\}$ of positive reals with the property

(3) $\sum_n 1/a_n < \infty$

and for any $f \in L_1$ it holds $T^nf/a_n \longrightarrow 0$, a.e.

ii) the condition (3) is the best possible to assure the a.e. convergence of T^nf/a_n to zero.

Proof.

i) Denote $U = \sum_{i=0}^{\infty} T^i/a_i$, $a_0 = 1$. Then

$$U = \sum_{i=0}^{\infty} ((i+1)M_{i+1} - iM_i)/a_i = \sum_{i=0}^{\infty} (i+1)M_{i+1}(1/a_i - 1/a_{i+1}) ,$$

$$\|U\|_1 \le 1 + M\sum_{i=1}^{\infty} (i+1)(1/a_i - 1/a_{i+1}) = 1 + M\sum_{i=1}^{\infty} 1/a_i < \infty ,$$

so that U is a well defined linear operator on L_1. This implies directly $T^n f/a_n \longrightarrow 0$ a.e. , for every $f \in L_1$.

ii) Let the condition (3) does not hold, that is $\sum_n 1/a_n = \infty$. Then there exist a mean bounded operator T on some L_1 and $f \in L_1$ for which $T^n f/a_n \longrightarrow 0$,a.e., does not hold. It is clear / after Example 1 / that we can concentrate ourselves to the case $a_n > n$. Modifying the Davis's proof of his Lemma on p. 148 in [4] we obtain for iid $\{f_n\}$ that $G(|f_1|) \notin L_1$ implies $\sup_n |f_n/a_n| \notin L_1$, where $G(a_n) = \sum_{k=1}^{n} (a_n - a_k)/a_k$, $G = G(a_n)$ on $\langle a_n, a_{n+1})$. / The condition $f.\log^+ f \notin L_1$ for $a_n = n$ is an immediate consequence of $G(n) \sim (n+1)\log(n+1)$ ./

For $a_n \ge n$, $\sum_n 1/a_n = \infty$ we have $\limsup_n G(a_n)/a_n \ge \limsup_n (\sum_{k=1}^{n-1} 1/a_k) - 1 = \infty$, so that there exists $f \in L_1$ such that $G(|f|) \notin L_1$. From now on, we can continue as in Example 1.

<u>Corollary.</u> Let T be a power bounded linear operator on L_1, i.e. $0 < \limsup_n \|T^n\|_1 < \infty$. Then i) and ii) of Theorem 2. hold.

<u>Remark 2.</u> Theorem 2. solves also another problem of clasic ergodic theory: what conditions on $\{a_n\}$ assure

(4) $\sup_n |S^n f/a_n| \in L_1$

for all measure preserving transformations S on (X, Σ , m), $f \in L_1(X, \Sigma , m)$. It is easy to see that for convergent $\sum_n 1/a_n$ does (4) hold. The proof of part ii) of Theorem 2. shows that for divergent $\sum_n 1/a_n$ the condition (4) may be false!

For a general linear operator T on L_1 with $0 < \limsup_n \|T^n\|_1/n^k < \infty$ we can easily generalize the part i) of Theorem 2. We are so far unable to generalize or modify the part ii) .

<u>Theorem 3.</u> Let $0 < \limsup_n \|T^n\|_1/n^k < \infty$ / or let $0 < \limsup_n \|M_n\|_1/n^k < \infty$ /. Then for any increasing sequence $\{a_n\}$ of positive reals with the property

(5) $\sum_n n^k/a_n < \infty$

and for any $f \in L_1$ it holds $T^n f/a_n \longrightarrow 0$, a.e.

<u>Conjecture.</u> The condition (5) in Theorem 3. is best possible.

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