Rainer Linde

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In: Zdeněk Frolík and Vladimír Souček and Jiří Vinárek (eds.): Proceedings of the 13th Winter School on Abstract Analysis, Section of Analysis. Circolo Matematico di Palermo, Palermo, 1985. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 10. pp. [83]–110.

Persistent URL: http://dml.cz/dmlcz/701866

s-numbers of diagonal operators and Besov embeddings

by Rainer Linde

Friedrich-Schiller-Universität Jena, Sektion Mathematik, DDR
69 Jena, UHH

Abstract

In recent years, many papers have appeared in which the asymptotic behaviour of concrete s-numbers of the identity map from $l_p(m)$ into $l_q(m)$ had been calculated. We shall present a survey of these results for estimating the asymptotic behaviour of s-numbers of diagonal operators and Besov embeddings. For this end we shall prove a new finitization method.
Moreover we want to give some historical remarks.

0. Terminology

The class of all (bounded linear) operators between arbitary Banach spaces is denoted by L, while L(E,F) stands for the set of those operators acting from E into F. If M is a (closed linear) subspace of the Banach space E, then J_M^E denotes the natural injection from M into E and Q_M^E stands for the natural surjection from E onto E/M.
Let (ξ_k) and (η_k) be two sequences. Then $\xi_k \prec \eta_k$ means that there exists a positive constant c such that $\xi_k \leq c\, \eta_k$ for all k=1,2,... . The notation $\xi_k \asymp \eta_k$ is used, if simultaneously $\xi_k \prec \eta_k$ and $\eta_k \prec \xi_k$ hold.
Let $0 < r < \infty$ and $0 < w \leq \infty$. Then the Lorentz space $l_{r,w}$ consists of all bounded sequences $a=(\alpha_n)$ for which the

following expression is finite:

$$\| a | l_{r,w} \| := \begin{cases} \left(\sum_{n=1}^{\infty} (n^{1/r - 1/w} \alpha_n^*)^w \right)^{1/w} & \text{for } w < \infty \\ \sup_n n^{1/r} \alpha_n^* & \text{for } w = \infty, \end{cases}$$

where $\alpha_n^* = \inf \{ \sigma \geq 0 : \operatorname{card}(k : |\alpha_k| \geq \sigma) \leq n \}$ denotes the so-called non-increasing rearrangement of (α_n). In the case $|\alpha_1| \geq |\alpha_2| \geq \dots$ it turns out that $\alpha_n^* = |\alpha_n|$.

1. s-Numbers

An s-function is a map s which assingns to every operator $S \in L$ a scalar sequence $(s_n(S))$ such that the following conditions are satisfied:

(1) $\|S\| \geq s_1(S) \geq s_2(S) \geq \dots \geq 0$ for all $S \in L$.

(2) $s_{n+m-1}(S+T) \leq s_n(S) + s_m(T)$ for $S, T \in L(E,F)$ and $n,m = 1,2,\dots$

(3) $s_n(BSX) \leq \|B\| \, s_n(S) \, \|X\|$ for $X \in L(E_0, E)$, $S \in L(E,F)$, $B \in L(F, F_0)$ and $n = 1,2,\dots$.

(4) $s_n(S) = 0$ for $\operatorname{rank}(S) < n$.

(5) $s_n(I : l_2(n) \longrightarrow l_2(n)) = 1$, where I denotes the identical map.

We call $s_n(S)$ the n-th s-number of the operator S.

REMARK. In contrary to earlier definitions (cf. [62] 11.1.1 and 11.8.1) we assume in (2) already the additivity of the s-function.

Next we shall list some well-known facts about s-functions.

(1) From $S \in L(E,F)$ and $s_n(S) = 0$ it follows $\operatorname{rank}(S) < n$.

(2) The s-numbers are continuous functions, namely

$$|s_n(S) - s_n(T)| \leq \|S - T\|$$

for all $S, T \in L(E,F)$.

(3) On the class of all compact operators acting between Hilbert spaces there is only one s-function.

We now want to define the most usual examples of s-functions. Let $S \in L(E,F)$ and $n=1,2,\ldots$ Then we give the following definitions:

n-th approximation number

$$a_n(S)=\inf\left\{\| S-L\| : \operatorname{rank}(L)< n\right\},$$

n-th Gelfand number

$$c_n(S)=\inf\left\{\| SJ_M^E \| : M \subseteq E \text{ with } \operatorname{codim} M < n\right\},$$

n-th Kolmogorov number

$$d_n(S)=\inf\left\{\| Q_N^F S \| : N \subseteq F \text{ with } \dim N < n\right\},$$

n-th Weyl number

$$x_n(S)=\sup\left\{a_n(SX) : X\in L(l_2,E) \text{ with } \|X\| \leq 1\right\},$$

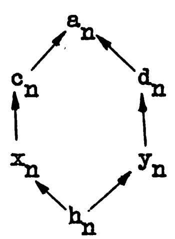
n-th Chang number

$$y_n(S)=\sup\left\{a_n(YS) : Y\in L(F,l_2) \text{ with } \|Y\| \leq 1\right\},$$

n-th Hilbert number

$$h_n(S)=\sup\left\{a_n(YSX) : X\in L(l_2,E), Y\in L(F,l_2) \text{ and } \|X\|, \|Y\| \leq 1\right\}.$$

The relationships between these s-functions will be indicated in the following diagram where the arrows point from the smaller s-numbers to the greater ones.



Furthermore there holds the inequality

$$h_n(S) \leq s_n(S) \leq a_n(S)$$

for any s-function s and all operators $S \in L$.

If we denote by S' the dual operator of S, then

$$x_n(S')=y_n(S),\ y_n(S')=x_n(S),\ h_n(S')=h_n(S)$$

for an arbitary operator, while the analogous equalities

$$c_n(S')=d_n(S),\ d_n(S')=c_n(S),\ a_n(S')=a_n(S)$$

are true, in general, for compact operators only.

Using s-functions one can define quasi-normed operator ideals (for the definition of quasi-normed operator ideals we refer to [62] 6.1.1).

Let $0<r<\infty$, $0<w\leq\infty$ and s be any s-function. Then the operator ideal $L^{(s)}_{r,w}(E,F)$ consists of all operators $S\in L(E,F)$ such that the sequence $(s_n(S))$ belongs to the Lorentz space $l_{r,w}$. In this case we have a quasi-norm given by

$$\| S|L^{(s)}_{r,w}\| := \|(s_n(S))|l_{r,w}\| .$$

REMARK. It is convenient to identify $L^{(s)}_{\infty,\infty}$ with L.

An important interpolation formula for the operator ideals $L^{(s)}_{r,w}$ is due to H. König [37] (cf. also [90]).

THEOREM 1. Let s be any s-function, $0<r_0<r_1\leq\infty$, $0<w,w_0,w_1\leq\infty$ and $0<\theta<1$. If $1/r=(1-\theta)/r_0+\theta/r_1$, then

$$(L^{(s)}_{r_0,w_0},\ L^{(s)}_{r_1,w_1})_{\theta,w} \subseteq L^{(s)}_{r,w} ,$$

and there exists a constant $c>0$, depending on the s-function and the numbers r,r_0,r_1,w,w_0,w_1 only such that

$$\| S|L^{(s)}_{r,w}\| \leq c\, \| S|L^{(s)}_{r_0,w_0}\|^{1-\theta} \| S|L^{(s)}_{r_1,w_1}\|^{\theta}$$

for all $S\in L(E,F)$.

It was shown by J. Peetre/G. Sparr [61] that in the case of approximation numbers the equality

$$(L^{(a)}_{r_0,w_0},\ L^{(a)}_{r_1,w_1})_{\theta,w} = L^{(a)}_{r,w}$$

holds with equivalent quasi-norms.

HISTORICAL REMARKS. At the beginning of the century E. Schmidt [73] introduced the concept of singular numbers of integral operators between Hilbert function spaces. J. v. Neumann/R. Schatten [59] extended this concept to the setting of compact operators between Hilbert spaces. The n-th singular number $s_n(S)$ of such an operator S is defined to be the n-th eigenvalue $\lambda_n(|S|)$. Here $(\lambda_n(|S|))$ denotes the non-increasing sequence of all eigenvalues of the operator $|S| = (S^* S)^{1/2}$ counted according to their algebraic multiplicities. It turns out that the singular numbers yield the unique s-function on the class of all compact operators acting between Hilbert spaces. From attempts to classify compact operators between Hilbert spaces H. Triebel [88] gave the definition of the so-called Schatten classes

$$S_{r,w}(H,K) = \left\{ S \in L(H,K) \; : \; (s_n(S)) \in l_{r,w} \right\} .$$

For w=r this goes back to J. v. Neumann/R. Schatten [59].
The following remarkable inequality is due to H. Weyl [92].
For any Hilbert space H and every operator $S \in S_r(H,H)$ it is

$$\Big(\sum_{n=1}^{\infty} |\lambda_n(S)|^r \Big)^{1/r} \leq \Big(\sum_{n=1}^{\infty} s_n(S)^r \Big)^{1/r} .$$

Therefore inequalities of this type nowadays are called also Weyl inequalities. The following results are known (cf. [37], [67]).
THEOREM 2. Let $0 < r < \infty$, $0 < w \leq \infty$ and s be one of the s-functions a, c, d, x, y. Then there exists a positive constant $c_{r,w}$ such that

$$\| (\lambda_n(S)) | l_{r,w} \| \leq c_{r,w} \| S | L^{(s)}_{r,w} \|$$

for all Banach spaces E and all $S \in L^{(s)}_{r,w}(E,E)$.
An important step in extending the concept of singular numbers of compact operators between Hilbert spaces to operators between

Banach spaces was the result of D. E. Alachverdiev [1] (cf. also [10]). He proved that the singular numbers of a compact operator between Hilbert spaces coincide with its approximation numbers. Later on A. Pietsch [63] defined the operator ideals $L_r^{(a)} := L_{r,r}^{(a)}$ as an extension of the Schatten classes.

The definition of Kolmogorov numbers was given by I. A. Novoselskij [60]. This concept is based on a paper of A. N. Kolmogorov [36] dealing with the so-called diameters or widths.

An axiomatic approach to the theory of s-function is due to A. Pietsch [66]. He also gave the definition of the Gelfand numbers as a dual concept to the Kolmogorov numbers and as corresponding quantities to Gelfand diameters which were defined by B. S. Mitjagin/V. M. Tichomirov [57].

A survey of the essential properties of the approximation numbers and the Kolmogorov numbers was already given by B. S. Mitjagin/A. Pełczyński [56].

The Hilbert numbers were introduced by W. Bauhardt [3], while A. Pietsch [67] defined the Weyl and Chang numbers.

2. s-Numbers of diagonal operators

Let $a=(\alpha_n)$ be any sequence belonging to the Lorentz space $l_{t,w}$. Then we consider the corresponding diagonal operator D_a, which is given by $D_a(\xi_n) = (\alpha_n \xi_n)$. If $1 \leq p,q \leq \infty$ and $1/t > (1/q - 1/p)_+$, then D_a is an operator from l_p into l_q. In the special case where $\alpha_n = n^{-1/t}$ we shall denote the corresponding diagonal operator by D_t. Since the l_p spaces are symmetric, without loss of generality, we can assume that $\alpha_1 \geq \alpha_2 \geq \dots \geq 0$.

Let s be any s-function. Then we ask the question for which r the diagonal operator D_a belongs to $L_{r,w}^{(s)}(l_p, l_q)$. The following theorem enables us to reduce this problem to the investigation of the asymptotic behaviour of the expression

$$\| I: l_p(m) \longrightarrow l_q(m) \mid L_{r,w}^{(s)} \| .$$

THEOREM 3. Let s be any s-function, and let $1 \leq p,q \leq \infty$. Further assume that the exponents r and t are related via the linear equation

$$1/r = \mu/t + \gamma$$

for $0 \leq t_0 < t < t_1 \leq (1/q - 1/p)_+^{-1}$, where the parameters μ and γ may depend upon p and q. Then the following are equivalent:

(1) If $0 < w \leq \infty$ and $t_0 < t < t_1$, then

$$a \in l_{t,w} \text{ implies } D_a \in L_{r,w}^{(s)}(l_p, l_q).$$

(2) If $t_0 < t < t_1$, then

$$s_n(D_t : l_p \longrightarrow l_q) \prec n^{-1/r}$$

for n=1,2,... .

(3) If $0 < w \leq \infty$ and $t_0 < t < t_1$, then

$$\| I: l_p(m) \longrightarrow l_q(m) \mid L_{r,w}^{(s)} \| \prec m^{1/t}$$

for m=1,2,... .

(4) The estimat (3) holds for $w=\infty$.

P r o o f: (1) $\longrightarrow$ (2): The assertion follows easily from the relation

$$n^{1/r} s_n(D_t : l_p \longrightarrow l_q) \leq \| D_t : l_p \longrightarrow l_q \mid L_{r,\infty}^{(s)} \|$$

for n=1,2,... .

(2) $\longrightarrow$ (3): Let $a \in l_{t,\infty}$. Then

$$s_n(D_a : l_p \longrightarrow l_q) \leq \sup_k | k^{1/t} \alpha_k | \, s_n(D_t : l_p \longrightarrow l_q)$$

implies $D_a \in L_{r,\infty}^{(s)}(l_p, l_q)$. According to the closed graph theorem there exists a constant $c > 0$ such that

$$\| D_a : l_p \longrightarrow l_q | L^{(s)}_{r,\infty} \| \leq c \, \| a | l_{t,\infty} \|$$

for all $a \in l_{t,\infty}$. This yields

$$\| I : l_p(m) \longrightarrow l_q(m) \,|\, L^{(s)}_{r,\infty} \| \leq c \, m^{1/t} .$$

Finally, the interpolation formula given in theorem 1 complete this step.

(3) $\longrightarrow$ (4): trivial.

(4) $\longrightarrow$ (1): Let t with $t_0 < t < t_1$. Since every quasi-norm is equivalent to an u-norm (cf. [62] 6.2.5) we may assume that $L^{(s)}_{r,\infty}$ is u-normed. Without loss of generality let $u \leq t$. Further we define for $k=0,1,\ldots$ the sets

$$N_k = \left\{ n \in N \, : \, 2^k \leq n < 2^{k+1} \right\} .$$

If $a \in l_{t,u}$, then it follows

$$\| a | l_{t,u} \|^u = \sum_{n=1}^{\infty} (n^{1/t - 1/u} \alpha_n)^u = \sum_{k=0}^{\infty} \sum_{n \in N_k} (n^{1/t - 1/u} \alpha_n)^u$$

$$\geq \sum_{k=0}^{\infty} 2^k (2^{(k+1)(1/t - 1/u)} \alpha_{2^{k+1}})^u = \frac{1}{2} \sum_{k=0}^{\infty} (2^{(k+1)/t} \alpha_{2^{k+1}})^u .$$

Let D_k be the operator defined by

$$D_k(\xi_n) = \begin{cases} \alpha_n \xi_n & \text{for } n \in N_k \\ 0 & \text{for } n \notin N_k \end{cases} .$$

Then $D_a = \sum_{k=0}^{\infty} D_k$. Hence the trivial fact $\| D_k : l_p \longrightarrow l_p \| = \alpha_{2^k}$ and the assumption lead to

$$\| D_a | L^{(s)}_{r,\infty} \| \leq \Big(\sum_{k=0}^{\infty} \| D_k | L^{(s)}_{r,\infty} \|^u \Big)^{1/u} \leq c \Big(\sum_{k=0}^{\infty} (2^{k/t} \alpha_{2^k})^u \Big)^{1/u}$$

$$\leq c \, \alpha_0 + c \Big(\sum_{k=0}^{\infty} (2^{(k+1)/t} \alpha_{2^{k+1}})^u \Big)^{1/u} \leq c \, \alpha_0 + 2^{1/u} c \, \| a | l_{t,u} \| < \infty .$$

Since this is true for all $t \in (t_0, t_1)$, again using the interpolation formula, we obtain statement (1).

In the sequel we shall apply this theorem to that s-functions introduced in section 1. Because of the duality relations it is enough to consider the s-functions a, c, x and h.

As recently proved by <u>E. D. Gluskin</u> [13],

$$a_n(I : l_p(m) \to l_q(m)) \asymp \max\left\{c_n(I : l_p(m) \to l_q(m)), d_n(I : l_p(m) \to l_q(m))\right\}.$$

So we can restrict ourselves to the s-functions c, x and h. First we shall review the known estimates of $s_n(I : l_p(m) \to l_q(m))$. For abbreviation we shall denote the identity operator I from $l_p(m)$ into $l_q(m)$ by $I^m_{p,q}$.

3. Gelfand numbers

The following theorem is taken from <u>E. D. Gluskin</u> [12], [13] (cf. also [66], [78]).

THEOREM 4.

(1) Let $1 \le q \le p \le \infty$. Then

$$c_n(I^m_{p,q}) = (m-n+1)^{1/q - 1/p} \qquad \text{for} \quad 1 \le n \le m .$$

(2.1) Let $2 \le p \le q \le \infty$. Then

$$c_n(I^m_{p,q}) \asymp \begin{cases} \left(\frac{m-n+1}{m}\right)^{\frac{1/p - 1/q}{1 - 2/q}} & \text{for} \quad 1 \le n \le m - m^{2/q} \\ m^{1/q - 1/p} & \text{for} \quad m - m^{2/q} \le n \le m . \end{cases}$$

(2.2) Let $1 < p \le 2 \le q \le \infty$. Then

$$c_n(I^m_{p,q}) \asymp \begin{cases} \left(\frac{m-n+1}{m}\right)^{1/2} & \text{for} \quad 1 \le n \le m^{2/p'} \\ \left(\frac{m-n+1}{m}\right)^{1/2} m^{1/p'} n^{-1/2} & \text{for} \quad m^{2/p'} \le n \le \frac{m}{m^{2/q - 1} + 1} \\ m^{1/q - 1/p} & \text{for} \quad \frac{m}{m^{2/q - 1} + 1} \le n \le m . \end{cases}$$

(2.3) Let $1 \le p \le q \le 2$. Then

$$c_n(I^m_{p,q}) \asymp \begin{cases} 1 & \text{for } 1 \le n \le m^{2/p'} \\ (m^{1/p'} n^{-1/2})^{\frac{1/p-1/q}{1/p-1/2}} & \text{for } m^{2/p'} \le n \le m \end{cases} .$$

In order to apply theorem 3, we deduce from the previous theorem the following

LEMMA 1. Let $1 \le p,q \le \infty$ and $0 < r < \infty$. Then

$$\| I^m_{p,q} \mid L^{(c)}_{r,\infty} \| \asymp m^{1/t}$$

with the following values of $1/t$:

(1) Let $1 \le q \le p \le \infty$. Then

$$1/t = 1/r - 1/p + 1/q .$$

(2.1) Let $2 \le p \le q \le \infty$. Then

$$1/t = 1/r .$$

(2.2) Let $1 < p \le 2 \le q \le \infty$. Then

$$1/t = \begin{cases} 1/r - 1/p + 1/2 & \text{for } 0 < r \le 2 \\ 2/(p'r) & \text{for } 2 < r < \infty . \end{cases}$$

(2.3) Let $1 \le p \le q \le 2$. Then

$$1/t = \begin{cases} 1/r - 1/p + 1/q & \text{for } 0 < r \le 2\,\frac{1/p-1/2}{1/p-1/q} \\ 2/(p'r) & \text{for } 2\,\frac{1/p-1/2}{1/p-1/q} < r < \infty . \end{cases}$$

To illustrate the following results, we shall indicate the algebraic expressions of $1/r$ as a function of t, p and q, in the unit square with the coordinates $1/p$ and $1/q$. On the ploted lines one, in general, only has $D_a \in L^{(s)}_{u,\infty}$ for all u with $u > r$.

THEOREM 5. If $1 \le p,q \le \infty$, $1/t > (1/q - 1/p)_+$, $0 < w \le \infty$ and $a \in l_{t,w}$, then D_a belongs to $L^{(c)}_{r,w}(l_p, l_q)$, where $1/r$ takes values as indicated in the following diagrams.

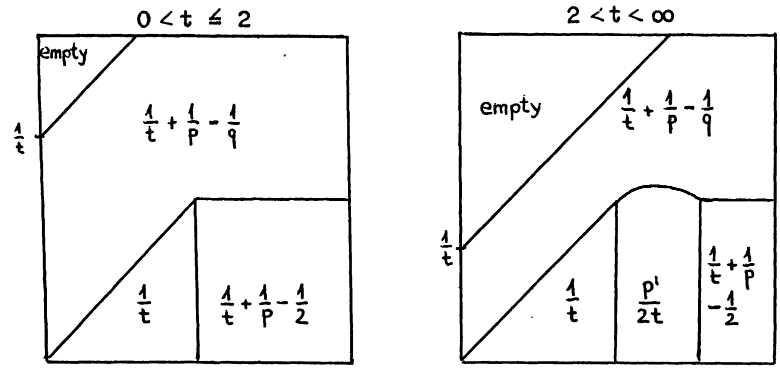


Figure 1

REMARK. The equation of the crooked line in the right diagram is given by $1/t = \frac{1/p - 1/q}{p'(1/p - 1/2)}$.

4. Approximation numbers

The next result follows immediately from the above mentioned relation between approximation numbers, Gelfand and Kolmogorov numbers.

THEOREM 6. If $1 \leq p,q \leq \infty$, $1/t > (1/q - 1/p)_+$, $0 < w \leq \infty$ and $a \in l_{t,w}$, then D_a belongs to $L^{(a)}_{r,w}(l_p, l_q)$, where $1/r$ takes values as indicated in the following diagrams.

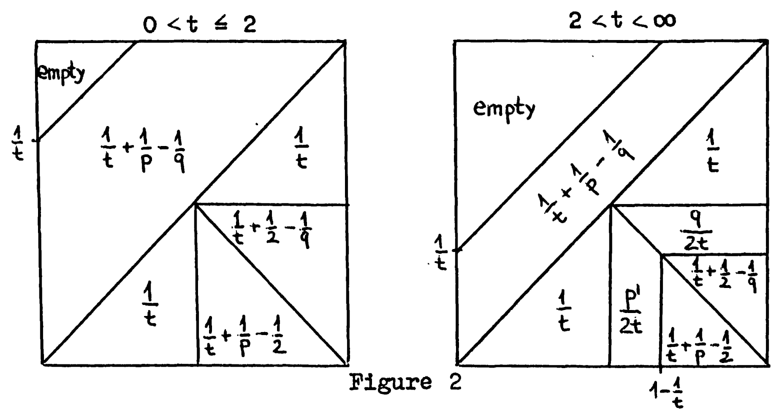


Figure 2

5. Weyl numbers

The following is known from [41] , [68].

THEOREM 7.

(1) Let $1 \leq p < \max(2,q) \leq \infty$ and $1 \leq n \leq m/2$. Then

$$x_n(I^m_{p,q}) \asymp \begin{cases} 1 & \text{for } 2 \leq p \leq q \leq \infty \\ n^{1/q-1/p} & \text{for } 1 \leq p \leq q \leq 2 \\ n^{1/2-1/p} & \text{for } 1 \leq p \leq 2 \leq q \leq \infty \\ m^{1/q-1/p} & \text{for } 1 \leq q \leq p \leq 2 \end{cases} .$$

(2) Let $\max(2,q) \leq p$. Then

$$x_n(I^m_{p,q}) \asymp \begin{cases} m^{1/q-1/p} & \text{for } 1 \leq n \leq m^{2/p} \\ m^{1/q}\, n^{-1/2} & \text{for } m^{2/p} \leq n \leq m \end{cases} .$$

From this we deduce the following behaviour of the corresponding quasi-norms.

LEMMA 2. Let $1 \leq p,q \leq \infty$ and $0 < r < \infty$. Then

$$\| I^m_{p,q} \mid L^{(x)}_{r,\infty} \| \asymp m^{1/t}$$

with the following values of $1/t$:

(1) Let $1 \leq p < \max(2,q) \leq \infty$. Then

$$1/t = \begin{cases} 1/r & \text{for } 2 \leq p \leq q \leq \infty \\ (1/r-1/p+1/q)_+ & \text{for } 1 \leq p \leq q \leq 2 \\ (1/r-1/p+1/2)_+ & \text{for } 1 \leq p \leq 2 \leq q \leq \infty \\ 1/r-1/p+1/q & \text{for } 1 \leq q \leq p \leq 2 \end{cases} .$$

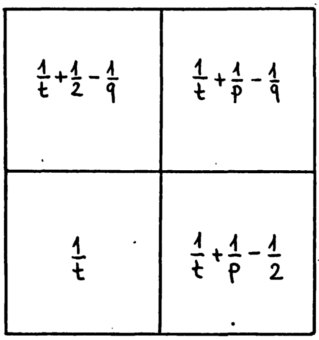
(2) Let $\max(2,q) \leq p$. Then

$$1/t = \begin{cases} 1/r-1/p+1/q & \text{for } 0 < r \leq 2 \\ 2/(pr)-1/p+1/q & \text{for } 2 < r < \infty . \end{cases}$$

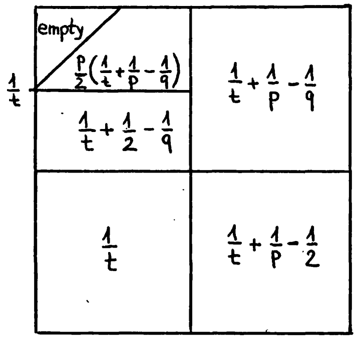
Therefore we get

THEOREM 8. If $1 \leq p,q \leq \infty$, $1/t > (1/q - 1/p)_+$, $0 < w \leq \infty$ and $a \in l_{t,w}$, then D_a belongs to $L^{(x)}_{r,w}(l_p, l_q)$, where $1/r$ takes values as indicated in the following diagrams.

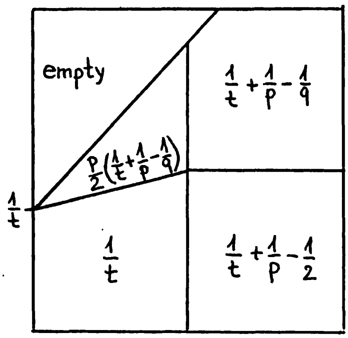
$0 < t \leq 1$



$1 < t \leq 2$



$2 < t < \infty$



6. Hilbert numbers

There are recent results concerning the asymptotic behaviour of Hilbert numbers of $I: l_p(m) \longrightarrow l_q(m)$. Since $h_n(S') = h_n(S)$, we can restrict ourselves to the case $1 \leq p' \leq q \leq \infty$. Then we have the well-known estimates [5], [18].

THEOREM 9.

(1) Let $1 \leq p' \leq q \leq 2$. Then

$$h_n(I^m_{p,q}) \asymp \begin{cases} m^{1/q-1/p} & \text{for} \quad 1 \leq n \leq m^{2/p} \\ m^{1/q}\, n^{-1/2} & \text{for} \quad m^{2/p} \leq n \leq m^{2/q'} \\ m\, n^{-1} & \text{for} \quad m^{2/q'} \leq n \leq m \end{cases} .$$

(2) Let $2 \leq q \leq p \leq \infty$. Then

$$h_n(I^m_{p,q}) \asymp \begin{cases} m^{1/q-1/p} & \text{for} \quad 1 \leq n \leq m^{2/p} \\ m^{1/q}\, n^{-1/2} & \text{for} \quad m^{2/p} \leq n \leq m \end{cases} .$$

(3) Let $2 \leq p' \leq q \leq \infty$. Then

$$h_n(I^m_{p,q}) \asymp n^{1/q-1/p} \quad \text{for} \quad 1 \leq n \leq m .$$

Using entropy numbers, for the open case $2 < p < q < \infty$, <u>B. Carl</u> (unpublished) could prove the following

PROPOSITION 1. Let $2 < p < q < \infty$. Then

$$h_n(I^m_{p,q}) \prec \begin{cases} \left(\frac{\log(m/n+1)}{n}\right)^{p/2(1/p-1/q)} & \text{for} \quad 1 \leq n \leq m^{2/p} \\ m^{1/q}\, n^{-1/2} & \text{for} \quad m^{2/p} \leq n \leq m \end{cases} .$$

It is also known that in the case $2 < p < q < \infty$ the inequality

$$h_n(I^m_{p,q}) \succ \begin{cases} m^{1/q-1/p} & \text{for} \quad 1 \leq n \leq m^{2/p} \\ m^{1/q}\, n^{-1/2} & \text{for} \quad m^{2/p} \leq n \leq m \end{cases}$$

is true (cf. [18]). This yields

LEMMA 3. Let $1 \leq p,q \leq \infty$ and $0 < r < \infty$. Then

$$\| I^m_{p,q} \mid L^{(h)}_{r,\infty} \| \asymp m^{1/t}$$

with the following values of $1/t$:

(1) Let $1 \leq p' \leq q \leq 2$. Then

$$1/t = \begin{cases} 1/r & \text{for } 0<r \leq 1 \\ 2/(q'r) - 1 + 2/q & \text{for } 1<r \leq 2 \\ 2/(pr) - 1/p + 1/q & \text{for } 2<r<\infty . \end{cases}$$

(2) Let $2 \leq p,q \leq \infty$. Then

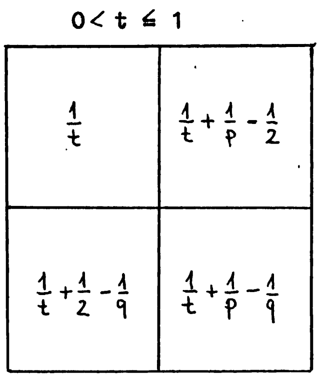
$$1/t = \begin{cases} 1/r - 1/2 + 1/q & \text{for } 0<r \leq 2 \\ (2/(pr) - 1/p + 1/q)_+ & \text{for } 2<r<\infty . \end{cases}$$

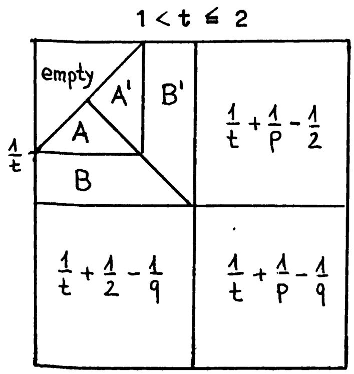
(3) Let $2 \leq p' \leq q \leq \infty$. Then

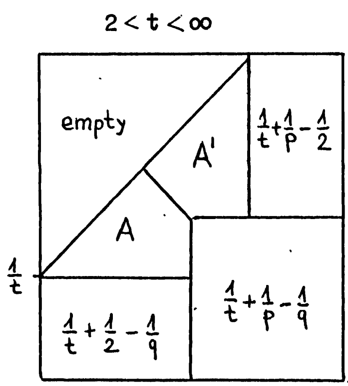
$$1/t = (1/r - 1/p + 1/q)_+ \qquad \text{for } 0<r<\infty .$$

Therefore we get

THEOREM 10. If $1 \leq p,q \leq \infty$, $1/t > (1/q - 1/p)_+$, $0<w \leq \infty$ and $a \in l_{t,w}$, then D_a belongs to $L^{(h)}_{r,w}(l_p,l_q)$, where $1/r$ takes values as indicated in the following diagrams.







$$A = \frac{p}{2}\left(\frac{1}{t} + \frac{1}{p} - \frac{1}{q}\right)$$

$$A' = \frac{q'}{2}\left(\frac{1}{t} + \frac{1}{p} - \frac{1}{q}\right)$$

$$B = \frac{q'}{2}\left(\frac{1}{t} + 1 - \frac{2}{q}\right)$$

$$B' = \frac{p}{2}\left(\frac{1}{t} + \frac{2}{p} - 1\right)$$

Figure 4

HISTORICAL REMARKS. The first results about s-numbers of diagonal operators are due to soviet mathematicians. These results are formulated in the language of diameters of subsets of normed spaces.

Already in 1954 S. B. Stečkin [81] showed that

$$d_n(I:l_1(m) \longrightarrow l_2(m)) = \left(\frac{m-n+1}{m}\right)^{1/2}$$

Later on M. I. Stesin [80] was able to prove

$$d_n(I:l_p(m) \longrightarrow l_q(m)) = (m-n+1)^{1/q-1/p}$$

for $1 \leq q \leq p \leq \infty$. After the axiomatic approach to the s-number theory in 1974, many further estimates of this kind were obtained. First A. Pietsch [66] extended the result of Stesin to approximation numbers and Gelfand numbers in the following way. If $1 \leq q \leq p \leq \infty$, then

$$a_n(I^m_{p,q}) = c_n(I^m_{p,q}) = d_n(I^m_{p,q}) = (m-n+1)^{1/q-1/p} .$$

The behaviour of these three s-functions in the case $1 \leq p < q \leq \infty$ had been unknown for a long time. Using number theoretical methods R. S. Ismagilov [24] obtained the estimate

$$d_n(I:l_p(m) \longrightarrow l_\infty(m)) \prec \left(\frac{m^{1/2}}{\quad}\right)^{1/p}$$

for $1 \leq p \leq \infty$. With the help of random matrices B. S. Kashin [32] and B. S. Mitjagin [55] improved this for p=1 as follows

$$a_n(I:l_1(m) \rightarrow l_\infty(m)) = d_n(I:l_1(m) \rightarrow l_\infty(m)) \prec n^{-1/2}(\ln \tfrac{em}{n})^{1/2} .$$

It turned out that random matrices are an useful tool for these investigations. So B. Carl/A. Pietsch [5] could prove the estimate

$$a_n(I:l_p(m) \rightarrow l_{p'}(m)) \prec n^{-3/2+1/p} m^{2/p'} (\ln(2m))^{1/p-1/2}$$

for $1 \leq p \leq 2$. By deducing this problem to a combinatorical one, K. Höllig [19] was able to improve the result of Carl and Pietsch.

He obtained the estimate

$$a_n(I:l_p(m) \to l_{p'}(m)) \prec n^{-3/2 + 1/p} m^{2/p'} \left(\frac{\ln m}{\ln n}\right)^{2/p - 1}$$

for $1 \leq p \leq 2$. Recently E. D. Gluskin [13] succeded in solving the whole problem. He determind the exact asymptotic behaviour of $s_n(I:l_p(m) \to l_q(m))$ for s equal to a, c or d.
Moreover his results imply the relation

$$a_n(I:l_p(m) \to l_q(m)) \asymp \max\left\{c_n(I:l_p(m) \to l_q(m)), d_n(I:l_p(m) \to l_q(m))\right\}$$

already mentioned above.
The concept of Weyl and Chang numbers was developed by A. Pietsch [67] in 1977. He also gave the first estimates for these numbers. Then it was C. Lubitz [41] who determined all r such that the diagonal operator D_a with $a \in l_{t,w}$ belongs to $L^{(x)}_{r,w}(l_p, l_q)$. For this purpose he used the results of Pietsch and Gluskin.
The first estimates of Hilbert numbers go back to W. Bauhardt [3] and B. Carl/A. Pietsch [5]. Recently by results of S. Heinrich/R. Linde [18] and B. Carl, it was possible to calculate the asymptotic behaviour of Hilbert numbers of diagonal operators.

7. s-Numbers of Besov embeddings

Let Ω be a sufficiently smooth region of R^N. The main theorem in the theory of Besov spaces is the following.

THEOREM 11. Let $1 \leq p,q \leq \infty$ and $\varrho, \sigma \in R$ such that $(\varrho - \sigma)/N > (1/p - 1/q)_+$. Then $B^\varrho_p(\Omega)$ is continously embedded in $B^\sigma_q(\Omega)$. Moreover the embedding operator is compact.

For the definition of $B^\varrho_p(\Omega) := B^\varrho_{p,p}(\Omega)$ we refer to the monograph of H. Triebel [91].
Using the results of Z. Ciesielski [6], S. Ropela [71] and Z. Ciesielski/T. Figiel [7] about spline bases, it turns out

that up to finite dimensional perturbation there exist two invertible operators U and V such that the following diagram becomes commutative:

$$\begin{array}{ccc} B_p^{\varrho}(\Omega) & \xrightarrow{\quad J \quad} & B_q^{\sigma}(\Omega) \\ U \downarrow \uparrow U^{-1} & & V^{-1} \uparrow \downarrow V \\ l_p & \xrightarrow{\quad D_t \quad} & l_q \end{array}$$

Here the number t is determined by the relation

$$1/t = (\varrho - \sigma)/N - 1/p + 1/q .$$

So it is possible to carry over all results concerning diagonal operators to Besov embeddings.

HISTORICAL REMARKS. As already mentioned, A. N. Kolmogorov [36] introduced in 1936 the concept of the n-th (Kolmogorov) diameter $d_n(K,F)$ of a subset K of a normed space F,

$$d_n(K,F) = \inf_{\substack{N \subseteq F \\ \dim N < n}} \ \sup_{x \in K} \ \inf_{y \in N} \| x-y | F \| .$$

To compute Kolmogorov diameters of the unit ball of Sobolev spaces V. E. Majorov [42] used a discretization method (cf. also E. D. Gluskin [11]).

In this paper the method of discretization (finitization) with the help of quasi-norms is presented for the first time.

It holds the following relation between the n-th Kolmogorov number and the n-th Kolmogorov diameter. If $S \in L(E,F)$, then

$$d_n(S) = d_n(S(U_E),F).$$

In this notation the first results are due to Kolmogorov himself. He proved that

$$d_n(J: W_2^{\varrho}[0,1] \to L_2[0,1]) \asymp n^{-\varrho} .$$

In the sequel the main interest was concentrated on the so-called Sobolev embedding $J:W_p^{\varrho}(\Omega) \longrightarrow L_q(\Omega)$, which exists for $\varrho/N > (1/p - 1/q)_+$. The following table contains the asymptotic behaviour of $d_n(J:W_p^{\varrho}[0,1] \longrightarrow L_q[0,1])$.
Simultaneously it gives a historical order of these results. Let $1 \le p,q \le \infty$ and $\varrho > (1/p - 1/q)_+$. Then

$$d_n(J:W_p^{\varrho}[0,1] \longrightarrow L_q[0,1]) \asymp n^{-f(\varrho,p,q)}$$

p,q	ϱ	$f(\varrho,p,q)$	author
$p=q=2$	$0<\varrho<\infty$	ϱ	Kolmogcrov [36]
$p=\infty$, $q=\infty$ $q=2$	$0<\varrho<\infty$	ϱ	Stečkin [81]
$1\le p=q\le\infty$	$0<\varrho<\infty$	ϱ	Babažanov/ Tichomirov [2]
$1\le p\le 2$, $q=2$	$\frac{1}{p}-\frac{1}{2}<\varrho<\infty$	$\varrho-\frac{1}{p}+\frac{1}{2}$	Ismagilov [23]
$p=1$, $q=\infty$	$1<\varrho<\infty$	$\varrho-\frac{1}{2}$	Joffe/ Tichomirov [28]
$p=\infty, 1\le q\le\infty$	$0<\varrho<\infty$	ϱ	Makovoz [47]
$1\le q\le p\le\infty$	$0<\varrho<\infty$	ϱ	Tichomirov [86]
$1\le p\le q\le 2$	$\frac{1}{p}-\frac{1}{q}<\varrho<\infty$	$\varrho-\frac{1}{p}+\frac{1}{q}$	Ismagilov [24]
$1\le p\le 2\le q\le\infty$	$1<\varrho<\infty$	$\varrho-\frac{1}{p}+\frac{1}{2}$	Kashin [32]
$2\le p\le q\le\infty$	$\frac{1}{2}<\varrho<\infty$	ϱ	
$p=1, 2\le q\le\infty$	$1-\frac{1}{q}<\varrho<1$	$\frac{q}{2}(\varrho-1+\frac{1}{q})$	Kashin [35]
$1\le p\le 2\le q\le\infty$	$\frac{1}{p}<\varrho\le 1$	$\varrho-\frac{1}{p}+\frac{1}{2}$	Gluskin [12] Lubitz [41]
	$\frac{1}{p}-\frac{1}{q}<\varrho\le\frac{1}{p}$	$\frac{q}{2}(\varrho-\frac{1}{p}+\frac{1}{q})$	
$2\le p\le q\le\infty$	$\frac{\frac{1}{p}-\frac{1}{q}}{1-\frac{2}{q}}<\varrho\le\frac{1}{2}$	ϱ	
	$\frac{1}{p}-\frac{1}{q}<\varrho\le\frac{\frac{1}{p}-\frac{1}{q}}{1-\frac{2}{q}}$	$\frac{q}{2}(\varrho-\frac{1}{p}+\frac{1}{q})$	

References

[1] Alachverdiev, D. E.: On the degree of approximation of completely continous operators by finite dimensional operators (Russian), Scientific publ. of state univ. of Aserbaidshan 2 (1957), 27-37.

[2] Babadžanov, S. B./Tichomirov, V. M.: Diameters of a function class in an L_p-space (Russian), Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk (1967), 24-30.

[3] Bauhardt, W.: Hilbert-Zahlen von Operatoren in Banachräumen, Math. Nachr. 79 (1977), 181-187.

[4] Birman, M. S./Solomjak, M. Z.: Approximation of functions of the W_p^{α}-classes by piecewice-polynomial functions (Russian), Mat. Sb. 73 (1967), 331-355.

[5] Carl, B./Pietsch, A.: Some contributions to the theory of s-numbers, Comment. Math. Prace Mat. 21 (1978), 65-76.

[6] Ciesielski, Z.: Constructive function theory and spline systems, Studia Math. 53 (1975), 277-302.

[7] Ciesielski, Z./Figiel, T.: Spline approximation and Besov spaces on compact manifolds, Studia Math. 75 (1982), 13-36.

[8] Edmunds, D. E.: Embedding of Sobolev spaces, Nonlinear analysis, function spaces and applications (Proc. Spring School, Horni Bradlo, 1978), pp. 38-58, Leipzig 1979.

[9] Edmunds, D. E./Moscatelli, V. B.: Fourier approximation and embeddings of Sobolev spaces, Diss. Math. 145 (1977).

[10] Fiedler, M./Pták, V.: Sur la meilleure approximation des transformations linéaires par des transformation de rang prescrit, C. R. Acad. Sci. Paris 254 (1962), 3805-3807.

[11] Gluskin, E. D.: On a problem concerning diameters (Russian), Dokl. Akad. Nauk SSSR 219 (1974), 527-530.

[12] Gluskin, E. D.: On some finite dimensional problems of the theory of diameters (Russian), Vestnik Leningr. Univ. 13 (1981), 5-10.

[13] Gluskin, E. D.: Norms of random matrices and diameters of finite dimensional sets (Russian), Mat. Sb. 120 (1983), 180-189.

[14] Gohberg, I. C./Krein, M. G.: Introduction to the theory of linear nonselfadjoint operators in Hilbert spaces, Moskau 1965, Providence 1969, Paris 1971.

[15] Gohberg, I. C./Krein, M. G.: The basic propositions on defect numbers, root numbers and indices of linear operators (Russian), Uspehi Mat. Nauk 12 (2) (1957), 43-118.

[16] Golitschek, M. von: On n-widths and interpolation by polynomial splines, J. Approx. Theory 26 (1979), 132-141.

[17] Ha, Chung-Wei: Approximation numbers of linear operators and nuclear spaces, J. Math. Anal. Appl. 46 (1974), 292-311.

[18] Heinrich, S./Linde, R.: On the asymptotic behaviour of Hilbert numbers, Math. Nachr. 119 (1984), 117-120.

[19] Höllig, K.: Approximationszahlen von Sobolev-Einbettungen, Math. Ann. 242 (1979), 273-281.

[20] Höllig, K.: Diameters of classes of smooth functions, Quantitative approximation (Proc. Internat. Sympos., Bonn 1979), pp. 163-175, New York, 1980.

[21] Hutton, C. V.: On the approximation numbers of an operator and its adjoint, Math. Ann. 210 (1974), 277-280.

[22] Hutton, C. V./Morrell, J. S./Retherford, J. R.: Diagonal operators, approximation numbers, and Kolmogorov diameters, J. Approx. Theory 16 (1976), 48-80.

[23] Ismagilov, R. S.: n-dimensional diameters of compacta in a Hilbert space (Russian), Funkcional. Anal. i Priložen. 2 (2) (1968), 32-39.

[24] Ismagilov, R. S.: Diameters of sets in linear normed spaces and approximation of functions by trigonometrical polynomials (Russian), Uspehi Mat. Nauk 29 (3) (1974), 161-178.

[25] Ismagilov, R. S.: Diameters of compact sets in linear normed spaces (Russian), Collection of papers on "Geometry of linear spaces and operator theory", pp. 75-113, Jaroslavl 1977.

[26] Ismagilov, R. S./Nasyrova, H.: Diameters of a class of smooth functions in the space L_2 (Russian), Mat. Zametki 22 (1977), 671-678.

[27] Jerome, J. W.: Asymptotic estimates of n-widths in Hilbert spaces, Proc. Amer. Math. Soc. 33 (1972), 367-372.

[28] Joffe, A. D./Tichomirov, V. M.: Duality of convex functions and extremal problems (Russian), Uspehi Mat. Nauk 23 (1968), 51-116.

[29] Kadlec, J./Korotkov, V. B.: Estimates of the s-numbers of imbedding operators and operators with increasing smoothness, Czechoslovak Math. J. 18 (1968), 678-699.

[30] Kashin, B. S.: Kolmogorov diameters of octahedra (Russian), Dokl. Akad. Nauk SSSR 214 (1974), 1024-1026.

[31] Kashin, B. S.: The diameters of octahedra (Russian), Uspehi Mat. Nauk 30 (4) (1975), 251-252.

[32] Kashin, B. S.: Diameters of some finite dimensional sets and of some classes of smooth functions (Russian), Izv. Akad. Nauk SSSR, Ser. Mat., 41 (1977), 334-351.

[33] Kashin, B. S.: On estimates of diameters, Quantitative approximation (Proc. Internat. Sympos., Bonn 1979), pp. 177-184, New York 1980.

[34] Kashin, B. S.: Some properties of the space of trigonometric polynomials with a uniform norm (Russian), Trudy Mat. Inst. Steklov 145 (1980), 111-116.

[35] Kashin, B. S.: On diameters of Sobolev classes of small order smoothness (Russian), Vestnik Moskov. Univ. 5 (1981), 50-54.

[36] Kolmogorov, A. N.: Über die beste Annäherung von Funktionen einer gegebenen Funktionenklasse, Ann. of Math. 37 (1936), 107-110.

[37] König, H.: s-Zahlen und Eigenwerte von Operatoren in Banachräumen, Thesis, Bonn 1977.

[38] König, H.: Approximation numbers of Sobolev imbeddings over unbounded domains, J. Functional Analysis 29 (1978), 74-87.

[39] König, H.: s-numbers of Besov-Lorentz imbeddings, Math. Nachr. 91 (1979), 389-400.

[40] König, H.: s-numbers, eigenvalues and the trace theorem in Banach spaces, Studia Math. 67 (1980), 157-171.

[41] Lubitz, C.: Weylzahlen von Diagonaloperatoren und Sobolev-Einbettungen, Thesis, Bonn 1982.

[42] Majorov, V. E.: Discretization of the problem of diameters (Russian), Uspehi Mat. Nauk 30 (1975), 179-180.

[43] Majorov, V. E.: About the best approximation of the class $W_1^r(I^s)$ in the space $L_\infty(I^s)$ (Russian), Mat. Zametki 19 (1976), 699-706.

[44] Majorov, V. E.: Linear diameters of Sobolev classes (Russian), Dokl. Akad. Nauk SSSR 243 (1978), 1127-1130.

[45] Majorov, V. E.: Extremal chains of subspaces for Kolmogorov and linear diameters (Russian), Funkcional. Anal. i Priložen. 13 (1979), 91-92.

[46] Majorov, V. E.: Linear diameters of Sobolev classes and chains of extremal subspaces (Russian), Mat. Sb. 113 (1980), 437-463.

[47] Makovoz, Ju. I.: Theorems about the best approximation and diameters of sets in Banach spaces (Russian), Thesis, Minsk 1969.

[48] Makovoz, Ju. I.: A certain method of obtaining lower estimates for diameters of sets in Banach spaces (Russian), Mat. Sb. 87 (1972), 136-142.

[49] Markus, A. S.: Eigenvalues and singular values of the sum and the product of linear operators (Russian), Uspehi Mat. Nauk 19 (4) (1964), 93-123.

[50] Melkman, A. A.: n-widths of octahedra, Quantitative approximation (Proc. Internat. Sympos., Bonn 1979), pp. 209-216, New York 1980.

[51] Micchelli, C. A./Pinkus, A.: The exact asymptotic value for the N-width of smooth functions in L^{∞}, Approximation Theory (Proc. Internat. Sympos. Univ. Texas 1976), pp. 469-474, New York 1976.

[52] Micchelli, C. A./Pinkus, A.: Total positivity and the exact n-width of certain sets in L^1, Pacific J. Math. 71 (1977), 499-515.

[53] Micchelli, C. A./Pinkus, A.: On n-widths in L^{∞}, Trans. Amer. Math. Soc. 234 (1977), 139-174.

[54] Mitjagin, B. S.: Approximative dimension and bases in nuclear spaces (Russian), Uspehi Mat. Nauk 16 (4) (1961), 63-132.

[55] Mitjagin, B. S.: Random matrices and subspaces (Russian). Collection of papers on "Geometry of linear spaces and operator theory", pp. 175-202. Jaroslavl 1977.

[56] Mitjagin, B. S./Pełczyński, A.: Nuclear operators and approximative dimension, Proc. of ICM, Moscow 1966, 366-372.

[57] Mitjagin, B. S./Tichomirov, V. M.: Asymptotical characteristics of compact sets in linear spaces (Russian), Trudy IV. Vsesojus. Mat. Sezda, Leningrad 1961, 2 (1964), 299-308.

[58] Mitjagin, B. S./Chenkin, G. M.: Inequalities between certain n-diameters (Russian), Trudy Sem. Funkt. An., Voroneš 7 (1963), 93-103.

[59] Neumann, J. von/Schatten, R.: The cross-space of linear transformations, Ann. of Math. 47 (1946), 608-630; 49 (1948), 557-582.

[60] Novoselskij, I. A.: About certain asymptotical characteristics of linear operators (Russian), Izv. Akad. Nauk Moldav. SSR 6 (1964), 85-90.

[61] Peetre, J./Sparr, G.: Interpolation of normed Abelian groups, Ann. Mat. Pura Appl. 42 (1972), 217-262.

[62] Pietsch, A.: Operator ideals, Berlin 1978.

[63] Pietsch, A.: Einige neue Klassen von kompakten linearen Abbildungen, Rev. Roumaine Math. Pures Appl. 8 (1963), 427-447.

[64] Pietsch, A.: Ideale von S_p-Operatoren in Banachräumen, Studia Math. 38 (1970), 59-69.

[65] Pietsch, A.: r-nukleare Sobolevsche Einbettungsoperatoren, Elliptische Differentialgleichungen II, pp. 203-215, Schriftenreihe Inst. Math. Deutsch. Akad. Wissensch. Berlin, Reihe A, Heft 8, Berlin 1971.

[66] Pietsch, A.: s-numbers of operators in Banach spaces, Studia Math. 51 (1974), 201-223.

[67] Pietsch, A.: Weyl numbers and eigenvalues of operators in Banach spaces, Math. Ann. 247 (1980), 149-168.

[68] Pietsch, A.: Eigenvalues of integral operators I, Math. Ann. 247 (1980), 169-178.

[69] Pinkus, A.: Matrices and n-wdths, Lin. Alg. Appl. 27 (1979), 245-278.

[70] Pinkus, A.: On n-widths of periodic functions, J. Analyse Math. 35 (1979), 209-235.

[71] Ropela, S.: Spline bases in Besov spaces, Bull. Acad. Polon. Sci., Ser. Math. Astr. Phys. 24 (1976), 319-325.

[72] Rosenberger, B.: Approximationszahlen, p-nukleare Abbildungen und Hilbertraumcharakterisierungen, Math. Ann. 213 (1975), 211-221.

[73] Schmidt, E.: Zur Theorie der linearen und nichtlinearen Integralgleichungen, Math. Ann. 63 (1907), 433-476; 64 (1907), 161-174.

[74] Scholz, R.: Abschätzungen linearer Durchmesser in Sobolev und Besov-Räumen, Manuscripta Math. 11 (1974), 1-14.

[75] Scholz, R.: Durchmesserabschätzungen für die Einheitskugel des Sobole-Raumes $W_q^r(\Omega)$ in $L_p(\Omega)$, Applicable Analysis 5 (1975/76), 257-264.

[76] Solomjak, M. Z./Tichomirov, V. M.: Geometric characteristics of the imbedding map from W_p^α in C (Russian), Izv. Vysš. Učebn. Zaved. Mat. 10 (1967), 76-82.

[77] Stehling, W.: Approximationszahlen für Diagonaloperatoren zwischen Orliczräumen, Math. Nachr. 93 (1979), 165-176.

[78] Stesin, M. I.: The Aleksandrov diameters of finite dimensional octahedra (Russian), Vestnik Moskov. Univ. Ser. Mat. Meh. 28 (3) (1973), 30-35.

[79] Stesin, M. I.: The Aleksandrov diameters of balls in Banach spaces, (Russian), Uspehi Mat. Nauk 28 (6) (1973), 219-220.

[80] Stesin, M. I.: Aleksandrov diameters of finite dimensional sets and classes of smooth functions, (Russian), Dokl. Akad. Nauk SSSR 220 (1975), 1278-1281.

[81] Stečkin, S. B.: On the best approximation of given classes of functions by arbitary polynomials (Russian), Uspehi Mat. Nauk 9 (1) (1954), 133-134.

[82] Stečkin, S. B.: Approximation properties of sets in normed linear spaces (Russian), Rev. Math. Pures Appl. 8 (1963), 5-18.

[83] Tichomirov, V. M.: Some problems on approximation theory (Russian), Moscow 1976.

[84] Tichomirov, V. M.: Diameters of sets in function spaces and theory of best approximation (Russian), Uspehi Mat. Nauk 15 (3) (1960), 81-120.

[85] Tichomirov, V. M.: A remark on n-dimensional diameters of sets in Banach spaces (Russian), Uspehi Mat. Nauk 20 (1) (1965), 227-230.

[86] Tichomirov, V. M.: Certain problems in approximation theory (Russian), Mat. Zametki 9 (1971), 593-607.

[87] Tichomirov, V. M./Ismagilov, R. S./Babadžanov, S. B.: Geometry of Banach spaces and diameters of sets (Russian), Izv. Akad. Nauk. UzSSR Ser. Fiz.-Mat. Nauk (4) (1979), 25-32.

[88] Triebel, H. Über die Verteilung der Approximationszahlen kompakter Operatoren in Sobolev-Besov-Räumen, Invent. Math. 4 (1967), 275-293.

[89] Triebel, H.: Über die Approximationszahlen der Einbettungsoperatoren, Arch. Math. (Basel) 19 (1968), 305-312.

[90] Triebel, H.: Interpolationseigenschaften von Entropie- und Durchmesseridealen kompakter Operatoren, Studia Math. 34 (1970), 89-107.

[91] Triebel, H.: Interpolation Theory, Function Spaces, Differential Operators, Berlin 1978.

[92] Weyl, H.: Inequalities between the two kinds of eigenvalues of a linear transformation, Proc. Nat. Acad. Sci. USA 35 (1949), 408-411.