Measure representations: Existence, Continuity and Axiom (D)

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In classical potential theory the sheaf of harmonic functions is associated to the Laplace operator:

$$h \text{ harmonic} \Leftrightarrow \Delta h = 0.$$

Similarly the sheaf of caloric functions is associated to the heat equation $\mathbf{\Delta} := \Delta - \frac{\partial}{\partial t} = 0$

$$h \text{ caloric} \Leftrightarrow \mathbf{\Delta} h = 0.$$

Conversely Bony [4] showed how harmonic structures on open subsets of $\mathbb{R}^n$ - satisfying some additional regularity assumptions - can be defined by a differential operator (at least on a dense subset).

In the general theory of harmonic spaces however the starting-point is an abstract sheaf of functions - called harmonic or hyperharmonic - without any intervention of a defining partial differential operator. Yet, in some situations it is useful or even necessary to associate operators - substitutes for the differential operators in $\mathbb{R}^n$ - to given harmonic structures. Obviously the following two conditions on the construction of such an operator σ are stringent

- the kernel of σ should consist exactly of the harmonic functions;
- σ should associate to potentials lying in its domain of definition their "mass", the notion of "mass" is clear for

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Newtonian potentials on $\mathbb{R}^n$, but must be defined precisely for potentials on arbitrary harmonic spaces.

In a series of papers (see [8] and the bibliography cited there) F.-Y. Maeda has developed a theory of Dirichlet integrals on those harmonic spaces X, to which such a defining operator can be associated, for which he coined the notion of ***measure representation*** . By definition a ***measure representation*** is a homomorphism

$$\sigma = (\sigma_U)_{U\in\mathfrak{U}} : \mathcal{R} = (\mathcal{R}(U))_{U\in\mathfrak{U}} \to \mathcal{M} = (\mathcal{M}(U))_{U\in\mathfrak{U}}$$

of the sheaf $\mathcal{R}$ of local differences of continuous superharmonic functions into the sheaf $\mathcal{M}$ of signed Radon measures such that

$$\sigma(f) \geq 0 \Leftrightarrow f \text{ is superharmonic} \qquad (f \in \mathcal{R}(U), U \in \mathfrak{U}),$$

($\mathfrak{U}$ denotes the family of all open subsets U of X).
Of course, the Laplace operator defines a measure representation of the classical harmonic sheaf. More generally, if $\mathcal{H}_L$ is the kernel of a differential operator L defined on an open subset X of $\mathbb{R}^n$ such that $(X,\mathcal{H}_L)$ is a harmonic space, then

$$\sigma : f \mapsto - Lf \text{ (in the distribution sense)}$$

defines a measure representation of this harmonic space.
Conversely Maeda has proved the following "Bony type" result - under the assumption of the existence of a measure representation - if X is a harmonic space, $X \subset \mathbb{R}^n$ open, such that the constant 1 and the coordinate functionals $\pi_1,\dots,\pi_n$ belong to $\mathcal{R}(X)$, then there exists a differential operator L whose coefficients are measures such that

$$L(h) = 0 \Leftrightarrow h \text{ harmonic on } U$$

for every C^2-function h defined on an open subset U of X.

In this survey paper I want to present the existence theorem of measure representations on arbitrary harmonic spaces (in the sense of [2] or [5] having a countable base of their topology) and sketch its proof (see also [15]), then study some continuity properties of σ and finally give a characterization of Axiom (D) via a "fine-local property" and the "bounded energy principle". The notation used is that of the book [5], but all harmonic spaces X will be assumed to have a ***countable base*** of their topology. In contrast with [5], $\mathcal{P}_o(U)$, $\mathcal{P}_c(U)$ and $\mathcal{P}_b(U)$ will stand for the cones of continuous potentials with compact support, continuous potentials and bounded potentials on an open subset U of X respectively. $\mathfrak{K}(U)$ denotes the space of all continuous functions with compact support on U.

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1. The existence theorem for measure representations

In [15] the following result was proved:

THEOREM. ***Every harmonic space admits a measure representation.***

In the sequel I shall recall the construction of a measure representation σ in the special case of a ***strong*** or $\mathfrak{P}$-harmonic space $(X,\mathcal{H}^*)$. A measure representation σ of an arbitrary harmonic space can subsequently be obtained by "glueing $\mathfrak{P}$-harmonic pieces together" using a continuous partition of the constant function 1.

According to the extension theorem ([2], p. 159 or [5], p. 46) for every open subset U of a $\mathfrak{P}$-harmonic space X a continuous function f on U belongs to $\mathcal{R}(U)$ iff the following condition

holds:
For every open subset V such that $\overline{V}$ is compact $\subset U$ there exist $u,v \in \mathcal{P}_o(X)$ such that

$$f = u - v \text{ on } V .$$

This remark indicates the steps to be carried out in the existence proof of measure representations:

1) a) Associate measures $\sigma(p)$ in a "reasonable way" to potentials $p \in \mathcal{P}_o(X)$.
   b) Verify the ***"measure representation property"***
   $\sigma(p) - \sigma(p') \geq o \Leftrightarrow p - p'$ is superharmonic (i.e. $p' \prec p$)

2) a) To $f \in \mathcal{R}(U)$ such that $f = u - v$ on V (where U,V open, $V \subset U$ and $u,v \in \mathcal{P}_o(X)$) associate a measure $\sigma(f)$ which equals $\sigma(u) - \sigma(v)$ on V.
   b) Verify that σ is well-defined and has the required properties.

In the following I want to concentrate on the first problem. Consider for a moment the special case that $(X,\mathcal{H}^*)$ admits a Green function G, i.e.

$$G : X \times X \to \overline{\mathbb{R}}_+$$

is continuous off the diagonal and every potential p can be represented as

$$p(\cdot) = G^{\mu}(\cdot) := \int G(\cdot,y)\mu(dy)$$

with a unique positive Radon measure $\mu = \mu_p$. Then - according to the uniqueness of the representing measures -

$$p \longmapsto \mu_p$$

is well-defined and satisfies the measure representation property for ***all*** potentials (not just for the continuous ones).

Let us return to the general situation and consider the following way of assigning measures: let $\overline{\mu}$ be a ***reference measure*** on X, i.e. a positive Radon measure such that

(*) $0 < \int p d\overline{\mu} < \infty$ for every $p \in \mathcal{P}_0(X)$, which is not identically 0,

or - more generally - a ***strictly positive H-integral*** (see [3]); i.e. an additive, positive-homogeneous, increasing functional

$$\overline{\mu} : \mathcal{S}_+(X) \to \overline{\mathbb{R}}_+$$

which is continuous in order from below and satisfies the positivity condition (*). (It will soon become clear, why this generalization to H-integrals is reasonable). For $f = p - p'$, where $p, p' \in \mathcal{S}_+(X)$, the symbols

$$\int f d\overline{\mu} \quad \text{or } \overline{\mu}(f) = \overline{\mu}(p) - \overline{\mu}(p')$$

are used interchangeably, provided that $\overline{\mu}(p') < \infty$.

REMARK. Such measures $\overline{\mu}$ always exist; take for example $\overline{\mu} = \sum_{n=1}^{\infty} \lambda_n \varepsilon_{x_n}$, where $\{x_n : n \in \mathbb{N}\}$ is a dense subset of X and $(\lambda_n)_{n \in \mathbb{N}}$ is a sequence of strictly positive real numbers such that $\sum_{n=1}^{\infty} \lambda_n p(x_n)$ converges for one strictly positive potential p.

To $p \in \mathcal{P}_b(X)$ a Radon measure $\sigma(p)$ is assigned by

$$\sigma(p)(\varphi) := \overline{\mu}(\varphi \circledcirc p), \ \varphi \in \mathcal{K}_+(X),$$

(measure representations in "***standard form***") or - more generally -

$$\sigma(p)(\varphi) := \overline{\mu}((g\varphi) \circledcirc p), \ \varphi \in \mathcal{K}_+(X),$$

where $\circledcirc$ denotes the specific multiplication (see [5], §§ 8.1,8.2) and g is a fixed strictly positive locally bounded Borel function.

REMARK. The definition of σ obviously depends on the choice of $\overline{\mu}$ and g. This arbitrariness stems from the fact that measure representations - like differential operators - are not uniquely determined by the harmonic structure.

The definition of $\sigma(f)$ makes sense more generally for $f = p - p' \in \mathcal{S}_+(X) - \mathcal{S}_+(X)$, provided that

$$\sigma(f) := \sigma(p) - \sigma(p')$$

belongs to the space $\mathcal{M}(X)$ of signed Radon measures on X. Simple examples (see Example 4 at the end of this section) show that in general σ does not satisfy the measure representation property for arbitrary $p,p' \in \mathcal{P}_b(X)$, i.e. it can happen that $\sigma(p) - \sigma(p')$ is a positive measure, but $p - p'$ is not superharmonic. Using tools of the theory of standard H-cones, due to N. Boboc, G. Bucur and A. Cornea [3], it can be shown that these problems do not occur for representation measures of ***continuous*** (or even regular) potentials:

LEMMA. *Let* $p,p' \in \mathcal{P}_c(X)$. *The restriction* $(\sigma(p)-\sigma(p'))_{|U}$ *of the measure* $\sigma(p)-\sigma(p')$ *to an open set* U *is positive, iff* $(p-p')_{|U} \in \mathcal{S}(U)$.

The following proof was communicated to me by G. Mokobodzki. It uses

MOTOO'S THEOREM. (see [5], Exercises 10.1.4, 10.2.1). *Let* p,p'' *be regular potentials such that* $p \prec p''$. *Then there exists a Borel function* f, $0 \le f \le 1$, *such that* $p = f \circledast p''$.

Proof of the Lemma (G. Mokobodzki). Set $p'' := p + p'$. Then according to Motoo's theorem there exist two Borel functions f, f' such that

$$0 \le f \le 1,\ p = f \odot p'' \text{ , hence } \sigma(p) = f \cdot \sigma(p'')$$
$$0 \le f' \le 1,\ p' = f' \odot p'' \text{ , hence } \sigma(p') = f' \cdot \sigma(p'') \text{ .}$$

But then
$$(\sigma(p)-\sigma(p'))_{|U} \ge 0 \Leftrightarrow f-f' \ge 0 \ \sigma(p'')_{|U} \text{ - a.e.} \Leftrightarrow (p-p')_{|U} \in \mathcal{S}(U) \text{ .}$$

EXAMPLES OF MEASURE REPRESENTATIONS.

1) Let $X =]-1,+1[$ be the harmonic space of the solutions of the one-dimensional Laplace equation on the real interval $]-1,+1[$. The measure representation in standard form corresponding to

$$\overline{\mu} = \text{Lebesgue measure on } X$$

is given by

$$\sigma(f) = -\frac{1-x^2}{2} f'' \text{ (in the distribution sense),}$$

the measure representation corresponding to the H-integral

$$\overline{\mu} : f \longmapsto \frac{1}{2}(f'_+(-1) - f'_-(1))$$

(left and right derivatives at the endpoints) is

$$\sigma(f) = -\frac{1}{2} f'' \text{ (in the distribution sense).}$$

This measure representation is associated to the symmetric Green function

$$G : (x,y) \longmapsto \min((1+x)(1-y),(1-x)(1+y))$$
$$X \times X \longrightarrow \mathbb{R}_+$$

in the following way

$$\bar{\mu}(G(\cdot,y)) = 1 \quad \text{for all } y \in X;$$
$$\sigma(G^\mu) = \mu \quad \text{for every } \mu \in \mathcal{M}_+(X).$$

2) Let X be a harmonic space with a symmetric Green function G and such that the constant 1 is hyperharmonic. Then - similarly as in the first example - there exists a strictly positive H-integral $\bar{\mu}$ (which is in general not a Radon measure on X) such that

$$\bar{\mu}(G(\cdot,y)) = \bar{\mu}(G(y,\cdot)) = 1 \quad \text{for all } y \in X.$$

The measure representation in standard form corresponding to this H-integral $\bar{\mu}$ reassigns to Greenian potentials

$$p = G^\mu = \int G(\cdot,y)\mu(dy)$$

their charge μ.

3) Let X be a harmonic space with a Green function G. It can be shown ([17], § 3) that there exist reference measures $\bar{\mu}$ such that

$$g : y \longmapsto \frac{1}{\bar{\mu}(G(\cdot,y))}$$

is *continuous*. Again the measure representation corresponding to $\bar{\mu}$ and g is associated to the given Green function G:

$$\sigma(G^\mu) = \mu,$$

since for $\varphi \in \mathcal{K}_+(X)$
$\sigma(G^\mu)(\varphi) = \overline{\mu}((g\varphi)\circ G^\mu) = \overline{\mu}(\int \frac{\varphi(y)}{\overline{\mu}(G(\cdot,y))} G(\cdot,y)\mu(dy)) = \int \varphi\, d\mu$,
but in general σ is not representable in standard form
(see [17], Example in § 2).

4) Let $X =]-1,+1[$. For every open interval $U \subset X$ let $\mathcal{H}(U)$ denote the space of all continuous functions $f : U \to \mathbb{R}$ such that

1) f is locally affine on $U \setminus \{0\}$
2) f is constant on $U \cap]-1,0]$, provided that $0 \in U$,

([5], Exercise 3.1.7).
The corresponding harmonic structure possesses two non-proportional potentials with the same superharmonic support $\{0\}$, namely

$$p : x \longmapsto 1 - |x|, \quad p' : x \longmapsto 1_{]0,1[}(x)(1-x);$$

especially there exists no Green function for X.
The measure representation corresponding to the Lebesgue measure $\overline{\mu}$ on $]-1,+1[$ is given by

$$\sigma(f) = - \varphi f'' + f'_-(o)\varepsilon_o, \quad f \in \mathcal{R}.$$

Here $f'_-(o)$ denotes the left derivative of f at o, ε_o is the Dirac measure at o, f'' denotes the second derivative in the distribution sense of f on $X \setminus \{o\}$ and $\varphi : X \to \mathbb{R}$ is defined by

$$\varphi(y) = \begin{cases} (1-\frac{y}{2})(1+y), & y < o \\ \frac{y}{2}(1-y), & y \geq o \end{cases}$$

(see also [15], Example (2.2.2); for this harmonic space F.-Y. Maeda found a similar measure representation, [8], Example 3.3.)

An easy calculation shows that for the two potentials p,p' defined above

$$\sigma(p) = \varepsilon_o, \ \sigma(p') = \frac{1}{2}\varepsilon_o,$$

hence $\sigma(p)-\sigma(p') \geq o$, but $p - p' \notin \mathcal{S}(X)$.

2. Continuous measure representations

Before studying continuity properties of measure representations consider the following example of a measure representation which is discontinuous with respect to the topology of uniform convergence.

EXAMPLE. Let $X =]-1,+1[$ be the harmonic space introduced in the last section, Example 4. Another measure representation $\tilde{\sigma}$ for this harmonic structure is given by

$$\tilde{\sigma}(f) := -\tilde{\varphi} \cdot f'' + f'_{-}(o)\varepsilon_o, \ f \in \mathcal{R},$$

where $\tilde{\varphi}(x) = |x|$ for $x \in X$ and f'' denotes the second derivative in the distribution sense. The sequence of continuous potentials

$$p_n(x) := \begin{cases} 1 - |x| & \text{for } \frac{1}{n} < |x| < 1 \\ 1 - \frac{1}{n} & \text{for } 0 \leq |x| \leq \frac{1}{n} \end{cases}, n \in \mathbb{N},$$

converges uniformly and increasingly to

$$p(x) = 1 - |x|;$$

but the corresponding sequence of measures

$$\tilde{\sigma}(p_n) = \frac{1}{n}\left(\varepsilon_{-\frac{1}{n}} + \varepsilon_{\frac{1}{n}}\right), \ n \in \mathbb{N},$$

converges vaguely to the zero measure, whereas $\tilde{\sigma}(p) = \varepsilon_o \neq 0$.

In [15], Theorem 3.5, it was shown that measure representations σ of the form

$$\sigma(p)(\varphi) = \overline{\mu}(g\,\varphi \circ p)\ ,\ p \in \mathcal{P}_b(X),\ \varphi \in \mathcal{K}_+(X),$$

where g is *continuous* and strictly positive, always define continuous maps $\sigma = \sigma_U$ from the cones $\mathcal{S}_c(U)$ of all continuous superharmonic functions on open sets U with respect to the topology of local uniform convergence to the spaces $\mathcal{M}_+(U)$ of positive Radon measures on U with the vague topology.

If the harmonic space satisfies Doob's convergence axiom and the condition (A) below - especially if there exists a Green function G on X (see [7] and [14]) - then - for a suitably chosen measure $\overline{\mu}$ - σ is continuous on $\mathcal{S}_+(U)$ even with respect to the topology of pointwise convergence.

A still coarser topology than the topology of pointwise convergence on $\mathcal{S}_+(U)$ is the topology of graph convergence studied by G. Mokobodzki (see [10] and [1]), which coincides with the natural topology τ_{nat}, introduced by N. Boboc, G. Bucur and A. Cornea on standard H-cones (see [3]). A sequence (p_n) in $\mathcal{S}_+(U)$ is convergent to $p \in \mathcal{S}_+(U)$ with respect to this topology iff for every subsequence $(p_{n_k})_{k\in\mathbb{N}}$

$$p = \widehat{\liminf_{k\to\infty} p_{n_k}} \quad \text{(lower semi-continuous regularization).}$$

The assertions mentioned above are contained in the following

THEOREM. *Assume that Doob's convergence axiom and the following condition* (A) *hold*

(A) *Every point* $x \in X$ *has an open neighbourhood* V_x *such that the smallest absorbing set* $A_{\complement V_x}$ *containing* $\complement V_x$ *is the whole space* X.

Then there exists a reference measure $\overline{\mu}$ *such that the measure representation* σ *associated to* $\overline{\mu}$ *and to an arbitrary strictly positive and continuous function* g *is naturally continuous on* $\mathcal{S}_+$*:*
For every open subset U

$$\sigma = \sigma_U : (\mathcal{S}_+(U), \tau_{nat}) \to (\mathcal{M}_+(U), \text{vague topology})$$

is a continuous map.
The proof is carried out in [17]. It relies on the following three lemmas valid under *Doob's convergence axiom.*

LEMMA 1. *Let* U *be an open and relatively compact subset of* X *and let* $L \subset U$ *be compact. Then the restriction map*

$$Q : \mathcal{P}_L(X) := \{p \in \mathcal{P}(X) : S(p) \subset L\} \to \mathcal{P}_{L \subset U}(U) := \{p \in \mathcal{P}(U) : S(p) \subset L\},$$
$$Q(\tilde{p}) := \textit{potential part of } \tilde{p}_{|U} = \tilde{p}_{|U} - R^{\complement U}_{\tilde{p}\,|U},$$

is a homeomorphism.

LEMMA 2. *Condition* (A) *is equivalent to*

(R) *There exists a reference measure* $\overline{\mu}$ *on* X *which is* τ_{nat} *- continuous on the cones* $\mathcal{P}_L(X)$, $L \subset X$ *compact.*

Especially Lemma 2 shows that in the presence of Doob's convergence axiom Condition (A) is even necessary for the continuity property of σ stated in the Theorem.

LEMMA 3. (see [16]). *The specific multiplication maps*

$$\mathcal{S}_+(U) \longrightarrow \mathcal{S}_+(U)$$
$$s \longmapsto f \cdot s$$

(for $f \in \mathcal{K}_+(U)$) *are* τ_{nat} *- continuous.*

3. Two characterizations of Axiom (D)

At the Oberwolfach meeting on axiomatic potential theory in 1984 I. Netuka suggested to characterize special properties of harmonic spaces by special properties of measure representations. As a first answer to this question two characterizations of Axiom (D) are presented in this section.

Axiom (D) was introduced in axiomatic potential theory by M. Brelot. It states that the domination principle is valid for all locally bounded potentials p, i.e.

$$\text{(D)} \qquad R_p^{S(p)} = p \quad \text{for every locally bounded potential } p.$$

In the following only measure representations σ in standard form are considered, determined by a strictly positive H-integral $\overline{\mu}$:

$$\sigma(p)(\varphi) = \overline{\mu}(\varphi \circledast p), \ p \in \mathcal{P}_b(X), \ \varphi \in \mathcal{K}_+(X).$$

The harmonic spaces are assumed to be strong or $\mathfrak{P}$-harmonic and the positive constants are hyperharmonic.

The characterizing conditions of Axiom (D) are the ***bounded energy principle***

$$\text{(E)} \qquad E(f) := \int f d\sigma(f) \geq 0 \quad \text{for every } f \in \mathcal{P}_b(X) - \mathcal{P}_b(X)$$

and the ***fine local principle*** (FL).

DEFINITION. ***A measure representation*** σ ***is said to satisfy the fine local principle*** (FL) ***or*** σ ***is said to be a fine local operator, iff, whenever*** $f \in \mathcal{P}_b(X) - \mathcal{P}_b(X)$ ***vanishes on a finely open set*** V, ***then the restriction*** $\sigma(f)_{|V}$ ***of*** $\sigma(f)$ ***to*** V ***is the zero measure.***

REMARK. If σ is the canonical measure representation associated to the Newtonian kernel for the classical harmonic sheaf, then E(f) is the usual Newtonian energy for $f \in \mathcal{P}_b(X) - \mathcal{P}_b(X)$. The bounded energy principle and the fine local principle are both satisfied by σ but neither of them is valid for the heat equation.

Another example where all these principles are violated is the following:

EXAMPLE. Axiom (D) is not valid in the harmonic space $X =]-1,1[$ introduced in § 1, Example 4. The measure representation σ associated to the Lebesgue measure $\overline{\mu}$ on $]-1,1[$ satisfies neither the bounded energy principle (E) nor the fine local property (FL). Indeed:
Let $p : x \mapsto 1 - |x|$ and $p' : x \mapsto (1-x)1_{]0,1[}(x)$.
Then
$E(4p'-p) = \int(4p'-p)d\sigma(4p'-p) = (4p'-p)(0) = -1 < 0$;
hence (E) is violated.
The potential p' vanishes on the finely open set $]-1,0]$, but the restriction of the measure

$$\sigma(p') = \frac{1}{2}\,\varepsilon_0$$

to $]-1,0]$ is not the zero measure; hence (FL) is violated.

The following theorem is proved in [17], § 1, 2:

THEOREM. *Let* $(X,\mathcal{H}^*)$ *be a* $\mathfrak{P}$*-harmonic space such that* $1 \in \mathcal{H}^*_+(X)$ *and let* σ *be a measure representation in standard form on* X. *Then the following conditions are equivalent:*

1) $(X,\mathcal{H}^*)$ *satisfies Axiom (D).*
2) σ *satisfies the bounded energy principle* (E).
3) σ *is a fine local operator.*

SOME COMMENTS ON THE PROOF.

1) ➡ 2). The implication 1) ➡ 2) has been proved by P. A. Meyer in the framework of Hunt processes with absolutely continuous resolvents ([9], p. 441; see also [11]) using additive functionals and martingale theory. A proof within the frame of axiomatic potential theory relies on a result of F.-Y. Maeda ([8], p. 39) according to which for every bounded $f \in \mathcal{P}_c(X) - \mathcal{P}_c(X)$

$$2f \circ f - f^2 \in \mathcal{S}(X) \qquad (\text{even} \in \mathcal{P}_c(X)),$$

i.e. $2f\,\sigma(f) - \sigma(f^2) \geq 0$

hence

$$E(f) = \int 1 \cdot f d\sigma(f) \geq \frac{1}{2} \int 1 d\sigma(f^2) = \frac{1}{2}\,\overline{\mu}(f^2) \geq 0.$$

This proves (E) for bounded differences of continuous potentials (without assuming Axiom (D)). The general case follows then by an approximation of bounded potentials by finite sums of continuous ones using Axiom (D).

2) ➡ 1). The implication 2) ➡ 1) has been proved by M. Rao [13] for a class of Lévy processes. His proof carries over to our situation provided that $E(\cdot)$ is definite on positive functions,

i.e. $E(f) = 0,\ f \geq 0 \Rightarrow f = 0.$

This is for instance true if $\overline{\mu}$ charges every finely open set. M. Rao's idea - translated to our situation - is the following:

Let $p \in \mathcal{P}_b(X)$ and let $u \in \mathcal{P}_b(X)$ such that $u = p$ on $S(p)$, $u \leq p$. Then (E) for $p - u$ implies

$$0 \leq \int (p-u)(d\sigma(p) - d\sigma(u)) = -\int (p-u) d\sigma(u) \leq 0$$

(since $u = p$ on $S(p) = \operatorname{Supp} \sigma(p)$ and $p - u \geq 0$), hence

$$\int (p-u)\,d\sigma(p-u) = 0.$$

According to the extra assumption of definiteness we have $p = u$. These considerations show

$$R_p^{S(p)} = p \qquad \text{for every } p \in \mathcal{P}_b(X),$$

a condition which is equivalent to Axiom (D) for $\mathfrak{P}$-harmonic spaces X with hyperharmonic positive constants.

Without the extra assumption of definiteness the proof is more involved; details can be found in [17].

3) $\Rightarrow$ 1). Again the proof is much simpler using the extra assumption of definiteness and is presented here; the general case is carried out in [17].
Let $p \in \mathcal{P}_b(X)$ and let $u \in \mathcal{P}_b(X)$ such that $u = p$ on $S(p)$. For every $\varepsilon > 0$ the potential

$$p_\varepsilon := \inf(u+\varepsilon, p)$$

coincides with p on a fine neighbourhood V_ε of $S(p)$. According to the fine local property of σ

$$\sigma(p_\varepsilon)_{|S(p)} = \sigma(p)_{|S(p)}$$

hence

$$\int 1_{S(p)}\, d\sigma(p_\varepsilon) \le \overline{\mu}(1_{S(p)} \circledast p_\varepsilon) + \overline{\mu}(1_{X\setminus S(p)} \circledast p_\varepsilon) = \overline{\mu}(p_\varepsilon) \le \overline{\mu}(p)$$

$$= \int 1\, d\sigma(p) = \int 1_{S(p)} d\sigma(p) = \int 1_{S(p)} d\sigma(p_\varepsilon).$$

But then $\int 1_{X\setminus S(p)} d\sigma(p_\varepsilon) = \overline{\mu}(1_{X\setminus S(p)} \circledast p_\varepsilon) = 0$, i.e.

$$\sigma(p_\varepsilon)_{|X\setminus S(p)} = 0 = \sigma(p)_{|X\setminus S(p)}.$$

The assumption of definiteness allows now to conclude

$$\int (p-p_\varepsilon)(d\sigma(p)-d\sigma(p_\varepsilon)) = 0 \Rightarrow p = p_\varepsilon,$$

hence

$$p \leq u + \varepsilon \quad \text{for every } \varepsilon > 0$$

and finally

$$p \leq u.$$

For the proof of the implication 1) $\Rightarrow$ 3) see [17].

REMARK. (Existence of fine measure representations) For harmonic spaces satisfying Axiom (D) B. Fuglede developed a "fine potential theory". In this case σ can be extended to a defining operator of the sheaf $\hat{\mathcal{R}}$ of fine-local differences of finite finely superharmonic functions:
For every finely open set V and $f : V \to \mathbb{R}$
$f \in \hat{\mathcal{R}}(V) \underset{def}{\Leftrightarrow}$ for every point $x \in V$ there exist a fine neighbourhood W and two finite finely superharmonic functions u,v on W such that $f = u - v$ on W.

$\Leftrightarrow$ for every point $x \in V$ there exist a fine neighbourhood W and $u,v \in \mathcal{P}_b(X)$ such that $f = u - v$ on W
(due to the "local extension property", [6], 9.9).

To $f \in \hat{\mathcal{R}}(V)$ a representing measure is assigned in the usual way:
If $f = u - v$ on a finely open set W, $u,v \in \mathcal{P}_b(X)$, then define

$$\hat{\sigma}(f)_{|W} := \sigma(u)_{|W} - \sigma(v)_{|W}.$$

According to the fine local property $\hat{\sigma}$ is well-defined and has the measure representation property

$$\hat{\sigma}(f)_{|V} \geq 0 \Leftrightarrow f \text{ is finely superharmonic on } V.$$

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