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DISCONNECTEDNESSES AND CLOSURE OPERATORS (*)

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Abstract

Closure operators which characterize disconnectednesses and relative disconnectednesses are introduced. Such operators are used to find conditions under which a relative disconnectednes is a disconnectedness.

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§1. Preliminaries (**)

In this paper we denote by **T** the class of all topological spaces, by $\mathbf{T}_i$ (i = 0,1,2) the classes of T_i-spaces, by **Sing** the class of spaces which have at most one point. Moreover we denote by **P** an arbitrary nonempty subclass of **T** and by $\underline{\mathbf{P}}$ the category of spaces of **P** and continuous functions. Of course $\underline{\mathbf{P}}$ is a full subcategory of $\underline{\mathbf{T}}$.

Let X be a space and $x \in X$.

1.1 DEFINITION. We call **P**-component of x in X the largest subspace Y of X containing x such that for each $P \in \mathbf{P}$ and for each $f: Y \to P$, f is constant (see [11], p.297).

1.2 DEFINITION. We call **P**-quasicomponent of x in X the largest subspace Y of X containing x such that for each $P \in \mathbf{P}$ and for each

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(**) Notations and definitions not explicitly given are from [6]. Moreover, the functions we consider are always continuous functions between topological spaces.

f: X $\to$ P, f|Y is constant (see [11], p.297).

1.3. DEFINITION. A space X is called totally **P**-disconnected if its **P**-components are singletons, totally **P**-separated if its **P**-quasicomponents are singletons (see [11], p.297).

We denote by **UP** the class of all totally **P**-disconnected spaces and by **QP** the class of all totally **P**-separated spaces.

It follows immediately from the definitions that

1.4 $$\mathbf{P} \subset \mathbf{QP} \subset \mathbf{UP} .$$

1.5 DEFINITION. A class **P** of spaces is called disconnectedness if **P** = **UP**, and relative disconnectedness if **P** = **QP**.

§2. The closure operators E_X and K_X.

Let f: A $\to$ B be a continuous function.

2.1 DEFINITION. f is said to be **P**-cancellable if for every P$\in$**P** and for every g_1, g_2: B $\to$ P such that $g_1 f = g_2 f$, we have $g_1 = g_2$.

Suppose now X is a space containing B as subspace.

2.2 DEFINITION. f is said to be **P**-cancellable rel X if for every P$\in$**P** and for every g_1, g_2: X $\to$ P such that $(g_1|B)f = (g_2|B)f$, we have $g_1|B = g_2|B$.

2.3. PROPOSITION. If **P'** is a class of spaces such that **P** $\subset$ **P'** $\subset$ **QP**, we have that f: A $\to$ B is **P'**-cancellable (or **P'**-cancellable rel X) iff f is **P**-cancellable (or **P**-cancellable rel X).

PROOF. Since **P** $\subset$ **P'** if f is **P'**-cancellable it is obvious that f is **P**-cancellable too.

Conversely, suppose f: A $\to$ B is **P**-cancellable. Let P'$\in$**P'** and g_1, g_2: B $\to$ P' be functions such that $g_1 f = g_2 f$. Then for every P$\in$**P** and for every h: P' $\to$ P we have $hg_1 f = hg_2 f$ and therefore $hg_1 = hg_2$. Since P'$\in$**QP**, the class of all continuous functions from P' whose range is in **P** distinguishis the points (see [10], 3.3). It follows that $g_1 = g_2$.

Similar arguments can be used to prove the proposition when the function is cancellable rel X.

From now on we denote by X an arbitrary topological space and by A an arbitrary subspace of X.

2.4 DEFINITION. By $E_X^{\mathbf{P}}(A)$ we denote the largest subspace Y of X such that $A \subset Y$ and the inclusion of A into Y is **P**-cancellable.

2.5 DEFINITION. By $K_X^{\mathbf{P}}(A)$ we denote the largest subspace Y of X such that $A \subset Y$ and the inclusion of A into Y is **P**-cancellable rel X.

It can be easily proved that the operators $E_X^{\mathbf{P}}$ and $K_X^{\mathbf{P}}$ are Moore closures and that if $f\colon X \to Y$ is a continuous function we have:

2.6 $E_X^{\mathbf{P}}(A) \subset K_X^{\mathbf{P}}(A)$;

2.7 $K_X^{\mathbf{P}}(A) = X \iff E_X^{\mathbf{P}}(A) = X$;

2.8 $f(E_X^{\mathbf{P}}(A)) \subset E_Y^{\mathbf{P}}(f(A))$; $f(K_X^{\mathbf{P}}(A)) \subset K_Y^{\mathbf{P}}(f(A))$;

2.9 the followings are equivalent:

(i) f is **P**-cancellable;

(ii) $E_Y^{\mathbf{P}}(f(X)) = Y$;

(iii) $K_Y^{\mathbf{P}}(f(X)) = Y$.

The operator $K_X^{\mathbf{P}}$ was introduced in [12] and studied in [4]. The operator $E_X^{\mathbf{P}}$ coincides with the epiclosure defined in [2] when **P** is productive, hereditary and $X \in \mathbf{P}$.

When there is no confusion about the class **P**, we indicate the introduced operators only by E_X and by K_X.

2.10 PROPOSITION. If **P'** is a class of spaces such that $\mathbf{P} \subset \mathbf{P'} \subset \mathbf{QP}$, we have

$$E_X^{\mathbf{P}} = E_X^{\mathbf{P'}} \; ; \; K_X^{\mathbf{P}} = K_X^{\mathbf{P'}} \; .$$

PROOF. It follows immediately from 2.3.

2.11 PROPOSITION. Let $x \in X$. We have:

(a) $E_X^{\mathbf{P}}(\{x\})$ is the **P**- component of x in X;

(b) $K_X^{\mathbf{P}}(\{x\})$ is the **P**-quasicomponent of x in X.

PROOF. (a) It follows immediately from the fact that if V is a subspace of X such that $x \in X$, the inclusion $j\colon \{x\} \to V$ is **P**-cancellable iff for each $P \in \mathbf{P}$ the functions from V to P are all constant.

(b) It can be proved in a similar way as (a).

2.12 COROLLARY. (a) **UP** is the class of all spaces X whose points are $E_X^{\mathbf{P}}$-closed.

(b) **QP** is the class of all spaces X whose points are $K_X^{\mathbf{P}}$-closed.

PROOF. It follows from 2.11.

2.13 PROPOSITION. The followings are equivalent:

(a) $\mathbf{P} \subset \mathbf{T}_2$;

(b) $\overline{A} \subset E_X^{\mathbf{P}}(A)$;

(c) $\overline{A} \subset K_X^{\mathbf{P}}(A)$.

PROOF. (a) $\Rightarrow$ (b) It follows from the fact that the inclusion $j: A \to \overline{A}$ is $\mathbf{T}_2$-cancellable and therefore **P**-cancellable.

(b) $\Rightarrow$ (c) It follows from 2.6.

(c) $\Rightarrow$ (a) See [12] (p.555).

2.14 LEMMA. A space X belongs to **QP** iff the diagonal Δ_X is $K_{X\times X}^{\mathbf{P}}$-closed.

PROOF. If $X \in \mathbf{QP}$, the projections p_1, p_2: $X\times X \to X$ coincide exactly on Δ_X, and therefore (see 2.3) $K_{X\times X}^{\mathbf{P}}(\Delta_X) = \Delta_X$.

Conversely, suppose $K_{X\times X}^{\mathbf{P}}(\Delta_X) = \Delta_X$. Then there are two functions f,g: $X\times X \to P$, with $P \in \mathbf{QP}$, such that $f|\Delta_X = g|\Delta_X$ and $f(x,y) \neq g(x,y)$ whenever $x \neq y$. If z is an arbitrary point of X, we consider the embedding $j: X \to X\times X$ defined by $j(x) = (x,z)$. We have: $fj(z) = gj(z)$ and $fj(t) \neq gj(t)$ for every $t \in X-\{z\}$. Hence $K_X^{\mathbf{P}}(\{z\}) = \{z\}$, and from 2.12 $X \in \mathbf{QP}$.

2.15 PROPOSITION. The followings are equivalent:

(a) $\mathbf{QP} \subset \mathbf{QP'}$;

(b) $K_X^{\mathbf{P}}(A) \supset K_X^{\mathbf{P'}}(A)$.

PROOF. (a) $\Rightarrow$ (b) It follows easily from the definitions and 2.3.

(b) $\Rightarrow$ (a) If (b) holds for each space X we have

$$K_{X\times X}^{\mathbf{P'}}(\Delta_X) \subset K_{X\times X}^{\mathbf{P}}(\Delta_X).$$

If $X \in \mathbf{QP}$, by 2.14 we have $K^{\mathbf{P}}_{X\times X}(\Delta_X) = \Delta_X$ and so Δ_X is $K^{\mathbf{P'}}_{X\times X}$-closed. By 2.14 again we have $X \in \mathbf{QP'}$.

2.16 COROLLARY. The followings are equivalent:

(a) $\mathbf{P} \subset \mathbf{T}_0$;

(b) $b_X(A) \subset E^{\mathbf{P}}_X(A)$;

(c) $b_X(A) \subset K^{\mathbf{P}}_X(A)$.

PROOF. (a) $\Longrightarrow$ (b) It follows from the fact that the inclusion $j: A \to b_X(A)$ is $\mathbf{T}_0$-cancellable (see [13]) and therefore $\mathbf{P}$-cancellable.

(b) $\Longrightarrow$ (c) It follows from 2.6.

(c) $\Longrightarrow$ (a) It follows from 2.15 and [12] (p.557).

<u>**Examples**</u>.

Let S be a singleton, C the two-points indiscrete space, D the Sierpinski dyad, I the real interval [0,1].

If $\mathbf{P} = \{S\}$ then $\mathbf{QP} = \mathbf{UP} = \mathbf{Sing}$ and $E^{\mathbf{P}}_X(A) = K^{\mathbf{P}}_X(A) = X$.

If $\mathbf{P} = \{C\}$ then $\mathbf{QP} = \mathbf{UP} = \mathbf{T}$ and $E^{\mathbf{P}}_X(A) = K^{\mathbf{P}}_X(A) = A$.

If $\mathbf{P} = \{D\}$ then $\mathbf{QP} = \mathbf{UP} = \mathbf{T}_0$ and $E^{\mathbf{P}}_X(A) = K^{\mathbf{P}}_X(A) = b_X(A)$, where $b_X(A)$ is the b-closure of A in X (see [13] , 2.5; [12], p.557).

If $\mathbf{P} = \{D_2\}$, where D_2 is the two-points discrete space, then $\mathbf{QP}$ is the class of all totally separated spaces and $\mathbf{UP}$ is the class of all totally disconnected spaces. Moreover

$$K^{\mathbf{P}}_X(A) = \bigcap\{B \mid A \subset B \subset X,\ B \text{ is clopen in } X\}.$$

If $\mathbf{P} = \{I\}$ then $\mathbf{QP}$ is the class of all functionally Hausdorff spaces. Moreover

$$K^{\mathbf{P}}_X(A) = \bigcap\{B \mid A \subset B \subset X,\ B \text{ is a zeroset in } X\} .$$

We observe that when $\mathbf{P} = \{I\}$ and in many other cases it is not easy to know how the operator $E^{\mathbf{P}}_X$ works and how the class $\mathbf{UP}$ is.

§3. Disconnectednesses and relative disconnectednesses.

UP and **QP** are subcategories of **T** closed under products and injective functions. Therefore they are extremal epireflective in **T** (see [8]).

We indicate by $R: \mathbf{T} \to \mathbf{UP}$, $S: \mathbf{T} \to \mathbf{QP}$ the corresponding epireflectors and by $r_X: X \to RX$ and $s_X: X \to SX$ the epireflection maps associated to R and S respectively. We remind that r_X is the quotient map which identifies the points of each **P**-component (see [1], Th.3.7) and s_X is the quotient map which identifies the points of each **P**-quasicomponent (see [10], p.304).

3.1. PROPOSITION. A function $f: X \to Y$ is **P**-cancellable iff $Sf: SX \to SY$ is an epimorphism in **QP**.

PROOF. Let $f: X \to Y$ be **P**-cancellable, $P \in \mathbf{QP}$ and $f_1, f_2: SY \to P$ such that $f_1(Sf) = f_2(Sf)$. Then $f_1(Sf)s_X = f_2(Sf)s_X$. Since $(Sf)s_X = s_Y f$ we have $f_1 s_Y f = f_2 s_Y f$. By 2.3 f is **QP**-cancellable, hence $f_1 s_Y = f_2 s_Y$. Since s_Y is an epimorphism in **T**, we obtain $f_1 = f_2$.

Conversely, let Sf be an epimorphism in **QP**. If $P \in \mathbf{P}$ and $f_1, f_2: Y \to P$ are functions such that $f_1 f = f_2 f$, there exist two functions $g_1, g_2: SY \to P$ such that $g_1 s_Y = f_1$, $g_2 s_Y = f_2$. Thus $g_1 s_Y f = g_2 s_Y f$, and therefore $g_1(Sf)s_X = g_2(Sf)s_X$. Since s_X is an epimorphism in **T** and Sf is an epimorphism in **QP**, we obtain $g_1 = g_2$.

3.2 PROPOSITION. $K_X^{\mathbf{P}}(A) = s_X^{-1}(K_{SX}^{\mathbf{P}}(s_X(A)))$.

PROOF. By 2.8 we have $K_X(A) \subset s_X^{-1}(K_X(s_X(A)))$. Suppose there exists a point $y \in s_X^{-1}(K_X(s_X(A))) - K_X(A)$. Then we can find two functions $f_1, f_2: X \to P$, with $P \in \mathbf{P}$, such that $f_1|A = f_2|A$ and $f_1(y) \neq f_2(y)$. If we consider the functions $g_1, g_2: SX \to P$, such that $g_1 s_X = f_1$, $g_2 s_X = f_2$, we obtain $g_1 s_X(y) \neq g_2 s_X(y)$. Since $g_1|s_X(A) = g_2|s_X(A)$ we deduce that $s_X(y) \notin K_{SX}(s_X(A))$, and this is absurd.

REMARK. We do not know whether an analogous proposition for the operator $E_X^{\mathbf{P}}$ and the epireflection map r_X holds. By 2.13 it could only be proved that such equality holds when $\mathbf{P} \subset \mathbf{T}_2$.

We remind that if **P** is productive and hereditary and $X \in \mathbf{P}$, for each $A \subset X$ the inclusion $j: A \to X$ is an extremal monomorphism iff $E_X^{\mathbf{P}}(A) = A$, and j is a regular monomorphism iff $K_X^{\mathbf{P}}(A) = A$ (see [2]).

As a consequence of this fact and of corollaries 3.5, 3.6 in [3], we obtain the following

3.3 PROPOSITION. If **P** if a disconnectedness contained in $\mathbf{T}_1$ and different from **Sing** we have

$$E_X^{\mathbf{P}}(A) = K_X^{\mathbf{P}}(A) = A .$$

3.4 PROPOSITION. The following conditions are equivalent:

(a) $\mathbf{UP} = \mathbf{QP}$;

(b) $E_X^{\mathbf{P}} = K_X^{\mathbf{P}}$ for each $X \in \mathbf{T}$;

(c) $E_X^{\mathbf{P}}(\{x\}) = K_X^{\mathbf{P}}(\{x\})$ for each $X \in \mathbf{T}$ and $x \in X$;

(d) $K_X^{\mathbf{P}}(A) = K_B^{\mathbf{P}}(A)$ for each X,A,B such that $A \subset B \subset X$ and $K_X^{\mathbf{P}}(B) = B$.

PROOF. (a) $\Rightarrow$ (b) If **QP** coincides with $\mathbf{T}$, $\mathbf{T}_0$ or $\mathbf{Q}\{\mathbf{S}\}$, then $\mathbf{QP} = \mathbf{UP}$ and $E_X^{\mathbf{P}} = K_X^{\mathbf{P}}$ (see examples in §2).

Moreover the only disconnectednesses which are not contained in $\mathbf{T}_1$ are $\mathbf{T}$ and $\mathbf{T}_0$ (see [1], Prop. 2.10). Thus we have only to consider the case $\mathbf{QP} = \mathbf{UP} \subset \mathbf{T}_1$, with $\mathbf{QP} \neq \mathbf{Q}\,\mathbf{S}$. If X is a space and $A \subset X$, by 3.3 we have $K_{SX}(s_X(A)) = s_X(A)$ and by 3.2

$$K_X(A) = s_X^{-1}(K_{SX}(s_X(A))) = s_X^{-1}(s_X(A)) .$$

It follows

$$K_X(A) = s_X^{-1}(s_X(A)) = r_X^{-1}(r_X(A)) = \bigcup\{E_X(\{x\}) \mid x \in A\} \subset E_X(A) ,$$

hence, by 2.6, $K_X(A) = E_X(A)$.

(b) $\Rightarrow$ (c) Obvious.

(c) $\Rightarrow$ (a) It follows immediately from 2.12.

(c) $\Leftrightarrow$ (d) It can be proved in a similar way as in Prop.1.8 in [2], even though the present assertion is more general.

REMARKS. (a) If **P** is a class of Hausdorff spaces, the operators $E_X^{\mathbf{P}}$ and $K_X^{\mathbf{P}}$ coincide if and only if $\mathbf{QP} = \mathbf{Sing}$. For if $E_X^{\mathbf{P}} = K_X^{\mathbf{P}}$

and **QP** $\neq$ **Sing**, for every $X \in \mathbf{P}$ and $A \subset X$ we have by 2.13 and 3.3:

$$\overline{A} \subset E_X^{\mathbf{P}}(A) = K_X^{\mathbf{P}}(A) = A .$$

This would imply that every $X \in \mathbf{P}$ is discrete and this is not possible. As a consequence we get again that in $\mathbf{T}_2$ there are no disconnectednesses different from **Sing** (see [1]).

(b) The notions given in this paper can be introduced in a topological category. In particular Preuß introduced and studied the relative disconnectednesses in this more general setting ([10]). The situation seems to be a little more complicated for the disconnectednesses. A reason is that in **T** the quotient space obtained by identifying the points of each **P**-component is **P**-totally disconnected and this fact is not always true in a topological category. For instance this is not true in the bireflective hull in **T** of the Hausdorff spaces.

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