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Some more κ-fully normal spaces

In: Zdeněk Frolík and Vladimír Souček and Jiří Vinárek (eds.): Proceedings of the 13th Winter School on Abstract Analysis, Section of Topology. Circolo Matematico di Palermo, Palermo, 1985. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 11. pp. (55)–59.

Persistent URL: `http://dml.cz/dmlcz/701879`

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0. Introduction

In [Ma] Mansfield introduced the notions of κ-full and almost-κ-full normality. One of the problems still left unanswered is the solution of the equation:

κ-full normality = almost-κ-full normality + $\mathcal{P}$.

In [Ju] Junnila essentially showed that almost-κ-fully normal orthocompact spaces are κ-fully normal, so the problem was raised whether orthocompactness might be a solution of the above equation (it was the only known candidate). The purpose of this note is to show that this is, at least consistently, not the case. We find under GCH (actually somewhat weaker assumptions suffice) that for every κ there is a κ-fully normal space which is not orthocompact.

1. Definitions and preliminaries

1.0. Covering properties

Let X be a topological space. Let $\mathcal{U}$ and $\mathcal{V}$ be (open) covers of X . Let $\kappa \geqslant 2$ be a cardinal. We say that $\mathcal{V}$ is

- a κ-star refinement of $\mathcal{U}$ iff whenever $\mathcal{V}' \subseteq \mathcal{V}$ has cardinality $\leq \kappa$ and $\bigcap \mathcal{V}' \neq \emptyset$, there is a $U \in \mathcal{U}$ with $\bigcup \mathcal{V}' \subseteq U$;
- an almost κ-star refinement of $\mathcal{U}$ iff whenever $x \in X$ and $A \subseteq St(x, \mathcal{V})$ has cardinality $\leq \kappa$, there is a $U \in \mathcal{U}$ with $A \subseteq U$.

Also an open cover $\mathcal{O}$ of X is interior-preserving iff for every $\mathcal{O}' \subseteq \mathcal{O}$, $\bigcap \mathcal{O}'$ is open.

We call a space X

- (almost-)κ-fully normal iff every open cover of X has an open (almost-)κ-star refinement;
- orthocompact iff every open cover of X has an open interior-preserving refinement.

As noted in the introduction (almost-)κ-full normality was defined by Mansfield in [Ma] , for more information on orthocompactness see [Sc].

1.1. Functions from ω to ω

As usual ω is the set of finite ordinals, and ${}^{\omega}\omega$ is the set of functions from ω to ω.

Let $f, g \in {}^{\omega}\omega$ we define

$$f <_* g \text{ iff } \{n \in \omega : g(n) \leq f(n)\} \text{ is finite.}$$

A set $\mathcal{D} \subseteq {}^{\omega}\omega$ is said to be dominating iff

$$\forall f \in {}^{\omega}\omega \, \exists g \in \mathcal{D} : f <_* g .$$

A scale in ${}^{\omega}\omega$ is a dominating subset of ${}^{\omega}\omega$ which is well-ordered by $<_*$.

Scales exist in some models of ZFC and don't exist in others. It is easy to show that CH implies the existence of an ω_1-scale (i.e. a scale of order-type ω_1), on the other hand adding ω_2 Cohen reals to any model of ZFC will produce a model without scales. For more information on these matters the reader is refered to [vD]. All other undefined notions can be found in [En] or [Ku2].

2. A 2-fully normal space which is not orthocompact

The space of this section was described first by Burke and van Douwen in [BuvD], later Fletcher and Künzi [FlKü] showed that it is not orthocompact. Let $\langle f_\alpha : \alpha \in \lambda \rangle$ be a scale in ${}^{\omega}\omega$, with λ a regular cardinal. Of course we assume that $\alpha \in \beta \in \lambda \rightarrow f_\alpha <_* f_\beta$. We also assume that each f_α is strictly increasing.

We topologize the set $\lambda \cup \omega \times \omega$ (disjoint union) as follows:

- the points of $\omega \times \omega$ will be isolated, and for convenience $0 \in \lambda$ will be isolated too.
- for $\beta \in \alpha \in \lambda$ and $n \in \omega$ we let
  $$U(\alpha, \beta, n) = (\beta, \alpha] \cup \{\langle k, l \rangle : f_\beta(k) < l \leq f_\alpha(k) \text{ and } k \geq n\}$$
  then for $\alpha \in \lambda \setminus \{0\}$ $\{U(\alpha, \beta, n) : \beta \in \alpha \text{ and } n \in \omega\}$ will be a local base at α.

The basic properties of this space were investigated in [BuvD], we show here that it is 2-fully normal but not orthocompact. To this end we first prove a lemma.

2.0. Lemma

Let for every $\alpha \in \lambda \setminus \{0\}$ a basic neighbourhood $U(\alpha, \beta_\alpha, n_\alpha)$ be given. Then there are a stationary set $S \subseteq \lambda$, $n \in \omega$ and ordinals $b_0 > \ldots > b_m = 0$ such that

(i) $\alpha \in S \rightarrow \beta_\alpha = b_0$ and $k \geq n \rightarrow f_{b_0}(k) < f_\alpha(k)$

(ii) $i \in m \to (k \geq n \to f_{b_{i+1}}(k) < f_{b_i}(k))$ and $n \geq n_{b_i}$

$\square$ W.L.O.G. Assume that for $k \geq n_\alpha$ $f_{\beta_\alpha}(k) < f_\alpha(k)$ whenever $\alpha \in \lambda \setminus \{0\}$. Then by the pressing-down Lemma [Ku;II.6.15] we can find a stationary $S \subseteq \lambda$, $b_o \in \lambda$ and $n_o \in \omega$ such that for $\alpha \in S$, $\beta_\alpha = b_o$ and $n_\alpha = n_o$. Now simply let, in case $b_i > 0$, $b_{i+1} = \beta_{b_i}$. Then

$b_o > b_1$ ... so we can find $m \in \omega$ with $b_m = 0$. Now because $f_{b_{i+1}} <_* f_{b_i}$ for $i \in m$, it is easy to find $n \geq n_o$ such that

(ii) (and (i)) is satisfied $\square$

Using an observation one to Kunen [Ku1] we can even get a little more:

2.1. Lemma

The assumptions are as in 2.0. The additional conclusion is

(iii) $\forall g \in {}^\omega\omega\ \exists \alpha \in S : (k \geq n \to g(k) < f_\alpha(k))$.

$\square$ If not then for every $n \in \omega$ we can find $g_n \in {}^\omega\omega$ such that $\forall \alpha \in S$ $\exists k \geq n$ such that $g_n(k) \geq f_\alpha(k)$. Define $g \in {}^\omega\omega$ by $g(n) = \max_{i \leq n} g_i(n) + 1$. Now $\langle f_\alpha : \alpha \in S \rangle$, being a cofinal subset of a scale, is dominating. So pick $\alpha \in S$ and $m \in \omega$ such that for $k \geq m$ $g(k) < f_\alpha(k)$. By assumption we can find $k \geq m$ with $f_\alpha(k) \leq g_m(k) < g(k) < f_\alpha(k)$ which is a contradiction. So we can enlarge our original $n \in \omega$ somewhat to get (iii). $\square$

We now verify 2-full normality and non-orthocompactness.

2.2. Our space is 2-fully normal

$\square$ Let $\mathcal{O}$ be an open cover of our space.

For every $\alpha \in \lambda \setminus \{0\}$ pick $\beta_\alpha \in \alpha$, $n_\alpha \in \omega$ and $O_\alpha \in \mathcal{O}$ such that $U(\alpha, \beta_\alpha, n_\alpha) \subseteq O_\alpha$. Apply Lemmas 2.0 and 2.1. to obtain S, $b_o, \ldots, b_m$ and n satisfying (i), (ii) and (iii). Let

$\mathcal{U} = \{U(b_i, b_{i+1}, n) : i \in m\} \cup \{U(\alpha, b_o, n) : \alpha \in S\} \cup \{\{x\} : x \in \omega \times \omega \cup \{0\}\}$.

Clearly $\mathcal{U}$ is an open refinement of $\mathcal{O}$.

Let $U_1, U_2 \in \mathcal{U}$ satisfy $U_1 \cap U_2 \neq \emptyset$.

If U_1 or U_2 consists of one point then there is nothing to prove. It then follows from (i) and (ii) that there are $\alpha_1, \alpha_2 \in S$ with $U_i = U(\alpha_i, b_o, n)$ $(i = 1,2)$, in the other case.

Set $g(k) = \max\{f_{\alpha_1}(k), f_{\alpha_2}(k)\}$ $(k \in \omega)$ and find $\alpha_3 \in S$ such that $k \geq n$ $g(k) < f_{\alpha_3}(k)$.

Then $U_1 \cup U_2 \subseteq U(\alpha_3, b_0, n) \subseteq O_{\alpha_3}$.

Thus $\mathcal{U}$ is a 2-star refinement of $\mathcal{O}$. □

2.3. Our space is not orthocompact

□ Let $\mathcal{V}$ be an interior-preserving open cover of our space.
For $\alpha \in \lambda \setminus \{0\}$ set $V_\alpha = \bigcap \{V \in \mathcal{V} : \alpha \in V\}$ and pick $\beta_\alpha \in \alpha$ and $n_\alpha \in \omega$ with $U(\alpha, \beta_\alpha, n_\alpha) \subseteq V_\alpha$.
Find S , $b_0, \ldots, b_m$ and n as above.
Let $C = \{\delta \in \lambda : \forall \gamma \in \delta\ \forall m \in \omega\ \exists \beta \in \delta \cap S (k \geqslant n \rightarrow f_\gamma(k) + m < f_\beta(k))\}$.
As usual C is a closed and unbounded subset of . For unboundedness use (iii).
Fix $\delta \in S \cap C$, then for $\alpha \in S \cap (b_0, \delta]$, $\alpha \in U(\delta, \beta_0, n) \subseteq V_\delta$, so $U(\alpha, b_0, n) \subseteq V_\alpha \subseteq V_\delta$. Thus $\bigcup \{U(\alpha, b_0, n) : \alpha \in (b_0, \delta] \cap S\} \subseteq V_\delta$.
Using the fact that $\delta \in C$ it is straightforward to verify that $F = \{\langle k, l\rangle : f_{b_0}(k) < l\} \subseteq \bigcup \{U(\alpha, b_0, n) : \alpha \in (b_0, \delta] \cap S\} \subseteq V_\delta$.

Now for no $\alpha \in \lambda \setminus \{0\}$ does $U(\alpha, 0, 0)$ contain F .
It follows that $\mathcal{U} = \{U(\alpha, 0, 0) : \alpha \in \lambda \setminus \{0\}\} \cup \{\{x\} : x \in \omega \times \omega \cup \{0\}\}$ does not have an interior-preserving open refinement □

3. Generalizations

It is straightforward to generalize the construction from section 2 to higher cardinals.
Let κ be an infinite cardinal number, and let $\mu = \kappa^+$ (the cardinal succesor of κ).
Define a pre-order $<_*$ on ${}^\mu\mu$ as before:

$$f <_* g \quad \text{iff} \quad \exists \alpha \in \mu \ (\beta \geqslant \alpha \rightarrow f(\beta) < g(\beta)) .$$

We let $\langle f_\alpha : \alpha \in \lambda \rangle$ (λ regular) be a scale in ${}^\mu\mu$, such a scale exists if e.g. $2^\mu = \mu^+$ is assumed, and $\lambda = \mu^+$ in this case.
We topologize $\lambda \cup \mu \times \mu$ as above.
The proofs in section 2 go through almost verbatim, for κ -full normality observe that if $\mathcal{G} \subseteq {}^\mu\mu$ has cardinality $\leqslant \kappa$ then $g(\alpha) = \sup \{f(\alpha) + 1 : f \in \mathcal{G}\}$ ($\alpha \in \mu$) is well-defined.
We leave the details to the reader.
The result is a κ -fully normal space which is not orthocompact.

4. Question

We know now that orthocompactness does not work. But the problem remains as to whether there is a nontrivial topological property $\mathcal{P}$ such that κ -full normality = almost- κ -full normality + $\mathcal{P}$.

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