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In: Zdeněk Frolík and Vladimír Souček and Jiří Vinárek (eds.): Proceedings of the 13th Winter School on Abstract Analysis, Section of Topology. Circolo Matematico di Palermo, Palermo, 1985. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 11. pp. [89]–103.

Persistent URL: http://dml.cz/dmlcz/701883

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A. P. Šostak

0. Introduction

Fuzzy topological spaces where defined in 1968 by C.Chang [2] and later redefined in a somewhat different way by R.Lowen [8] and by B.Hutton [7]. These definitions are based on the fundamental concept of a fuzzy set introduced by L.Zadeh [17]. In the last fifteen years a vast literature devoted to various kinds of fuzzy topological spaces has appeared (see e.g. [5] - [16] and others). However so far as we know in all the works of this area fuzzy are only sets (sometimes they are fuzzy even of higher orders as e.g. in [12]). But the so called fuzzy topology is always crisp - we mean it is a crisp subfamily of some family of fuzzy sets. The aim of this paper is to define and to begin the study of <u>fuzzy structures</u> of topological type.

To formulate our program and general ideas more precisely, recall first that the objects of the category Top of topological spaces are pairs $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a family of its subsets, i.e. $\mathcal{T} \in 2^{2^X}$, satisfying the well-known axioms. The objects of the category Fuz of "usual" fuzzy topological spaces are pairs (X,T) where X is a set and T essentially is a family of its fuzzy subsets, i.e. $T \in 2^{I^X}$, satisfying some natural axioms (see e.g. [2], [8], [7]). Therefore both in Top and in Fuz for every subset (in the second case also for every fuzzy subset) of a space it is precisely known whether it is open or not. The idea of this paper is to allow fuzzy subsets (specifically also usual subsets) to be open to some degree, and this degree may range from 1 ("completely open sets") to 0 ("completely non-open sets"). Thus, a fuzzy topological space will be understood here as a pair (X, τ) where X is a set and $\tau \in I^{I^X}$, i.e.

a fuzzy topology is a function $\mathcal{T} : I^X \longrightarrow I$ (satisfying some axioms) which assigns to every fuzzy subset of X the real number, which shows "to what extent" this set is open (definition (2.1)). This idea seems to be rather natural also because for a fuzzy set, i.e. for a set without distinct boundaries, it is not always suitable to ask whether it is "completely" open.

This approach will lead us to a category FT , the objects of which are fuzzy topological spaces as they are sketched above and morphisms are appropriately defined (see definition (2.9)).

Since our principal interest in this paper is the category FT , the terms "a fuzzy topological space" and " a fuzzy continuous mapping" will always mean respectively an object and a morphism of this category. In the case when working with fuzzy topological spaces and fuzzy continuous mappings in the sense of C.Chang, i.e. objects and morphisms of the category Fuz , this will be always explicitly stated.

The structure of the paper is as follows.

In the first section we "fuzzify" such notions of set theory as "inclusion", "equality", "intersection" and "union". In section 2 the category FT of fuzzy topological spaces and fuzzy continuous mappings is defined. The products and coproducts of this category are studied in section 3.

The forth section is devoted to the so called induced fuzzy topological spaces. They form a subcategory IFT of FT which appears to be isomorphic in a natural way to the category Top .

In section 5 we discuss the role of the category Fuz of "classical" fuzzy topological spaces as a full subcategory of FT.

The last, sixth section is devoted to the notion of the compactness degree of a fuzzy subset of a fuzzy topological space.

The following notations will be used in the paper. If X is a set and $A \subset X$ then $A^c = X \setminus A$. Analogously, if μ is a fuzzy subset of X , i.e. $\mu : X \longrightarrow I$, then $\mu^c = 1 - \mu$. As usual, $I = [0,1]$ and $2 = \{0,1\}$. The set of all fuzzy subsets (all subsets) of a given set X will be denoted I^X (respectively, 2^X). The symbols $\bigvee$ and $\bigwedge$ will be used respectively for the supremum and the infimum of a family of fuzzy sets . The category of topological spaces is denoted Top .

1. Preliminaries

The aim of this section is to "fuzzify" such basic set-theoretic notions as inclusion, equality, intersection and union. Here we essentially use the ideas of Z.Diskin [3].

Let X be a set and let $A \subset X$. Then A can be identified with its characteristic function. We shall use the same notation for this function, i.e. $A: X \to \{0,1\} = 2$ where $A(x) = 1$ iff $x \in A$.

The inclusion $A \subset B$ for subsets of X can be written as $A \leq B$ and generalizing this fact for fuzzy subsets μ and ν of X the inclusion $\mu \subset \nu$ is usually understood as the inequality $\mu \leq \nu$. Our approach however is different. The inclusion $A \subset B$ can also be written as $A^c \cup B = X$ or $(A^c \vee B)(x) = 1$ for all $x \in X$ and hence as $\inf_X (A^c \vee B)(x) = 1$. On the other hand $\inf_X (A^c \vee B)(x) = 0$ iff $A \not\subset B$. "Fuzzifying" this observation we come to the following

(1.1) Definition (Fuzzy inclusion). For fuzzy subsets μ and ν of a set X let $(\mu \mathrel{\tilde{\subset}} \nu) = \inf_X (\mu^c \vee \nu)(x)$.

(1.2) Remark. Thus the fuzzy inclusion may be considered as a function $\tilde{\subset}\, I^X \times I^X \to I$. The real number $\mu \mathrel{\tilde{\subset}} \nu$ shows "to what extent" the fuzzy set μ is contained in the fuzzy set ν. Notice also that the restriction of $\tilde{\subset}$ to $2^X \times 2^X$, i.e. to the family of pairs of crisp subsets of the set X is just the usual inclusion. More precisely, $A \mathrel{\tilde{\subset}} B = 1$ iff $A \subset B$ and $A \mathrel{\tilde{\subset}} B = 0$ iff $A \not\subset B$.

(1.3) Remark. It may look strange at first that for a proper fuzzy set μ the number $\mu \mathrel{\tilde{\subset}} \mu$ is never equal to 1. However this will not seem unnatural if one thinks of μ as of a relation between fuzzy sets and not just as of a relation between functions. Thus, μ being a fuzzy set may "contain some points of X only partly" and therefore we cannot require that μ should be completely contained in itself.

Recalling that $A = B$ iff $A \subset B$ and $B \subset A$ we propose to "fuzzify" the relation of equality in the following way:

(1.4) Definition (Fuzzy equality). For fuzzy subsets μ and ν of a set X let $(\mu \cong \nu) = (\mu \mathrel{\tilde{\subset}} \nu) \wedge (\nu \mathrel{\tilde{\subset}} \mu)$.

(1.5) Remark. The fuzzy equality can be considered as a function $\cong I^X \times I^X \to I$ extending the usual equality $= : 2^X \times 2^X \to 2$, which is defined in the natural way.

A crisp family $\mathfrak{A}$ of crisp subsets of a given set X can be realized as a function $\mathfrak{A}: 2^X \to 2$ indicating which subsets belong to it. The intersection of all subsets from $\mathfrak{A}$ can be realized as a function $\wedge \mathfrak{A} : X \to 2$ defined by the equality $\wedge \mathfrak{A}(x) = 1$ iff $x \in A$ for all $A \in \mathfrak{A}$. Formally this can be written as follows

$$(\wedge \mathfrak{A})(x) = \inf_{A \in 2^X} ((\mathfrak{A}(A))^c \vee A(x)).$$

Generalizing this formula to the case of a fuzzy family of fuzzy subsets of X we obtain the following

(1.6) <u>Definition</u>. Let $\mathfrak{A} : I^X \to I$. (Such a mapping will be understood as a fuzzy family of fuzzy subsets of X). The intersection of this fuzzy family is a function $\wedge\mathfrak{A}: X \to I$ defined by the equality

$$(\wedge \mathfrak{A})(x) = \inf_{\mu \in I^X} ((\mathfrak{A}(\mu))^c \vee \mu(x)).$$

For a crisp family $\mathfrak{A}$ of crisp subsets of X the union of all elements of $\mathfrak{A}$ can be defined by the equality

$$(\vee \mathfrak{A})(x) = \sup_{A \in 2^X} (\mathfrak{A}(A) \wedge A(x)).$$

Generalizing this approach for the fuzzy case we get the following

(1.7) <u>Definition</u>. Let $\mathfrak{A} : I^X \to I$. The union of this fuzzy family is a function $\vee\mathfrak{A} : X \to I$ defined by the equality

$$(\vee \mathfrak{A})(x) = \sup_{\mu \in I^X} (\mathfrak{A}(\mu) \wedge \mu(x)).$$

2. <u>Fuzzy topological spaces and fuzzy continuous mappings</u>

(2.1) <u>Definition</u>. Let X be a set. By a <u>fuzzy topology</u> on X we call a function $\tau : I^X \to I$ satisfying the following three axioms:

(1) if $\mu, \nu \in I^X$, then $\tau(\mu \wedge \nu) \geq \tau(\mu) \wedge \tau(\nu)$;

(2) if $\mu_i \in I^X$ for all $i \in \mathcal{J}$, then $\tau(\bigvee_i \mu_i) \geq \bigwedge_i \tau(\mu_i)$;

(3) $\tau(0) = \tau(1) = 1$.

The real number $\tau(\mu)$ will be called <u>the degree of openess</u> of the fuzzy set μ.

(2.2) <u>Definition</u>. <u>A fuzzy topological space</u> is a pair (X, τ) where X is a set and τ is a fuzzy topology on it.

(To shorten the expression we shall often omit the word "topological" and say just "a fuzzy space".)

(2.3) <u>Remark</u>. The intuitive motivation for these definition is as follows. Speaking informally, the axiom (1) states that the intersection of two fuzzy sets is not "less open" than the minimum of"openess" of these sets. The axiom (2) requires that the degree of openess of the union of any crisp family of fuzzy sets should be not less than the "smallest" degree of openess of these sets. The last,axiom (3) just states, that the empty set and the whole space are "absolutely open".

The main examples of fuzzy spaces considered in this paper are the so called induced fuzzy spaces (section 4) and fuzzy topological spaces in the sense of C.Chang and in the sense of R.Lowen (section 5). We hope that these special but important kinds of fuzzy spaces will also help to justify our definitions.

(2.4) Definition. If there are two fuzzy topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on the same set X ,we say that $\mathcal{T}_1$ is stronger than $\mathcal{T}_2$ if $\mathcal{T}_1(\mu) \geqslant \mathcal{T}_2(\mu)$ for every $\mu \in I^X$.

(2.5) Definition. Let $(X,\mathcal{T})$ be a fuzzy topological space. We define the mapping $\mathcal{T}^* : I^X \to I$ by the equality $\mathcal{T}^*(\mu) = \mathcal{T}(\mu^c)$ for every $\mu \in I^X$. The number $\mathcal{T}^*(\mu)$ will be called the degree of closedness of a fuzzy set μ .

From the definitions (2.1) and (2.5) one easily gets the following

(2.7) Proposition. The mapping $\mathcal{T}^* : I^X \to I$ has the following properties:

$(1)^*$ if $\mu, \nu \in I^X$, then $\mathcal{T}^*(\mu \vee \nu) \geqslant \mathcal{T}^*(\mu) \wedge \mathcal{T}^*(\nu)$,

$(2)^*$ if $\mu_i \in I^X$ for all $i \in \mathcal{J}$, then $\mathcal{T}^*(\bigwedge_i \mu_i) \geqslant \bigwedge_i \mathcal{T}^*(\mu_i)$;

$(3)^*$ $\mathcal{T}^*(0) = \mathcal{T}^*(1) = 1$.

(2.8) Remark. It is clear that a fuzzy topological space can be equivalently defined as a pair $(X,\mathcal{T}^*)$ where $\mathcal{T}^* : I^X \to I$ satisfies the properties $(1)^*$, $(2)^*$,$(3)^*$ and is understood as the degree of closedness of fuzzy subsets. The corresponding fuzzy topology is to be defined by the equality $\mathcal{T}(\mu) = \mathcal{T}^*(\mu^c)$.

(2.9) Definition. Let $(X,\mathcal{T})$ and (Y,σ) be fuzzy spaces and $f : X \to Y$ is a mapping. This mapping is called fuzzy continuous if $\mathcal{T}(f^{-1}(\nu)) \geqslant \sigma(\nu)$ for every $\nu \in I^Y$.

Speaking informally, fuzzy continuous mappings are the ones which do not diminish the degree of openess of fuzzy subsets in the direction of preimage.

(2.10) Proposition. A mapping $f : X \to Y$ of fuzzy spaces $(X,\mathcal{T})$ and (Y,σ) is fuzzy continuous iff $\mathcal{T}^*(f^{-1}(\nu)) \geqslant \sigma^*(\nu)$, for every $\nu \in I^Y$.

The proof is direct and therefore omitted.

(2.11) Proposition. Let (X,τ), (Y,σ), (Z,ρ) be fuzzy spaces and $f: X \to Y$, $g: Y \to Z$ fuzzy continuous mappings. Then the composition $g \circ f: X \to Z$ is also fuzzy continuous.

The proof is obvious.

Since the composition is associative and the identity mapping $e: X \to X$ is fuzzy continuous with respect to any fuzzy topology on X the following definition is justified.

(2.12) Definition. By FT we denote the category the objects of which are fuzzy topological spaces and the morphisms are fuzzy continuous mappings between them.

Some properties of this category will be considered in the next section.

3. Products and coproducts of fuzzy topological spaces

The main aim of this section is to show that there exist products and coproducts in the category FT and to construct them explicitly.

(3.1) The initial fuzzy topology for a mapping.

Let X be a set, (Y,σ) a fuzzy topological space and $f: X \to Y$ is a mapping. By the initial fuzzy topology for this mapping we understand the weakest fuzzy topology τ on X such that the mapping $f: (X,\tau) \to (Y,\sigma)$ is fuzzy continuous.

To construct such a fuzzy topology consider the set $M = \{\mu = f^{-1}(\nu) : \nu \in I^Y\}$ of fuzzy subsets of X. For a given $\mu \in M$ let $P_\mu = \{\nu : \nu \in I^Y, \mu = f^{-1}(\nu)\}$ and define $\tau(\mu) = \sup\{\sigma(\nu) : \nu \in P_\mu\}$. It is obvious that $\cup\{P_\mu : \mu \in I^X\} = I^Y$ and $\tau(f^{-1}(\nu)) \geq \sigma(\nu)$ for every $\nu \in I^Y$.

Let $\mu_1, \mu_2 \in M$, then $\mu = \mu_1 \wedge \mu_2 \in M$ and moreover, $P_\mu \supset \{\nu_1 \wedge \nu_2 : \nu_1 \in P_{\mu_1}, \nu_2 \in P_{\mu_2}\}$. Therefore $\tau(\mu) = \sup\{\sigma(\nu) : \nu \in P_\mu\} \geq \sup\{\sigma(\nu_1 \wedge \nu_2) : \nu_1 \in P_{\mu_1}, \nu_2 \in P_{\mu_2}\} \geq \sup\{\sigma(\nu_1) \wedge \sigma(\nu_2) : \nu_1 \in P_{\mu_1}, \nu_2 \in P_{\mu_2}\} = \sup\{\sigma(\nu_1) : \nu_1 \in P_{\mu_1}\} \wedge \sup\{\sigma(\nu_2) : \nu_2 \in P_{\mu_2}\} = \tau(\mu_1) \wedge \tau(\mu_2)$, and hence

$$\tau(\mu) \geq \tau(\mu_1) \wedge \tau(\mu_2) \quad \text{for } \mu_1, \mu_2 \in M \qquad (1).$$

In a similar way we can show that for any subfamily $\{\mu_i : i \in \mathcal{J}\}$ of M

$$\tau(\bigvee_i \mu_i) \geq \bigwedge_i \{\sup \sigma(\nu_i) : \nu_i \in P_{\mu_i}\} = \bigwedge_i \tau(\mu_i) \qquad (2)$$

Moreover, it is obvious that $0 = f^{-1}(0) \in M$, $1 = f^{-1}(1) \in M$ and

$$\tau(0) = \tau(1) = 1 \qquad (3)$$

Thus $\tau : M \to I$ satisfies the axioms of definition (2.1).

Now we extend $\mathcal{T}$ to a mapping $\mathcal{T}: I^X \to I$ by letting $\mathcal{T}(\mu) = 0$ for all $\mu \notin M$. It is easy to check that the function $\mathcal{T}$ thus defined is indeed a fuzzy topology. Moreover, from the construction it is clear that $\mathcal{T}$ is the weakest fuzzy topology on X making the mapping $f: (X,\mathcal{T}) \longrightarrow (Y,\sigma)$ fuzzy continuous.

(3.2) Initial fuzzy topology for a family of mappings.

Let now $\{(Y_a, \sigma_a) : a \in \mathcal{A}\}$ be a family of fuzzy spaces and consider for each $a \in \mathcal{A}$ a mapping $f_a: X \to Y_a$. Let $\mathcal{T}_a: I^X \to I$ be the initial fuzzy topology on X for f_a, and let the mapping $\mathcal{T}: I^X \to I$ be defined by the equality $\mathcal{T}(\mu) = \inf_a \mathcal{T}_a(\mu)$ where $\mu \in I^X$. Since $\mathcal{T}(\mu_1 \wedge \mu_2) = \inf_a \mathcal{T}_a(\mu_1 \wedge \mu_2) \geqslant \inf_a (\mathcal{T}_a(\mu_1) \wedge \mathcal{T}_a(\mu_2))$ $\geqslant \inf_a \mathcal{T}_a(\mu_1) \wedge \inf_a \mathcal{T}_a(\mu_2) = \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$ and

$$\mathcal{T}(\bigvee_i \mu_i) = \inf_a \mathcal{T}_a(\bigvee_i \mu_i) \geqslant \inf_a \bigwedge_i \mathcal{T}_a(\mu_i) = \bigwedge_i \inf_a \mathcal{T}_a(\mu_i) = \bigwedge_i \mathcal{T}(\mu_i)$$

for any collection of fuzzy subsets μ_i of X, one can easily conclude that $\mathcal{T}$ is a fuzzy topology on X. Moreover, it is clear from (3.1) and from the construction of $\mathcal{T}$ that it is the weakest fuzzy topology on X for which all mappings $f_a: (X,\mathcal{T}) \to (Y_a,\sigma_a)$ are fuzzy continuous. This fuzzy topology $\mathcal{T}$ will be called the initial fuzzy topology for the family of mappings $\{f_a : X \to Y_a, a \in \mathcal{A}\}$. The existence of such a fuzzy topology allows us to state the following theorem.

(3.3) Theorem. FT is a complete category. In particular, FT contains products and inverse limits.

(3.4) Product of fuzzy topological spaces.

To construct the product in FT explicitly consider a family $\{(X_a, \mathcal{T}_a) : a \in \mathcal{A}\}$ of fuzzy topological spaces. The product of this family can be defined as a pair $(X,\mathcal{T})$, where X denotes the product of all sets X_a and $\mathcal{T}$ is the initial fuzzy topology generated on X by the family $\{p_a : X \to X_a, a \in \mathcal{A}\}$ of all projections.

(3.5) Product of fuzzy subsets.

Let $(X_1, \mathcal{T}_1)$, $(X_2, \mathcal{T}_2)$ be two fuzzy spaces and let $(X,\mathcal{T})$ denote their product. If $\mu_1 \in I^{X_1}, \mu_2 \in I^{X_2}$, then for a fuzzy set $\mu = \mu_1 \times \mu_2 \in I^X$ (which is defined as $\mu(x_1, x_2) = \mu_1(x_1) \wedge \mu_2(x_2)$) we have $\mathcal{T}(\mu) = \mathcal{T}(p_1^{-1}(\mu_1) \wedge p_2^{-1}(\mu_2)) \geqslant \mathcal{T}(p_1^{-1}(\mu_1)) \wedge \mathcal{T}(p_2^{-1}(\mu_2))$ $\geqslant \mathcal{T}_1(\mu_1) \wedge \mathcal{T}_2(\mu_2)$. Hence the degree of openess of the product of two fuzzy sets in the product space is not less than the minimal degree of openess of these sets in the corresponding fuzzy spaces.

Now let $\{(X_a, \mathcal{T}_a) : a \in \mathcal{A}\}$ be a family of fuzzy spaces and let $(X, \mathcal{T})$ denote their product. Take $\mu_a \in I^{X_a}$ for every a and let $\mu \in I^X$ denote the product of all μ_a (i.e. $\mu(x) = \bigwedge_a \mu_a(x_a)$ where $x \in X$ and x_a denotes the a-est coordinate). Quite similarly as above one can show that $\mathcal{T}^*(\mu) \geqslant \bigwedge_a \mathcal{T}_a^*(\mu_a)$, and hence the degree of closedness of the product of fuzzy sets is not less than the degrees of the degrees of closedness of the factors.

The rest of this section is devoted to the concept of coproduct (or direct sum) of fuzzy topological spaces and to some closely related notions.

(3.6) Final fuzzy topology for a mapping.

Let $(X, \mathcal{T})$ be a fuzzy space and Y is a set. Consider a mapping $f: X \to Y$ and for every $\nu \in I^Y$ let $\sigma(\nu) = \mathcal{T}(f^{-1}(\nu))$. It is easy to check that σ is a fuzzy topology on Y and moreover, it is the strongest fuzzy topology on Y for which the mapping $f: (X, \mathcal{T}) \longrightarrow (Y, \sigma)$ is fuzzy continuous.

(3.7) Final fuzzy topology for a family of mappings.

Let $\{(X_a, \mathcal{T}_a) : a \in \mathcal{A}\}$ be a family of fuzzy topological spaces and for every a consider a mapping $f_a: X_a \to Y$ where Y is a set. Let σ_a denote the final topology on Y for f_a. Define $\sigma: I^Y \longrightarrow I$ by the equality $\sigma(\nu) = \inf_a \sigma_a(\nu)$ for $\nu \in I^Y$. Quite similarly as in (3.2) one can show that σ is a fuzzy topology on Y. Moreover, it is easy to notice, that it is the strongest fuzzy topology on Y for which all the mappings $f_a: X_a \longrightarrow Y$ are fuzzy continuous.

From (3.7) immediately follows such a theorem:

(3.8) Theorem. The category FT is cocomplete. Specifically, it contains coproducts and direct limits.

(3.9) Coproduct in FT .

To construct the coproduct in FT explicitly consider a family $\{(X_a, \mathcal{T}_a) : a \in \mathcal{A}\}$ of fuzzy spaces and let $X = \bigoplus X_a$ denote the direct sum of the corresponding sets. The space $(X, \mathcal{T})$ where $\mathcal{T}$ is the final topology for the family of all inclusions $i_a: X_a \longrightarrow X$ is just the coproduct of these fuzzy spaces. Moreover, it is easy to notice, that $\mathcal{T}(\mu) = \inf_a \mathcal{T}_a(\mu_a)$ and $\mathcal{T}^*(\mu) = \inf \mathcal{T}_a^*(\mu_a)$ where μ_a denotes the restriction of $\mu \in I^X$ to X_a.

4. The induced fuzzy topology

Let $(X,\mathcal{T})$ be a (usual) topological space. In this section we show how the topology $\mathcal{T}$ induces a fuzzy topology on the same set. More precisely, a topology $\mathcal{T}: 2^X \to 2$ will be extended in a definite way to a fuzzy topology $\tau: I^X \to I$.

Thus, let $\mathcal{T}: 2^X \to 2$ be a topology on X and let $\mathcal{T}^*: 2^X \to 2$ be the corresponding closed topology (i.e. $\mathcal{T}^*(A) = \mathcal{T}(A^c)$ for $A\in 2^X$) The closure of the set A in $(X,\mathcal{T})$ can be defined by the equality $\bar{A} = \{x : \forall U \in \mathcal{T}_x, \exists y \in A \cap U\}$ where $\mathcal{T}_x$ denotes the family of all neighbourhoods of x in $(X,\mathcal{T})$. Observing $\bar{A}$ as a function, we can write also $\bar{A}(x) = \inf\limits_{U\in\mathcal{T}_x} \sup\limits_{y\in U} A(y)$. Analogously, the interior of A can be defined as $A^o(x) = \sup\limits_{U\in\mathcal{T}_x} \inf\limits_{y\in U} A(y)$. This observation justifies the following

(4.1) Denotation. For every $\mu\in I^X$ let

$$\bar{\mu}(x) = \inf_{U\in\mathcal{T}_x} \sup_{y\in U} \mu(y) \quad \text{and} \quad \mu^o(x) = \sup_{U\in\mathcal{T}_x} \inf_{y\in U} \mu(y) ,$$

where $\mathcal{T}_x$ denotes the family of all neighbourhoods of x in $(X,\mathcal{T})$.

Since $\mathcal{T}_x$ can be ordered by inclusion,we can rewrite the previous formulae as follows:

$$(4.1)' \quad \bar{\mu}(x) = \lim_{U\in\mathcal{T}_x} \sup_{y\in U} \mu(y) \ , \ \mu^o(x) = \lim_{U\in\mathcal{T}_x} \inf_{y\in U} \mu(y) .$$

Applying a well-known theorem (see e.g. [1] , Chapter 4 , §6 or [4] , Chapter 2 , p.153) one obtains

(4.2) Proposition. For every $\mu\in I^X$ the function $\bar{\mu}$ is upper semicontinuous and the function μ^o is lower semicontinuous. Moreover, a function $\mu\in I^X$ is upper semicontinuous (lower semicontinuous) iff $\mu = \bar{\mu}$ (respectively $\mu = \mu^o$).

Again, applying the well-known facts (see e.g. [1], Chapter 4, §§5,6) one can easily prove the following

(4.3) Proposition. Let $\mu, \nu\in I^X$ and let $\alpha\in I^X$ be a constant. Then

(1) $\mu \le \bar{\mu}$;	(1)' $\mu^o \le \mu$;
(2) $\overline{\mu\vee\nu} = \bar{\mu}\vee\bar{\nu}$;	(2)' $(\mu\wedge\nu)^o = \mu^o\wedge\nu^o$;
(3) $\bar{\bar{\mu}} = \bar{\mu}$;	(3)' $(\mu^o)^o = \mu^o$;
(4) $\bar{\alpha} = \alpha$.	(4)' $\alpha^o = \alpha$.

Summing up, from (4.3) and (4.2) we get the following statement:

(4.4) Proposition. The family $\tilde{\mathcal{T}} = \{\mu : \mu = \mu^o , \mu\in I^X\}$ is a fuzzy topology in the sense of R.Lowen [8] - [10] (and hence also in the sense of C.Chang [2]). Thus $(X,\tilde{\mathcal{T}})$ is an object of the category Fuz_L and therefore also of the category Fuz. Moreover, $(X,\tilde{\mathcal{T}}) = \omega(X,\mathcal{T})$ where $\omega : \mathrm{Top} \longrightarrow \mathrm{Fuz}_L$ is R.Lowen's embedding functor [9],[10].

(In this connection see also (5.4) below.)

(4.5) Definition (cf. [3]). Let $(X,\mathcal{T})$ be a topological space and define the mapping $\tau : I^X \longrightarrow I$ by the equality $\tau(\mu) = (\mu \mathrel{\tilde{\subset}} \mu^o)$ for $\mu \in I^X$. The mapping τ will be called the fuzzy topology induced by the (crisp) topology $\mathcal{T}$.

The correctedness of this definition is assured by the following

(4.6) Theorem The mapping τ is really a fuzzy topology. Moreover, the restriction of τ to 2^X coincides with $\mathcal{T}$.

Proof. (1) Let $\mu, \nu \in I^X$. Then $\tau(\mu\wedge\nu) = \inf_X((\mu\wedge\nu)^c \vee (\mu\wedge\nu)^o)(x) = \inf_X((\mu^c\vee\nu^c)\vee(\mu^o\wedge\nu^o))(x) \geqslant \inf_X((\mu^c\vee\mu^o) \wedge(\nu^c\vee\nu^o))(x) \geqslant \inf_X(\mu^c\vee\mu^o)(x) \wedge \inf_X(\nu^c\vee\nu^o)(x) = \tau(\mu)\wedge\tau(\nu)$.

(2) Let $\mu_i \in I^X$ for all $i\in \mathcal{J}$. Then $\tau(\bigvee_i \mu_i) =$

$\inf_X((\bigvee_i\mu_i)^c \vee (\bigvee\mu_i)^o)(x) = \inf_X((\bigwedge_i\mu_i^c)\vee(\bigvee_i\mu_i^o))(x) \geqslant$

$\inf_X \bigwedge_i(\mu_i^c\vee\mu_i^o) = \bigwedge_i \inf_X(\mu_i^c\vee\mu_i^o) = \bigwedge_i \tau(\mu_i)$.

(3) The equality $\tau(0) = \tau(1) = 1$ is obvious.

The second part of the theorem follows directly from the definitions.

(4.7) Denotation. Let IFT denote the full subcategory of FT consisting of all fuzzy topological spaces (X,τ) the fuzzy topology τ of which is induced by some topology $\mathcal{T}$ on X.

(4.8) Lemma. For every fuzzy subset μ of a topological space $(X,\mathcal{T})$ it holds $(\bar{\mu} \mathrel{\tilde{\subset}} \mu) = (\mu^c \mathrel{\tilde{\subset}} (\mu^c)^o)$.

Proof. From (4.4) one can easily conclude that $\bar{\mu}^c = (\mu^o)^c$ for every $\mu\in I^X$. Therefore $(\mu^c \mathrel{\tilde{\subset}} (\mu^c)^o) = \inf_X((\mu^c)^c\vee(\mu^c)^o)(x)$ $\inf_X(\mu \vee (\mu^c)^o)(x) = \inf_X(\mu\vee\bar{\mu}^c) = (\bar{\mu} \mathrel{\tilde{\subset}} \mu)$.

From the above Lemma and the definition (2.5) we immediately obtain the following statement, the intuitive sense of which seems to be clear:

(4.9) Proposition. For any fuzzy subset μ of a topological space $(X,\mathcal{T})$ the equality $\tau^*(\mu) = (\bar{\mu} \mathrel{\tilde{\subset}} \mu)$ holds.

(4.10) Remark. It is worth noting that for every proper fuzzy set $\tau(\mu) < 1$ and $\tau^{*}(\mu) < 1$.

(4.11) Theorem (cf. [3]). Let $(X, \mathcal{T})$, $(Y, \mathcal{S})$ be topological spaces and let τ and σ be fuzzy topologies induced by $\mathcal{T}$ and $\mathcal{S}$ respectively. If a mapping $f: (X, \mathcal{T}) \to (Y, \mathcal{S})$ is continuous, then the mapping $f: (X, \tau) \to (Y, \sigma)$ is fuzzy continuous.

Proof. Let $\nu \in I^{Y}$, then $\sigma(\nu) = \inf\limits_{y} ((1 - \nu)(y) \vee \sup\limits_{V \in \mathcal{S}_y} \inf\limits_{y' \in V} \nu(y'))$

($\mathcal{S}_y$ denotes the family of open neighbourhoods of y in $(Y, \mathcal{S})$).
On the other hand $\tau(f^{-1}(\nu)) = \inf\limits_{x} ((1 - f^{-1}(\nu))(x) \vee$
$\vee \sup\limits_{U \in \mathcal{T}_x} \inf\limits_{x' \in U} f^{-1}(\nu)(x') = \inf\limits_{x} (1 - \nu f(x)) \vee \sup\limits_{U \in \mathcal{T}_x} \inf\limits_{x' \in U} \nu f(x')$.

Fix $x \in X$ and let $y = f(x)$. To prove the theorem, i.e. the inequality $\tau(f^{-1}(\nu)) \geq \sigma(\nu)$ it suffices to show that
$\sup\limits_{V \in \mathcal{T}_y} \inf\limits_{y' \in V} \nu(y') \leq \sup\limits_{U \in \mathcal{T}_x} \inf\limits_{x' \in U} f^{-1}(\nu)(x')$. But this follows from the obvious inequality $\inf\limits_{y' \in V} \nu(y') \leq \inf\limits_{x' \in f^{-1}(V)} \nu f(x')$ and the fact $f^{-1}(V) \in \mathcal{T}_x$ for every $V \in \mathcal{T}_y$.

(4.12) Functor Φ : Top $\to$ FT .

For every topological space $(X, \mathcal{T})$ let $\Phi(X, \mathcal{T}) = (X, \tau)$ where τ is the fuzzy topology induced by $\mathcal{T}$. The theorem (4.11) ensures that, if $f: (X, \mathcal{T}) \to (Y, \mathcal{S})$ is a morphism in Top then $\Phi(f) = f: \Phi(X, \mathcal{T}) \to \Phi(Y, \mathcal{S})$ is a morphism in FT . Thus we get a functor Φ : Top $\to$ FT. It is clear that Φ is an embedding functor and the image Φ (Top) is just the category IFT as defined in (4.7).

(4.13) Functor Ψ: FT $\to$ Top .

For a fuzzy topological space (X, τ) let $\mathcal{T} = \{U: U \in 2^{X}, \tau(U) = 1\}$. It is easy to notice that $\mathcal{T}$ is a topology on X . Let $\Psi(X, \tau) = (X, \mathcal{T})$. If $f: (X, \tau) \to (Y, \sigma)$ is a fuzzy continuous mapping then one can easily check, that $f: (X, \mathcal{T}) \to (Y, \mathcal{S})$ where $(Y, \mathcal{S}) = \Psi(Y, \sigma)$ is a continuous mapping of the corresponding topological spaces. Thus by letting $\Psi(f) = f$ for every morphism of FT we obtain a functor Ψ: FT $\to$ Top . Moreover, it is clear that $\Psi \circ \Phi$: Top $\to$ Top is the identity functor.

(4.14) Proposition Functors Φ and Ψ are isotone.

Proof follows immediately from thedefinitions.

5. FUZ as a subcategory of FT.

Let Fuz denote the category of fuzzy topological spaces in the sense of C.Chang [2] and let Fuz_L denote its subcategory consisting of all fuzzy topological spaces in the sense of R.Lowen [8]. If (X,T) is an object of Fuz, then T can be considered as a mapping $T: I^X \longrightarrow 2$ and hence also as a mapping $T: I^X \longrightarrow I$. Moreover, from Chang's definition [2] it immediately follows that T satisfies the axioms of the definition (2.1). It is easy to check also that if $f:(X,T) \longrightarrow (Y,S)$ is a morphism in Fuz, then $f: (X,T) \longrightarrow (Y,S)$ is also fuzzy continuous in our sense (see definition (2.9)) and therefore it is a morphism of FT. Summing up we obtain the following statement:

(5.1) Proposition. The category Fuz and hence also the category Fuz_L are full subcategories of FT.

The next theorem strengthens this result.

(5.2) Theorem. Fuz is an epireflective subcategory of FT.

Proof. Let θ denote the inclusion functor of Fuz into FT. To construct the functor $\mathcal{P}$: FT $\longrightarrow$ Fuz which is a left ajoint for θ, take an object $(X,\mathcal{T})$ from FT and let $T = \{\mu : \mu \in I^X, \mathcal{T}(\mu) = 1\} \subset I^X$. It is easy to notice, that (X,T) is an object of Fuz. Let $\mathcal{P}(X,\mathcal{T}) = (X,T)$. One can easily check that if $f: (X,\mathcal{T}) \longrightarrow (Y,\sigma)$ is a morphism in FT then $f: (X,T) \longrightarrow (Y,S)$ is also a morphism in Fuz where $(X,T) = \mathcal{P}(X,\mathcal{T})$ and $(Y,S) = \mathcal{P}(Y,\sigma)$. Therefore, by letting $\mathcal{P}(f) = f$ we obtain a functor $\mathcal{P}$: FT $\longrightarrow$ Fuz. The composition $\mathcal{P}\circ\theta$: Fuz $\longrightarrow$ Fuz is obviously the identical functor.

It is easy to notice that for every fuzzy topological space $(X,\mathcal{T})$ the identical mapping $\rho : (X,\mathcal{T}) \longrightarrow (X,T)$ where $(X,T) = \mathcal{P}(X,\mathcal{T})$ is a morphism in FT. Furthermore, for any object (Y,S) of Fuz and any morphism $f: (X,\mathcal{T}) \longrightarrow (Y,S)$ there exists a morphism $g: (X,T) \longrightarrow (Y,S)$ in FT such that $f = \rho \circ g$. Thus we can conclude, that ρ is an epireflector in FT and hence Fuz is indeed an epireflective subcategory of FT.

(5.3) Remark. Since Top can be considered as a subcategory of Fuz, restricting the functor θ which is defined in (5.2) to Top, one gets an embedding functor θ' : Top $\longrightarrow$ FT. From the definition of ψ (see (4.13)) it is clear, that $\psi\circ\theta'$: Top $\longrightarrow$ Top is the identical functor.

(5.4) Remark. Let ω : Top $\longrightarrow$ Fuz_L be R.Lowen's embedding functor [9]. Then the composition $\theta\circ\omega$: Top $\longrightarrow$ FT is again an embedding of Top into FT such that the functor $\gamma\circ\theta\circ\omega$: Top $\longrightarrow$ FT is the identity.

Thus there are at least 3 embeddings of Top into Fuz which seem to be natural: they are realized by functors Φ , $\cdot\theta'$ and $\theta\circ\omega$ respectively.

6. The degree of compactness of a fuzzy set

In this section we introduce the notion of the degree of compactness for a fuzzy set in a fuzzy topological space. Some theorems concerning this notion are formulated. There are no proofs in this section: most of them are rather bulky and involve a special filter-type construction. The proofs as well as some details in this connection will be published elsewhere.

Let (X,τ) be a fuzzy topological space and $\mathcal{U}\subset I^X$ a crisp family of its subsets. Denote $\tau(\mathcal{U}) = \inf\{\tau(\mu) : \mu\in\mathcal{U}\}$. For a given $\mathcal{U}$ let $\mathcal{U}_0$ denote an arbitrary finite subfamily of $\mathcal{U}$.

(6.1) Definition. Let μ be a fuzzy subset of a fuzzy topological space (X,τ) and let $\alpha\in(0,1]$. The degree of compactness with respect to α-open sets is defined by the equality

$$c_\alpha(\mu) = \inf\{(\mu \mathrel{\tilde{\subset}} \vee\mathcal{U})^c \vee \sup\{(\mu \mathrel{\tilde{\subset}} \vee\mathcal{U}_0) : \mathcal{U}_0\subset\mathcal{U}\} : \tau(\mathcal{U})\geqslant\alpha\}.$$

(6.2) Definition. Let $c_0(\mu) = \inf\{c_\alpha(\mu) : \alpha\in(0,1]\}$.

(6.3) Example. Let $(X,\mathcal{T})$ be a topological space and let the function $\mathscr{C} : 2^X \longrightarrow \{0,1\}$ be defined by the equality

$$(6.1)' \quad \mathscr{C}(A) = \inf\{(A\subset\cup\mathcal{U})^c \vee \sup\{(\mu\subset\cup\mathcal{U}_0) : \mathcal{U}_0\subset\mathcal{U}\} : \mathcal{U}\subset\mathcal{T}\}.$$

It is easy to notice that $\mathscr{C}(A) = 1$ iff A is compact. On the other hand $c_\alpha(A) = \mathscr{C}(A)$ for any $\alpha\in[0,1]$. (In the left side of this equality A is understood as a subset of the fuzzy space $(X,\tau) = \Phi(X,\mathcal{T})$).

Therefore the definition (6.1) can be understood as a "fuzzification" of the equality (6.1)'.

(6.4) Proposition. If (X,τ) is a fuzzy topological space, $\mu,\nu\in I^X$ and $\alpha\in[0,1]$, then $c_\alpha(\mu\vee\nu) \geqslant c_\alpha(\mu)\vee c_\alpha(\nu)$.

(6.5) Theorem Let $f: (X,\tau)\rightarrow(Y,\sigma)$ be a fuzzy continuous mapping. Then for every $\mu\in I^X$ and every $\alpha\in I$ it holds $c_\alpha(\mu)\leq c_\alpha(f(\mu))$.

(6.6) Theorem. Let $\{(X_a,\tau_a): a\in\mathcal{A}\}$ be a family of fuzzy topological spaces and for every a μ_a is a fuzzy subset of X_a . Then $c_1(\prod\mu_a) = \bigwedge c_1(\mu_a)$.

Let $(X,\mathcal{T})$ be a topological space and $(X,\tau) = \Phi(X,\mathcal{T})$. The following result is obtained by Z.Diskin (see [3]).

(6.7) Theorem. (1) If $(X,\mathcal{T})$ is Hausdorff and $\mu \in I^X$ then $c_1(\mu) \leq \tau^*(\mu)$;

(2) if $(X,\mathcal{T})$ is compact, then $\tau^*(\mu) \leq c_1(\mu)$.

(6.8) Corollary. If $(X,\mathcal{T})$ is a compact Hausdorff space and $\mu \in I^X$ then $\tau^*(\mu) = c_1(\mu)$.

(6.9) Remark. The crisp prototypes of the statements (6.4) - (6.8) are well-known. For example, the theorem (6.7) when restricted to Top just states that compact subsets of a Hausdorff space are closed, and closed subsets of a compact space are compact.

There are some problems concerning the degree of compactness which we could not solve. Among them

(6.10) Problem. Do the statements of theorems (6.6) and (6.7) remain true also for α distinct from 1 ?

The essential part of this work was accomplished during the author's stay in Charles University of Prague in January and February 1985. The author is thankful to the administration of the Mathematical department of the University for giving him such an opportunity and good conditions for a work.

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