Zdzisław Pogoda
The regularity condition for some natural functors

In: Zdeněk Frolík and Vladimír Souček and Marián J. Fabián (eds.): Proceedings of the 14th Winter School on Abstract Analysis. Circolo Matematico di Palermo, Palermo, 1987. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 14. pp. [105]–115.

Persistent URL: http://dml.cz/dmlcz/701891

THE REGULARITY CONDITION FOR SOME NATURAL FUNCTORS

Zdzisław Pogoda

This paper is in final form and no version of it will be submitted for publication elsewhere.

In this note we present some generalizations of the notion of natural bundle, introduced by A.Nijenhuis in [3], studied by R.Palais , C.L.Terng [4] and D.B.A.Epstein, W.Thurston [1]. We define a natural functor from the category of affine bundles and their isomorphisms into the category of double bundles. We prove that the regularity condition for this natural functor, analoguous to the one given in the paper of Palais-Terng is a consequence of the other conditions. Additionally, an estimate of the rank of such a natural functor is given.

I. Basic definitions and examples.

Let $\mathcal{AB}_{n,k}$ be the category of affine bundles and their local isomorphisms, i.e. the category whose objects are fibre bundles (locally trivial) with the affine fibre of dimension k and the base manifold of dimension n. Morphisms are the following:
Let $\pi_1 : E_1 \longrightarrow M_1$ and $\pi_2 : E_2 \longrightarrow M_2$ be two affine bundles of $\mathcal{AB}_{n,k}$ and U_1 , U_2 be two open subsets of M_1 and M_2 ,respectively.Then a diffeomorphism $\Phi : \pi_1^{-1}(U_1) \longrightarrow \pi_2^{-1}(U_2)$ is called a local isomorphism of E_1 into E_2 if on each fibre it is an affine isomorphism. In practical terms, the affine fibre is the standard affine space $\mathbb{R}^k$. Each morphism of the trivial affine bundle $\mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$ into itself is of the form

$$\mathbb{R}^n \times \mathbb{R}^k \ni (x,v) \longmapsto (\varphi(x),\ f_x(v)) \in \mathbb{R}^n \times \mathbb{R}^k$$

where $\varphi : U_1 \longrightarrow U_2$ is a diffeomorphism of some open subsets U_1 and U_2 of $\mathbb{R}^n$ and $f : U_1 \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$ is a mapping such that for any x of U_1, $f(x,\cdot) = f_x$ is an affine isomorphism of $\mathbb{R}^k$ onto itself. We shall also consider the category $2\text{-}\mathcal{FB}$ of double bundles. The objects of this category are the triples $P \rightarrow E \rightarrow M$ where $P \rightarrow E$ and $E \rightarrow M$ are fibre bundles and morphisms are triples (F, f, f_0) for which the following diagram is commutative

$$\begin{array}{ccc} P_1 & \xrightarrow{\quad F \quad} & P_2 \\ \downarrow & & \downarrow \\ E_1 & \xrightarrow{\quad f \quad} & E_2 \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{\quad f_0 \quad} & M_2 \end{array}$$

We are interested in functors from the category $\mathcal{AB}_{n,k}$ into $2\text{-}\mathcal{FB}$

<u>Definition</u> 1. A functor $\mathcal{F}$ $(= \mathcal{F}_{n,k})$ from the category $\mathcal{AB}_{n,k}$ into $2\text{-}\mathcal{FB}$ is a natural functor if:

1) for any fibre bundle $E \rightarrow M$ of $\mathcal{AB}_{n,k}$, $\mathcal{F}E \rightarrow E \rightarrow M$ is a double bundle of $2\text{-}\mathcal{FB}$

2) for any morphism (local isomorphism) Φ of affine bundles, $\mathcal{F}\Phi$ is a morphism in the category $2\text{-}\mathcal{FB}$ covering i.e. the following diagram is commutative

$$\begin{array}{ccc} \mathcal{F}E_1 & \xrightarrow{\quad \mathcal{F}\Phi \quad} & \mathcal{F}E_2 \\ \downarrow & & \downarrow \\ E_1 & \xrightarrow{\quad \Phi \quad} & E_2 \end{array}$$

To be precise, $\mathcal{F}\Phi$ is a diffeomorphism of $\pi^{-1}_{E_1}(\mathrm{dom}\,\Phi)$ onto π^{-1}_{E}[illegible] where $\pi_{E_i} : \mathcal{F}E_i \longrightarrow E_i$, $i=1,2$ are the projections.

Moreover, an additional regularity condition is being formulated, which as it turns out, is a consequence of the other conditions.

3) Let U be an open subset of $\mathbb{R}$, E and $E' \in \mathcal{AB}_{n,k}$ and $\varphi : U \times E \quad E'$ be a mapping satisfying the following condition: for any $t \in U$, $\varphi_t = \varphi(t,\cdot)$ is a C^∞-morphism in $\mathcal{AB}_{n,k}$ then the mapping

$$U \times \mathcal{F}E \ni (t,x) \longmapsto (\mathcal{F}\varphi_t)(x) \in \mathcal{F}E'$$

is of class C^∞

Fundamental examples.

1) Let $E \in \mathcal{AB}_{n,k}$. Denote by $J(E,r)$, where $r \geqslant 1$, the space of r-jets at the point $(0,0)$ of local isomorphisms (morphisms in $\mathcal{AB}_{n,k}$) $\mathbb{R}^n \times \mathbb{R}^k \longrightarrow E$. The mapping $p_E \colon J(E,r) \longrightarrow E$, $j^r_{(0,0)}\varphi \longmapsto \varphi(0,0)$ is the natural projection. The space $J(E,r)$ has the structure of a fibre bundle and thus it is a principal fibre bundle with base and structure group $p^{-1}_{\mathbb{R}^n \times \mathbb{R}^k}(0,0) = AL^r_{n,k}$. The group $AL^r_{n,k}$ is the set of r-jets of local isomorphisms $\varphi \colon \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n \times \mathbb{R}^k$ which are morphisms in $\mathcal{AB}_{n,k}$ such that $\varphi(0,0) = (0,0)$. If $\Phi \colon E_1 \longrightarrow E_2$ is a local isomorphism of affine bundles, then the mapping $J^r\Phi \colon J(E_1,r) \longrightarrow J(E_2,r)$, $j^r_{(0,0)} f \longmapsto j^r_{(0,0)} \Phi \circ f$ is a morphism of the corresponding principal fibre bundles. The correspondence $E \longrightarrow J(E,r)$ defines a functor from the category $\mathcal{AB}_{n,k}$ into $2\text{-}\mathcal{FB}$

2) Let F be a smooth manifold with a countable basis on which the group $AL^r_{n,k}$ acts on the left. We are going to define a bundle FE associated to $J(E,r)$ with base E, fibre F and structure group $AL^r_{n,k}$. To be precise, FE is the space of orbits of the action of the group $AL^r_{n,k}$ on $J(E,r) \times F$ defined in the following way:

$$(f, \xi)g = (fg, g^{-1}\xi)$$

where $f \in J(E,r)$, $g \in AL^r_{n,k}$, $\xi \in F$.
The correspondence $E \longrightarrow FE$ defines a new functor from the category $\mathcal{AB}_{n,k}$ into $2\text{-}\mathcal{FB}$

The main theorem of this note is the following:

<u>Theorem A</u>. Let $\mathcal{F}$ be a natural functor from the category $\mathcal{AB}_{n,k}$ into $2\text{-}\mathcal{FB}$ (that is a functor satisfying the conditions 1 , 2 of Definition 1), then for some $r \geqslant 1$ and a manifold F on which the group $AL^r_{n,k}$ acts, the bundle $\mathcal{F}E$ is isomorphic with FE. Moreover, if $\pi^{-1}_{\mathbb{R}^n \times \mathbb{R}^k}(0,0)$ is an f- dimensional manifold, then

$$r \leqslant 2\dim \pi^{-1}_{\mathbb{R}^n \times \mathbb{R}^k}(0,0) + 1$$

In the language of the category theory Theorem A asserts that there exists a natural equivalence between the functor $\mathcal{F}$ and the corresponding functor of the associated fibre bundles (with fibre

F and structure group $AL^r_{n,k}$). Theorem A is a generalization of results of Palais-Terng [4] and Epstein, Thurston [1]. In the case of $\mathbb{R}^k = \{0\}$, i.e. k=0, we get the theorem of Epstein Thurston from [1].

II. Continuity of translations and actions.

Before passing to the proof of Theorem A we give some notations, lemmas and auxiliary theorems. Let $(x,y) \in \mathbb{R}^n \times \mathbb{R}^k \in \mathcal{AB}_{n,k}$. We denote by $\tau_{(x,y)}: \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n \times \mathbb{R}^k$ the translation given by the formula $\tau_{(x,y)}(u,v) = (x+u, y+v)$. Now we can define the following mapping

$$\mathcal{F}\tau : \mathbb{R}^n \times \mathbb{R}^k \times \mathcal{F}(\mathbb{R}^n \times \mathbb{R}^k) \longrightarrow \mathcal{F}(\mathbb{R}^n \times \mathbb{R}^k)$$
$$((x,y), v) \longmapsto (\mathcal{F}\tau_{(x,y)})(v)$$

One of the main steps of the proof of Theorem A is the following theorem.

<u>Theorem B</u>. The mapping $\mathcal{F}\tau$ is continuous.

The proof of this theorem consists in several lemmas and constructions. Let $\mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$ be a standard trivial affine bundle, and $\mathcal{F}(\mathbb{R}^n \times \mathbb{R}^k)$ be the corresponding double fibre bundle i.e.

$$\mathcal{F}(\mathbb{R}^n \times \mathbb{R}^k) \longrightarrow \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$$

Let us denote $\mathcal{F}\mathbb{R}^k = \pi^{-1}_{\mathbb{R}^n \times \mathbb{R}^k}(\{0\} \times \mathbb{R}^k)$ and let the projection $\pi : \mathcal{F}\mathbb{R}^k \longrightarrow \mathbb{R}^k$ be given by $\pi^{-1}_{\mathbb{R}^n \times \mathbb{R}^k}(0,v) \longrightarrow v$. Moreover, let $\tau_v = \tau_{(0,v)}$ and

$$\mathcal{F}^* : \mathbb{R}^k \times \mathcal{F}(\mathbb{R}^k) \longrightarrow \mathcal{F}(\mathbb{R}^k)$$
$$(v, x) \longmapsto \mathcal{F}\tau_v(x)$$

Then $\mathcal{F}^*$ defines an action of $\mathbb{R}^k$ on $\mathcal{F}(\mathbb{R}^k)$. The following Lemma is an immediate consequence of Theorem 5.1 of [1].

<u>Lemma 1</u>. The mapping

$$\mathbb{R}^n \times \mathcal{F}(\mathbb{R}^n \times \mathbb{R}^k) \ni (x,v) \longmapsto \mathcal{F}\tau_{(x,0)}(v) \in \mathcal{F}(\mathbb{R}^n \times \mathbb{R}^k)$$

is continuous.

<u>Proof</u>. For any smooth n-dimensional manifold M, let $\tilde{\mathcal{F}}M = \mathcal{F}(M \times \mathbb{R}^k)$ and $p_M : \tilde{\mathcal{F}}M \longrightarrow M$ be the composition of the projections

$$\pi_{M\times\mathbb{R}^k} : \mathcal{F}(M\times\mathbb{R}^k) \longrightarrow M\times\mathbb{R}^k \quad \text{and} \quad M\times\mathbb{R}^k \longrightarrow M$$

For any embedding $\varphi : M \longrightarrow N$ of two n-dimensional manifolds M and N let $\tilde{\mathcal{F}}\varphi = \mathcal{F}(\varphi \times \mathrm{id}_{\mathbb{R}^k})$. In this way we have obtained a natural bundle. Using the Epstein-Thurston theorem we get the lemma.

Now we shall state and prove two technical lemmas.

Lemma 2. Let $k>0$. Let us take a sequence $w_i \in \mathcal{F}\mathbb{R}^k$, $i = 1,2,\ldots$ having the following property: if $\{w_i\}$ $w \in \mathcal{F}\mathbb{R}^k$, then there exists a family of open neighbourhoods $\{V_i\}$ of 0 in $\mathbb{R}^k$ such that if $a \in \mathbb{R}^k$ and if $W \subset \mathcal{F}\mathbb{R}^k$ is an open neighbourhood of $\mathcal{F}\tau_a(w) \in \mathcal{F}\mathbb{R}^k$ then

$$\mathcal{F}^*\left((a+V_{2i})\times\{w_{2i}\}\right) \subset W$$

for i sufficiently large.

Proof. We shall prove that the neighbourhoods of the form

$$V_i = \{x \in \mathbb{R}^k : \ \|x\| < e^{-i}\} \qquad i = 1,2,\ldots$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^k$. Let us take $a \in \mathbb{R}^k$ and W an open neighbourhood of $\mathcal{F}\tau_a(w)$. Let us assume the contrary that is

$$\mathcal{F}^*\left((a+V_{2i})\times\{w_{2i}\}\right) \not\subset W$$

for an infinite number of i. Thus we can choose a sequence of points $c_i \in \mathbb{R}^k$ such that $c_i \in V_i$ for any i and $\mathcal{F}\tau_{a+c_{2i}}(w_{2i}) \notin W$ for an infinite number of i. We define a new sequence $u_{2i}=c_{2i}$ and $u_{2i+1}=0$. Using the Whitney theorem about extensions we get a smooth mapping $h : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ having the property that for i sufficiently large h is equal to u_i on $B(x_i, e^{-i})$ where $x_i = (\frac{1}{i}, 0, \ldots, 0) \in \mathbb{R}^n$. We define a local isomorphism H of $\mathbb{R}^n\times\mathbb{R}^k$ into $\mathbb{R}^n\times\mathbb{R}^k$ by the formula:

$$H : (x,\xi) \longmapsto (x, a+h(x)+\xi)$$

Then $\tau_{(-x_i,0)}\circ H \circ \tau_{(x_i,0)} = \tau_{(0,a+u_i)}$ on $B(0,e^{-i})\times\mathbb{R}^k$ for i even and is equal to $\tau_{(0,a)}$ on $B(0,e^{-i})\times\mathbb{R}^k$ for i odd. Of course, these equalities are true for i sufficiently large. Thus

$$\mathcal{F}\tau_{(-x_i,0)}\circ \mathcal{F}H \circ \mathcal{F}\tau_{(x_i,0)}(w_i) = \mathcal{F}\tau_a(w_i)$$

for i odd, and

$$\mathcal{F}\tau_{(-x_i,0)} \circ \mathcal{F}H \circ \mathcal{F}\tau_{(x_i,0)}(w_i) = \mathcal{F}\tau_{a+u_i}(w_i)$$

for i even and sufficiently large. Since the space $\mathcal{F}(\mathbb{R}^n \times \mathbb{R}^k)$ is Hausdorff, according to Lemma 1, the sequence $\mathcal{F}\tau_{a+u_{2i}}(w_{2i})$ is convergent to $\mathcal{F}\tau_a(w)$, thus a contradiction.

Lemma 3. Let $w_i \in \mathcal{F}\mathbb{R}^k$ $(k>0,\ i=1,2,\dots)$ be a sequence of points convergent to $w \in \mathcal{F}\mathbb{R}^k$. Let W be an open neighbourhood of ω in $\mathcal{F}\mathbb{R}^k$ Then there exists an open subset V in $\mathbb{R}^k$ satisfying the following condition:

$\mathcal{F}^*(V \times \{w\}) \subset W$ and $\mathcal{F}^*(V \times \{w_{2i}\}) \subset W$

for i sufficiently large.

Proof. Using the previous lemma for $a=0$, there we shall find an open neighbourhood U of 0 in $\mathbb{R}^k$ such that $\mathcal{F}^*(U \times \{w_i\}) \subset W$ Let $V_i \subset \mathbb{R}^k$ $(i=1,2,\dots)$ be neighbourhoods of 0 satisfying the condition: if $\mathcal{F}\tau_a(w) \in W$, then $\mathcal{F}^*((a+V_{2i}) \times \{w_{2i}\}) \subset W$ for i sufficiently large. Then

$$U \subset \bigcup_{k\in\mathbb{N}} \bigcap_{i\geq k} \{a \in \mathbb{R}^k : \mathcal{F}^*((a+V_{2i}) \times \{w_{2i}\}) \subset W\}$$

For any $k \in \mathbb{N}$ we put

$$H_k = \bigcap_{i\geq k} \{a \in U : \mathcal{F}^*((a+V_{2i}) \times \{w_{2i}\}) \subset W\}$$

thus $U = \bigcup_{k\in\mathbb{N}} H_k$

According to the Baire category theorem there exists k and a non-empty set $V \subset \mathbb{R}^k$ such that $V \subset \operatorname{int}(\overline{H}_k) \cap U$. We shall prove that V satisfies the conditions of lemma. Let $c \in V$ and $i \geq k$. We want to find $a \in H_k$ such that $a \in c+(-V_{2i})$, thus $c \in a+V_{2i}$, and hence $\mathcal{F}^*(c, w_{2i}) \in W$. Therefore $\mathcal{F}^*(V \times \{w_{2i}\}) \subset W$ for $i \geq k$. Since $U \supset V$, we get $\mathcal{F}^*(V \times \{w\}) \subset \mathcal{F}^*(U \times \{w\}) \subset W$, which ends the proof.

Now we can proceed with the proof of Theorem B. It is easy to check that for any $((x,v),z) \in (\mathbb{R}^n \times \mathbb{R}^k) \times \mathcal{F}(\mathbb{R}^n \times \mathbb{R}^k)$ we have:

$$\mathcal{F}\tau((x,v),z) =$$
$$= \mathcal{F}\tau_{(x+p\circ\pi_{\mathbb{R}^n\times\mathbb{R}^k}(z),0)} \circ \mathcal{F}^*(v, \mathcal{F}\tau_{(-p\circ\pi_{\mathbb{R}^n\times\mathbb{R}^k}(z),0)}(z))$$

where $p: \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$. According to Lemma 1 it is sufficient to show that $\mathcal{F}^*$ is continuous. Let $\xi_i \in \mathbb{R}^k$, $i=1,2,\dots$ be a sequence convergent to $\xi \in \mathbb{R}^k$, $w_i \in \mathcal{F}\mathbb{R}^k$ a sequence convergent to $w \in \mathcal{F}\mathbb{R}^k$. We shall prove that $\mathcal{F}^*(\xi_i, w_i) \longrightarrow \mathcal{F}^*(\xi, w)$. Without loosing generality we can assume that $\xi = \pi(w) = 0$. We shall prove that subsequence of the sequence $\mathcal{F}^*(\xi_i, w_i)$ contains a subsequence convergent to $\mathcal{F}^*(\xi, w)$. It is sufficient to achieve the demonstration of the theorem. Let $\{V_p\}_{p\in\mathbb{N}}$ be a basis of open subsets of the space $\mathcal{F}\mathbb{R}^k$. Let us denote

$$Q_p = \{u \in \mathbb{R}^k : \mathcal{F}^*(u+\xi_{2i}, w_{2i}) \in V_p \text{ for i sufficiently large}\}$$

$$\Omega_p = \{a \in \mathbb{R}^k : \mathcal{F}^*(a, w) \in V_p,\ a \notin Q_p\}$$

and

$$A = \{a \in \mathbb{R}^k : \mathcal{F}^*(a+\xi_{2i}, w_{2i}) \text{ is not convergent to } \mathcal{F}^*(a,w)\}$$

We shall show that $A \subset \bigcup_{p\in\mathbb{N}} \Omega_p$. Let us assume, that $\mathcal{F}(\xi_{2i}, w_{2i})$ is not convergent to w. Hence we get $A = \mathbb{R}^k$. From the Baire theorem there exists $p \in \mathbb{N}$ such that $\operatorname{int}(\overline{\Omega}_p) \neq \emptyset$. Let us take $a \in \Omega_p \cap \operatorname{int}(\overline{\Omega}_p)$, then

$$W = \pi^{-1}(-a + \operatorname{int}\overline{\Omega}_p) \cap \mathcal{F}^*(\{-a\} \times V_p)$$

is an open neighbourhood. According to Lemma 3 there exists an open subset $V \subset \mathbb{R}^k$ satisfying the conditions:

$*$) $\mathcal{F}^*(V \times \{w\}) \subset \pi^{-1}(-a + \operatorname{int}\overline{\Omega}_p)$

$**$) $\mathcal{F}^*(V \times \{w_{2i}\}) \subset \mathcal{F}^*(\{a\} \times V_p)$ for i sufficiently large.

$*$) implies that $a + V \subset \operatorname{int}\overline{\Omega}_p$, and $**$) that $\mathcal{F}^*((a+V) \times \{w_{2i}\}) \subset V_p$ for i sufficiently large, hence, as ξ_i tends to 0, $a + V \subset Q_p$. It is so, as $\eta \in a+V$ implies that $\eta + \xi_{2i} \in a+V$ for i sufficiently large, and then $\mathcal{F}^*(\eta + \xi_{2i}, w_{2i}) \subset V_p$ for i sufficiently large. Thus $\eta \in Q_p$
Moreover, $a + V \subset Q_p \cap \operatorname{int}(\overline{\Omega}_p) \subset (\mathbb{R} \setminus \Omega_p) \cap \operatorname{int}(\overline{\Omega}_p)$.
A contradiction.

As a consequence of the theory of Lie group we get (cf. [2])

Theorem C. Let $\mathcal{F}$ be a natural functor satisfying the conditions 1) and 2) of Definition 1. Then the action

$$\mathcal{F}\tau : \mathbb{R}^n \times \mathbb{R}^k \times \mathcal{F}(\mathbb{R}^n \times \mathbb{R}^k) \longrightarrow \mathcal{F}(\mathbb{R}^n \times \mathbb{R}^k)$$

is smooth (C^∞).

For $F = \pi^{-1}_{\mathbb{R}^n \times \mathbb{R}^k}(0,0)$ we have a following homeomorphism $(\mathbb{R}^n \times \mathbb{R}^k) \times F \to \mathcal{F}(\mathbb{R}^n \times \mathbb{R}^k)$ given by the formula

$$((x,\xi),v) \longmapsto \mathcal{F}\tau_{(x,\xi)}(v)$$

The inverse is defined as follows

$$u \longmapsto (\pi_{\mathbb{R}^n \times \mathbb{R}^k}(u), \mathcal{F}\tau_{-\pi_{\mathbb{R}^n \times \mathbb{R}^k}(u)}(u))$$

Both mappings are of class C^∞. Moreover, F is a smooth submanifold of $\mathcal{F}(\mathbb{R}^n \times \mathbb{R}^k)$

Now we are in the position to formulate a theorem which implies the regularity condition for a natural functor.

Theorem D. Let $\Phi_m : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n \times \mathbb{R}^k$ for m=1,... and $\Phi : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n \times \mathbb{R}^k$ be morphisms in the category $\mathcal{AB}_{n,k}$ such that the sequence $j^r_{(0,0)}\Phi_m$ is convergent to $j^r_{(0,0)}\Phi$. Moreover, let $v_m \in \pi^{-1}(0)$ be a sequence convergent to $v \in \pi^{-1}(0)$, where $\pi : \mathcal{F}\mathbb{R}^k \longrightarrow \mathbb{R}^k$. Then $\mathcal{F}\Phi_m(v_m)$ is convergent to $\mathcal{F}\Phi(v)$. In particular, if φ and ψ are two morphisms in $\mathcal{AB}_{n,k}$ having the same r-jet at (0,0) then $\mathcal{F}\varphi$ and $\mathcal{F}\psi$ are equal on $\pi^{-1}(0)$.

Proof. Substituting Φ_m by $\Phi^{-1} \circ \Phi_m$, we can assume $\Phi = \mathrm{id}$. Applying the form of Φ_m we can construct a sequence of smooth embeddings $\varphi_m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and a sequence of smooth mappings $h_m : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ such that

$$\Phi_m(x,\xi) = (\varphi_m(x), h_m(x) + \xi)$$

for m=1,... It is clear that the r-jets of (φ_m, h_m) at 0 are convergent to r-jet of $(\mathrm{id}_{\mathbb{R}^n}, 0)$, where $0 : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is a constant mapping given by the formula $x \longmapsto 0$. We shall show that any subsequence of the sequence $\mathcal{F}\Phi_m(v_m)$ contains a subsequence convergent to v. By taking a subsequence, we can assume that

$$\| D(\varphi_m - id_{\mathbb{R}^n}, h_m)(0) \| < e^{-m}$$

for any differential operator D obtained from partial differential relative to natural coordinates. Let us choose $\varepsilon_m < e^{-m}$, $\varepsilon_m > 0$ so that on a ε_m-neighbourhood of 0 $D(\varphi_m - id_{\mathbb{R}^n}, h_m)$ is smaller than e^{-m} for any differential operator D as above. Let $x_m = (\frac{1}{m}, 0, \ldots 0) \in \mathbb{R}^n$ According to the Whitney extension theorem there exists a smooth mapping $h: \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^k$ equal to 0 on $B(x_{2m+1}, \varepsilon_{2m+1})$ and to $(\varphi_{2m} - id_{\mathbb{R}^n}, h_{2m}) \circ \tau_{-x_{2m}}$ on $B(x_{2m}, \varepsilon_{2m})$ for m sufficiently large. The mapping h is of the form $(\bar{h}, \tilde{h})$ where $\bar{h}: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $\tilde{h}: \mathbb{R}^n \longrightarrow \mathbb{R}^k$ On some neighbourhood $U \times \mathbb{R}^k$ of the point (0,0) we define a morphism in $\mathcal{AB}_{n,k}$ by the formula $H: \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n \times \mathbb{R}^k$

$$H(x, \xi) = (\bar{h}(x) + x, \tilde{h}(x) + \xi)$$

Then H=id on $B(x_{2m+1}, \varepsilon_{2m+1}) \times \mathbb{R}^k$ and $H = \tau_{(x_{2m},0)} \circ \Phi_{2m} \circ \tau_{(-x_{2m},0)}$ on $B(x_{2m}, \varepsilon_{2m}) \times \mathbb{R}^k$ for m sufficiently large. Hence $\mathcal{F}\Phi_m(v_m) = v_m$ for m odd and $\mathcal{F}\Phi_m(v_m) = \mathcal{F}\tau_{(-x_m,0)} \circ \mathcal{F}H \circ \mathcal{F}\tau_{(x_m,0)}(v_m)$ for m even and sufficiently large. Then using Theorem B we get that $\mathcal{F}\tau_{(-x_m,0)} \circ \mathcal{F}H \circ \mathcal{F}\tau_{(x_m,0)}(v_m)$ tends to $\mathcal{F}H(v)$. Since the space $\mathcal{F}(\mathbb{R}^n \times \mathbb{R}^k)$ is Hausdorff, the sequence $\mathcal{F}\Phi_{2m}(v_{2m})$ is convergent to v. This ends the proof.

III. Natural equivalence of functors.

In this section the proof of Theorem A is outlined, but first of all we have to introduce some notions and notations.
Let us recall that $AL^r_{n,k}$ is the group of r-jets of local isomorphisms $\mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n \times \mathbb{R}^k$ at (0,0). $AL^r_{n,k}$ is a Lie group, as it is a closed subgroup of L^r_{n+k}, the group of all r-jets at 0 of isomorphisms of $\mathbb{R}^{n+k} \longrightarrow \mathbb{R}^{n+k}$ (not morphisms of $\mathcal{AB}_{n,k}$). We also define $AL^\infty_{n,k}$ – the set of infinite jets at (0,0) of morphisms of $\mathcal{AB}_{n,k}$ of $\mathbb{R}^n \times \mathbb{R}^k$ into itself. The group $AL^\infty_{n,k}$ is the projective limit of the system $\{AL^i_{n,k}, \pi^i_j\}$ where $\pi^i_j: AL^i_{n,k} \longrightarrow AL^j_{n,k}$ is the natural projection. We denote by $AN^i_{n,k}$ the kernel of the projection $AL^\infty_{n,k} \longrightarrow AL^i_{n,k}$. Exactly as in [5] we prove the following theorem

<u>Theorem E</u>. If F is a manifold (metric space) of dimension s and the action of $AL^\infty_{n,k}$ on F is continuous, then $AN^{2s+1}_{n,k}$ acts trivially on F.

Now we shall construct a natural equivalence between the natural functor $\mathfrak{F}$ and the functor described in Example 2. Let E be an object of $\mathcal{AB}_{n,k}$. The elements of the bundle FE are equivalence classes $\langle e,f\rangle$ where $e \in J(E,r)$ (for some r) and $f \in F$. The equivalence relation is the following:

$$\langle e,f\rangle = \langle e',f'\rangle \Leftrightarrow \exists\, g \in AL^r_{n,k} : e' = eg \ , \ f' = g^{-1}f$$

The projection $\overline{\pi}_E : FE \longrightarrow E$ is given by the formula

$$\langle j^r_{(0,0)}\varphi, f\rangle \longmapsto \varphi(0,0)$$

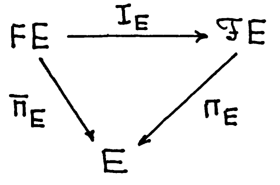
From the theorems proved up to now it follows that there exists $r > 0$ such that $AL^r_{n,k}$ acts on $F = \pi^{-1}_{\mathbb{R}^n \times \mathbb{R}^k}(0,0)$ and this action is continuous. Namely

$$g \cdot f = \mathfrak{F}\varphi(f)$$

where φ is a morphism in $\mathcal{AB}_{n,k}$ such that $j^r_{(0,0)}\varphi = g$. Moreover $r \leqslant 2\dim F + 1$. Thus for any object E of $\mathcal{AB}_{n,k}$ we can define isomorphism $I_E : FE \longrightarrow \mathfrak{F}E$ as follows:

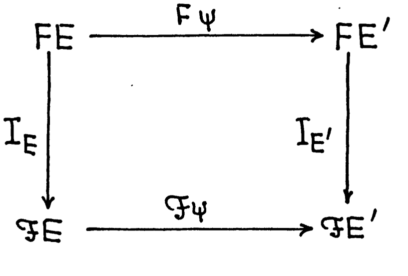
$$I_E(\langle j^r_0\varphi, f\rangle) = \mathfrak{F}\varphi(f)$$

The family of mappings $\{I_E\}$ defines a natural equivalence of functors $\mathfrak{F}$ and $F(.)$. The I_E are local diffeomorphisms. Indeed, if $\psi : E_1 \longrightarrow E_2$ is a local isomorphism and a morphism of $\mathcal{AB}_{n,k}$ then the following diagrams are commutative:



$$\pi_E \circ I_E = \overline{\pi}_E$$

and



$$I_{E'} \circ F\psi = \mathfrak{F}\psi \circ I_E$$

For example, we shall verify that the second diagram is commutative.

$$I_{E'} \circ F\varphi(\langle j^r_{(0,0)}\varphi, f\rangle) =$$

$$= I_{E'}(\langle j^r_{(0,0)}\psi\circ\varphi, f\rangle) = \mathfrak{F}\psi\circ\mathfrak{F}\varphi(f) =$$

$$= \mathfrak{F}\psi\circ I_E(\langle j^r_{(0,0)}\varphi, f\rangle)$$

The above remarks end the proof of Theorem A.

REFERENCES

1. EPSTEIN D.B.A., THURSTON W.P., Transformation groups and natural bundles, Proc. London Math. Soc. (3)38 (1979), 219-236.
2. MONTGOMERY D., ZIPPIN L., Topological transformation groups. Interscience, New York, 1955.
3. NIJENHUIS A., Natural bundles and their general properties. Symp. in honor of K.Yano, Kinokuniya Tokio(1972), 317-334.
4. PALAIS R.S., TERNG C.L., Natural bundles have finite order. Topology 16(1977), 271-277.
5. POGODA Z., Geometry of natural functors. Thesis.

INSTYTUT MATEMATYKI
UNIWERSYTET JAGIELLONSKI
UL.REYMONTA 4
30-059 KRAKÓW
POLAND