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NATURAL TRANSFORMATIONS OF AFFINE CONNECTIONS ON MANIFOLDS TO METRICS ON COTANGENT BUNDLES

Masami Sekizawa

Dedicated to Professor Shun-ichi Tachibana on the occasion of his 60th birthday

Let M be a smooth manifold and T^*M its cotangent bundle. There is a well-known "natural" construction which yields, for any affine connection ∇ on M, a pseudo-Riemannian metric $\bar{g}$ on T^*M, the so called <u>Riemann extension</u> of ∇ (see [6] - [10]). If a local coordinate system is given in M then the components of the metric $\bar{g}$ at each point $(x,w) \in T^*M$ depend only on the symmetrized components of the connection ∇ and the components of the given co-vector w. The more detailed analysis shows that this construction involves the geometry of the second order, and thus we can consider the Riemann extension as an example of "natural transformation of the second order".

The aim of this paper is to describe explicitly <u>all</u> second order natural transformations of a symmetric affine connection on a manifold into a metric(not necessarily regular) on its cotangent bundle. To solve this problem, we shall use the precise definitions as well as the general method established by D.Krupka[2] - [4], which reduces our geometric problem to the classification of corresponding "differential invariants" and then to solving a system of partial differential equations. The main result of this paper is the following

<u>Theorem 1.</u> A pseudo-Riemannian metric G (not necessarily regular) on T^*M comes from a second order natural transformation of a symmetric connection ∇ on M if and only if $G = a\bar{g} + b\theta^2$, where $\bar{g}$ is the Riemann extension of ∇, θ^2 is the tensor square of the canonical 1-form of T^*M, and a, b are constants.

Let us notice that the metric $G = a\bar{g} + b\theta^2$ is regular (of the signature (n,n)) if and only if $a \neq 0$.

O.Kowalski and the present author[1] have recently classified all second order natural transformations of a Riemannian metric g given on a base manifold M into a pseudo-Riemannian metric G given on the tangent bundle TM. Since the cotangent bundle T^*M over (M,g) is dual to TM, the analogous problem for T^*M is automatically settled through this duality.

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1. Canonical 1-form and Riemann extension

We shall adopt the Einstein summation convention in sections 1 and 2.

Let $(U;x^1,x^2,\ldots,x^n)$ and $(\bar{U};\bar{x}^1,\bar{x}^2,\ldots,\bar{x}^n)$ be two systems of local coordinates in a smooth manifold M of dimension n such that the domain $U\cap\bar{U}$ is not empty. The coordinate vector fields $E_i = \partial/\partial x^i$ and $\bar{E}_i = \partial/\partial\bar{x}^i$ $(1 \leq i \leq n)$ are related by the transformation formulas

$$(1.1)\quad E_i = A_i^a\bar{E}_a \quad\text{or}\quad \bar{E}_i = B_i^a E_a \quad (1 \leq i \leq n)$$

where $(A_i^a) = \left[\dfrac{\partial\bar{x}^a}{\partial x^i}\right]$ and $(B_i^a) = \left[\dfrac{\partial x^a}{\partial\bar{x}^i}\right]$ are the (mutually inverse) Jacobi matrices. If $w = w_h dx^h = \bar{w}_h d\bar{x}^h$ is a 1-form on $U\cap\bar{U}$ then we get by (1.1)

$$(1.2)\quad \bar{w}_h = B_h^a w_a \quad (1 \leq h \leq n).$$

Further if ∇ is an affine connection on M then its components Γ_{ij}^h and $\bar{\Gamma}_{ij}^h$ $(1 \leq h,i,j \leq n)$(i.e., $\nabla_{X_i}X_j = \Gamma_{ij}^a X_a$, $\nabla_{\bar{X}_i}\bar{X}_j = \bar{\Gamma}_{ij}^a\bar{X}_a$) are related by

$$(1.3)\quad \bar{\Gamma}^h_{ij} = A^h_a(B^b_i B^c_j \Gamma^a_{bc} + B^a_{ij}) \quad (1 \leqq h,i,j \leqq n),$$

where we put $B^h_{ij} = \partial^2 x^h/\partial\bar{x}^i\partial\bar{x}^j \quad (1 \leqq h,i,j \leqq n)$.

Now let us denote $p: T^*M \to M$ the natural projection of the cotangent bundle. Let $(p^{-1}U; x^1, x^2, \ldots, x^n, w_1, w_2, \ldots, w_n)$ and $(p^{-1}\bar{U}; \bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_n)$ be two systems of local coordinates in T^*M induced from $(U; x^1, x^2, \ldots, x^n)$ and $(\bar{U}; \bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)$, respectively. Then, whenever $U \cap \bar{U} \neq \phi$, the transformation law on $p^{-1}(U \cap \bar{U})$ is given by

$$\begin{cases} \bar{x}^i = \bar{x}^i(x^1, x^2, \ldots, x^n), \\ \bar{w}_i = B^a_i w_a, \quad (1 \leqq i \leqq n). \end{cases}$$

We set $X_i = \partial/\partial x^i$, $X^i = \partial/\partial w_i$ and $\bar{X}_i = \partial/\partial\bar{x}^i$, $\bar{X}^i = \partial/\partial\bar{w}_i$ for $1 \leqq i \leqq n$; then the two bases $\{X_1, X_2, \ldots, X_n, X^1, X^2, \ldots, X^n\}$ and $\{\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n, \bar{X}^1, \bar{X}^2, \ldots, \bar{X}^n\}$ are related to each other by

$$(1.4)\quad \begin{cases} \bar{X}_i = B^a_i X_a + B^c_i A^b_{ca} B^d_b w_d X^a, \\ \bar{X}^i = A^i_a X^a, \quad (1 \leqq i \leqq n), \end{cases}$$

where we put $A^h_{ij} = \partial^2 \bar{x}^h/\partial x^i \partial x^j \quad (1 \leqq h,i,j \leqq n)$.

Let now $G_{ij} = G(X_i, X_j)$, $G^h_i = G(X_i, X^h)$, $G^{ij} = G(X^i, X^j)$ be the local components of a symmetric (0,2)-tensor field G on T^*M. We shall always write such components in the block matrix form

$$(1.5)\quad G = \begin{bmatrix} G_{ij} & G^j_i \\ G^i_j & G^{ij} \end{bmatrix}.$$

Using (1.4), we obtain the following transformation formulas:

$$(1.6)\quad \begin{cases} \bar{G}_{ij} = B^a_i B^s_j G_{as} + B^c_i B^b_{ca} B^d_b w_d B^s_j G^a_s \\ \qquad + B^a_i B^u_j B^t_{us} B^v_t w_v G^s_a + B^c_i B^b_{ca} B^d_b w_d B^u_j A^t_{us} B^v_t w_v G^{as}, \\ \bar{G}^h_i = A^h_a B^s_j G^a_s + A^h_a B^u_j A^t_{us} B^v_t w_v G^{as}, \\ \bar{G}^{ij} = A^i_a A^j_s G^{as}, \quad (1 \leqq h,i,j \leqq n). \end{cases}$$

(In section 2, we shall show that the formulas (1.2), (1.3) and (1.6) define actions of the second order differential group L_n^2 on some vector spaces).

Let $p_*:T(T^*M)\longrightarrow TM$ be the differential of the natural projection $p:T^*M\longrightarrow M$ and let q be the natural projection of $T(T^*M)$ to T^*M, where "T" stands for the tangent bundle. Then the canonical 1-form θ on T^*M is defined by

$$\theta(\tilde{X}) = q(\tilde{X})(p_*\tilde{X})$$

for all $\tilde{X} \in T(T^*M)$. The exterior derivative $d\theta$ of θ is called the canonical 2-form on T^*M. In terms of the induced system of local coordinates in T^*M, θ and $d\theta$ are expressed as

$$\theta = w_i dx^i \quad \text{and} \quad d\theta = dw_i \wedge dx^i.$$

Let ∇ be an affine connection on the base manifold M. This induces a unique connection in the vector bundle T^*M, and thus each tangent space $(T^*M)_{(x,w)}$ splits into the horizontal and the vertical subspace:

$$(T^*M)_{(x,w)} = H_{(x,w)} \oplus V_{(x,w)}.$$

Let $\tilde{X} = h\tilde{X} + v\tilde{X}$ be the decomposition of a vector field $\tilde{X}$ on T^*M into the horizontal and the vertical part. The Riemann extension $\bar{g}$ of the affine connection ∇ on M to T^*M is a pseudo-Riemannian metric defined by

$$\bar{g}(\tilde{X},\tilde{Y}) = (d\theta)(v\tilde{X},h\tilde{Y}) + (d\theta)(v\tilde{Y},h\tilde{X})$$

for all vector fields $\tilde{X}$ and $\tilde{Y}$ on T^*M. If $\tilde{X} = \xi^i X_i + \xi_i X^i$, then we obtain easily

$$h\tilde{X} = \xi^i X_i + w_a\Gamma^a_{ib}\xi^b X^i \quad \text{and} \quad v\tilde{X} = (\xi_i - w_a\Gamma^a_{ib}\xi^b)X^i.$$

Thus the components of $\bar{g}$ with respect to the induced system of local coordinates are

$$\bar{g} = \begin{bmatrix} -w_a(\Gamma^a_{ij} + \Gamma^a_{ji}) & \delta^j_i \\ \delta^i_j & 0 \end{bmatrix},$$

where δ^i_j denotes the Kronecker's symbol. This shows that the components of $\bar{g}$ depend only on the symmetrized components of the affine connection and on the components of the co-vector w, and they do not depend on the local coordinates $(x^1,x^2,\dots,x^n)$ in M. In case of a symmetric connection we get

$$\bar{g} = \begin{bmatrix} -2w_a\Gamma^a_{ij} & \delta^j_i \\ \delta^i_j & 0 \end{bmatrix}.$$

Obviously, $\bar{g}$ is a pseudo-Riemannian metric with the signature (n,n). (See [6] - [10] for more information about Riemann extensions).

2. Differential invariants

Let us now recall the general theory of natural transformations due to D.Krupka. We refer to [2] - [4] for more details, and to [5] for the general philosophy of naturality.

Let L^r_n be the r-th order differential group of the n-dimensional Euclidean space R^n, that is, the Lie group of all r-jets of local diffeomorphisms of R^n with source and target at the origin $o \in R^n$, here r is any non-negative integer. Let P, Q be smooth manifolds on which the group L^r_n acts to the left. An r-<u>th order differential invariant</u> $f:P \to Q$ is an L^r_n-equivariant map of the left L^r_n-space P to the left L^r_n-space Q, i.e, a map satisfying $f(j^r_o\alpha.p) = j^r_o\alpha.f(p)$ for all $j^r_o\alpha \in L^r_n$ and all $p \in P$. Here the dot $.$ denotes the action of L^r_n on P (or on Q, respectively).

Further let F^rM denote the bundle of all frames of r-th order over M which carries a natural structure of a principal L^r_n-bundle $F^rM(M,L^r_n,\pi^r_n)$. We get a natural functor F^r from the category D_n of n-manifolds and injective immersions into the category of principal L^r_n-bundles and L^r_n-bundle morphisms. Here, for a given morphism $\varphi:M_1 \to M_2$ of D_n the corresponding morphism $F^r\varphi : F^rM_1 \to F^rM_2$ is given in a familiar way(see [5]).

Finally, for a left L^r_n-space P, let F^r_PM denote the fibre bundle with fibre P, associated to the principal L^r_n-bundle F^rM. We obtain a natural functor F^r_P from the category D_n into the category of fibre bundles and their morphisms. Here, for any morphism $\varphi:M_1 \to M_2$ of D_n the corresponding morphism $F^r_P\varphi:F^r_PM_1 \to F^r_PM_2$ is given by

$$F_P^r\varphi([y,p]) = [F^r\varphi(y),p]$$

for any $[y,p] \in F_P^r M_1$ ($[y,p]$ is the equivalence class of a pair $(y,p) \in F^r M_1 \times P$ with respect to the equivalence relation defined by the right action $(y,p).j_0^r\alpha = (y \circ j_0^r\alpha, j_0^r\alpha^{-1}.p)$ of L_n^r on $F^r M_1 \times P$).

For each manifold M and each differential invariant $f:P \to Q$ we can define a morphism $f_M: F_P^r M \longrightarrow F_Q^r M$ over the identity map $id: M \longrightarrow M$ by

$$f_M([y,p]) = [y,f(p)]$$

for all $[y,p] \in F_P^r M$. This morphism f_M is called the <u>realization of a differential invariant</u> f on the manifold M. Further, an <u>r-th order natural transformation</u> T of the functor F_P^r into the functor F_Q^r is a collection of bundle morphisms T_M over the identity map, where M is an object of D_n, such that the following diagram is commutative

$$\begin{array}{ccc} F_P^r M_1 & \xrightarrow{T_{M_1}} & F_Q^r M_1 \\ \downarrow{\scriptstyle F_P^r\varphi} & & \downarrow{\scriptstyle F_Q^r\varphi} \\ F_P^r M_2 & \xrightarrow{T_{M_2}} & F_Q^r M_2 \end{array}$$

for every morphism $\varphi: M_1 \longrightarrow M_2$ of D_n.

The following theorem due to Krupka[2] says that a problem to find all r-th order natural transformations of F_P^r to F_Q^r is equivalent to a problem to find all r-th order differential invariants f from P to Q.

<u>Theorem A.</u> Let $f:P \longrightarrow Q$ be an r-th order differential invariant. Then the correspondence $T_f: M \longrightarrow f_M$, where M is an object of D_n, is a natural transformation of the functor F_P^r to the functor F_Q^r. Moreover, the correspondence $f \longrightarrow T_f$ is a bijection between the set of all r-th order differential invariants from P to Q and the set of all r-th order natural transformations of F_P^r to F_Q^r.

Remark. Often P and Q are vector spaces and thus F_P^rM and F_Q^rM are vector bundles. Yet, a morphism $T_M : F_P^rM \to F_Q^rM$ need not be a morphism in the category of vector bundles because it may be non-linear on fibres.

In order to apply the method by Krupka to our problem, we shall restrict ourselves to second order differential invariants. We define functions A_i^h, A_{ij}^h $(1 \leqq h \leqq n,\ 1 \leqq i \leqq j \leqq n)$ on L_n^2 by

$$A_i^h(j_o^2\alpha) = D_i\alpha^h(o), \quad A_{ij}^h(j_o^2\alpha) = D_iD_j\alpha^h(o)$$

for any local diffeomorphism $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^n)$ with $\alpha(o) = o \in R^n$, here D_i denotes the partial derivative with respect to the i-th variable in R^n. The system of the canonical (global) coordinates of L_n^2 is a system of coordinates $\{B_i^h, B_{ij}^h\}$ $(1 \leqq h \leqq n,\ 1 \leqq i \leqq j \leqq n)$ of L_n^2 which are defined by

$$B_i^h(j_o^2\alpha) = A_i^h(j_o^2\alpha^{-1}), \quad B_{ij}^h(j_o^2\alpha) = A_{ij}^h(j_o^2\alpha^{-1})$$

$$(1 \leqq h \leqq n,\ 1 \leqq i \leqq j \leqq n).$$

The multiplication law $j_o^2\alpha \circ j_o^2\beta = j_o^2(\alpha \circ \beta)$ in L_n^2 is described in terms of the canonical coordinates as

$$\begin{cases} B_i^h(j_o^2\alpha \cdot j_o^2\beta) = B_a^h(j_o^2\alpha)B_i^a(j_o^2\beta), \\ B_{ij}^h(j_o^2\alpha \cdot j_o^2\beta) = B_{ab}^h(j_o^2\alpha)B_i^a(j_o^2\beta)B_j^b(j_o^2\beta) + B_a^h(j_o^2\alpha)B_{ij}^a(j_o^2\beta), \\ (1 \leqq h \leqq n,\ 1 \leqq i \leqq j \leqq n). \end{cases}$$

Since $B_i^h(j_o^2(id)) = \delta_i^h$, $B_{ij}^h(j_o^2(id)) = 0$ $(1 \leqq h \leqq n,\ 1 \leqq i \leqq j \leqq n)$ for the identity map id of R^n, we obtain that

$$(2.1) \quad B_a^h(j_o^2\alpha)A_i^a(j_o^2\alpha) = \delta_i^h \quad (1 \leqq h, i \leqq n),$$

$$(2.2) \quad A_{ab}^h(j_o^2\alpha)B_i^a(j_o^2\alpha)B_j^b(j_o^2\alpha) + A_a^h(j_o^2\alpha)B_{ij}^a(j_o^2\alpha) = 0,$$

$$(1 \leqq h \leqq n,\ 1 \leqq i \leqq j \leqq n)$$

for all $j_o^2\alpha \in L_n^2$. These formulas will be used in section 3.

Let us consider the vector space $P = R^{n*} \oplus (R^n \otimes (R^{n*} \odot R^{n*}))$ (of dimension $n+n^2(n+1)/2$) where R^{n*} is the dual space to R^n and $\odot$ denotes the symmetric product. We denote by $\{w_h\}(1 \leqq h \leqq n)$ and $\{\Gamma^h_{ij}\}(1 \leqq h \leqq n,\ 1 \leqq i \leqq j \leqq n)$ the canonical coordinates on R^{n*} and $R^n \otimes (R^{n*} \odot R^{n*})$, respectively. Then $\{w_h,\ \Gamma^h_{ij}\}(1 \leqq h \leqq n,\ 1 \leqq i \leqq j \leqq n)$ form a canonical system of coordinates on P. We define an action of L^2_n on P by the formulas

$$(2.3)\qquad \begin{cases} \bar{w}_h = B^a_h w_a, \\ \bar{\Gamma}^h_{ij} = A^h_a(B^b_i B^c_j \Gamma^a_{bc} + B^a_{ij}) \quad (1 \leqq h \leqq n,\ 1 \leqq i \leqq j \leqq n), \end{cases}$$

which are modelled according to (1.2) and (1.3).

Further, consider the vector space $Q = R^{n*} \oplus ((R^n \oplus R^{n*}) \odot (R^n \oplus R^{n*}))$ (of dimension $2n(n+1)$). Here we can define a canonical system of coordinates on Q in the form $\{z_h,\ G_{ij},\ G^h_i,\ G^{ij}\}(1 \leqq h \leqq n,\ 1 \leqq i \leqq j \leqq n)$. Then we define an action of L^2_n on Q by

$$(2.4)\qquad \begin{cases} \bar{z}_h = B^a_h z_a, \\ \bar{G}_{ij} = B^a_i B^s_j G_{as} + B^a_i A^b_{ca} B^d_b z_d B^s_j G^a_s \\ \qquad\quad + B^a_i B^u_j A^t_{us} B^v_t z_v G^s_a + B^c_i B^b_{ca} B^d_b z_d B^u_j A^t_{us} B^v_t z_v G^{as}, \\ \bar{G}^h_i = A^h_a B^s_i G^a_s + A^h_a B^u_i A^t_{us} B^v_t z_v G^{as}, \\ \bar{G}^{ij} = A^i_a A^j_s G^{as}, \quad (1 \leqq h \leqq n,\ 1 \leqq i \leqq j \leqq n), \end{cases}$$

which is in the agreement with (1.2) and (1.6).

One can see easily that, for the corresponding associated L^2_n-bundles $F^2_P M$ and $F^2_Q M$ over a manifold M, we always have

$$(2.5)\qquad F^2_P M = T^*M \oplus \mathcal{P}'_M, \quad F^2_Q = T^*M \oplus \mathcal{Q}'_M,$$

where $\mathcal{P}'_M$ and $\mathcal{Q}'_M$ are some vector bundles over M. (Here $\mathcal{P}'_M$ is an associated bundle to F^2M but $\mathcal{Q}'_M$ is not, as we see from the transformation rules (2.3) and (2.4)).

Now, <u>we define the problem to find all second order natural transformations of a symmetric affine connection on a manifold to a metric on its cotangent bundle as the problem to find all those natural transformations of F^2_P to F^2_Q which, with respect to the splittings (2.5), induce the identity map</u> $id: T^*M \to T^*M$ <u>for each</u> M.

According to Theorem A, our main Theorem 1 is reduced to the following

<u>Theorem 2.</u> All differential invariants $f : (w_h, \Gamma^h_{ij}) \mapsto (z_h, G_{ij}, G^h_i, G^{ij})$ from $P = R^{n*} \oplus (R^n \otimes (R^{n*} \odot R^{n*}))$ into $Q = R^{n*} \oplus ((R^n \oplus R^{n*}) \odot (R^n \oplus R^{n*}))$ such that $z_h = w_h$ $(1 \leqq h \leqq n)$ are given, in the canonical coordinates, by

$$G_{ij} = -2aw_s\Gamma^s_{ij} + bw_iw_j \quad (1 \leqq i \leqq j \leqq n),$$

$$G^h_i = a\delta^h_i \quad (1 \leqq h, i \leqq n),$$

$$G^{ij} = 0 \quad (1 \leqq i \leqq j \leqq n),$$

where a and b are constants.

3. Proof of Theorem 2

The method of the proof is that we attach to each equivariant map $P \to Q$ the corresponding Lie algebra homomorphism for the fundamental vector fields. These Lie algebra homomorphisms are then characterized by a system of differential equations to solve. We find all solutions and decide which of them really represent differential invariants.

First of all, it will be useful to extend the symbols Γ^h_{ij}, A^h_{ij}, B^h_{ij}, G_{ij}, G^{ij} also for the case $i \geqq j$ by putting $\Gamma^h_{ij} = \Gamma^h_{ji}$, $A^h_{ij} = A^h_{ji}$, and so on. Thus the range of all indices will be $\{1,2,\ldots,n\}$, and all indices will be independent. We note that, under this notation, we have to use the following conventions (cf. Krupka [3]):

$$\frac{\partial \Gamma^h_{ij}}{\partial \Gamma^p_{qr}} = \frac{1}{2}\delta^h_p(\delta^q_i\delta^r_j + \delta^q_j\delta^r_i), \quad \frac{\partial B^h_{ij}}{\partial B^p_{qr}} = \frac{1}{2}\delta^h_p(\delta^q_i\delta^r_j + \delta^q_j\delta^r_i),$$

$$\frac{\partial G_{ij}}{\partial G_{qr}} = \frac{1}{2}(\delta^q_i\delta^r_j + \delta^q_j\delta^r_i), \quad \frac{\partial G^{ij}}{\partial G^{qr}} = \frac{1}{2}(\delta^i_q\delta^j_r + \delta^i_r\delta^j_q)$$

for $1 \leqq h,i,j,p,q,r \leqq n$.

The fundamental vector fields on P relative to the action (2.3) are

$$\xi_p^q = \sum_{a,b,c} \frac{\partial \bar{\Gamma}^a_{bc}}{\partial B^p_q}(e)\frac{\partial}{\partial \Gamma^a_{bc}} + \sum_a \frac{\partial \bar{w}_a}{\partial B^p_q}(e)\frac{\partial}{\partial w_a}$$

$$= \sum_{a,b,c}(-\delta^a_p\Gamma^q_{bc} + \delta^q_b\Gamma^a_{pc} + \delta^q_c\Gamma^a_{bp})\frac{\partial}{\partial \Gamma^a_{bc}} + w_p\frac{\partial}{\partial w_q},$$

$$\xi_p^{qr} = \sum_{a,b,c} \frac{\partial \bar{\Gamma}^a_{bc}}{\partial B^p_{qr}}(e)\frac{\partial}{\partial \Gamma^a_{bc}} = \frac{\partial}{\partial \Gamma^p_{qr}}, \quad (1 \leqq p,q,r \leqq n),$$

where e denotes the identity element $j^2_0(id)$ of L^2_n. Here we also used the formula

$$\frac{\partial A^h_i}{\partial B^p_q}(e) = -\frac{\partial B^h_i}{\partial B^p_q}(e) = -\delta^h_p\delta^q_i \quad (1 \leqq h,i,p,q \leqq n)$$

which is derived from (2.1) by differentiation with respect to B^p_q.

The corresponding fundamental vector fields on Q relative to the action (2.4) are given by

$$\Xi^q_p = \sum_{a,b}\left(\frac{\partial \bar{G}_{ab}}{\partial B^p_q}(e)\frac{\partial}{\partial G_{ab}} + 2\frac{\partial \bar{G}^b_a}{\partial B^p_q}(e)\frac{\partial}{\partial G^b_a} + \frac{\partial \bar{G}^{ab}}{\partial B^p_q}(e)\frac{\partial}{\partial G^{ab}}\right) + \sum_a \frac{\partial \bar{z}_a}{\partial B^p_q}\frac{\partial}{\partial z_a}$$

$$= 2\sum_a\left(G_{ap}\frac{\partial}{\partial G_{aq}} + G^a_p\frac{\partial}{\partial G^a_q} - G^q_a\frac{\partial}{\partial G^p_a} - G^{aq}\frac{\partial}{\partial G^{ap}}\right) + z_p\frac{\partial}{\partial z_q},$$

$$\Xi^{qr}_p = \sum_{a,b}\left(\frac{\partial \bar{G}_{ab}}{\partial B^p_{qr}}(e)\frac{\partial}{\partial G_{ab}} + 2\frac{\partial G^b_a}{\partial B^p_{qr}}(e)\frac{\partial}{\partial G^b_a} + \frac{\partial G^{ab}}{\partial B^p_{qr}}(e)\frac{\partial}{\partial G^{ab}}\right)$$

$$= -z_p\sum_a\left(G^q_a\frac{\partial}{\partial G_{ar}} + G^a_p\frac{\partial}{\partial G^a_q} + G^r_a\frac{\partial}{\partial G_{aq}} + G^{ar}\frac{\partial}{\partial G^a_q}\right),$$

$(1 \leqq p,q,r \leqq n)$.

Here we also used the formula

$$\frac{\partial A^h_{ij}}{\partial B^p_{qr}}(e) = -\frac{\partial B^h_{ij}}{\partial B^p_{qr}}(e) = -\frac{1}{2}\delta^h_p(\delta^q_i\delta^r_j + \delta^q_j\delta^r_i) \quad (1 \leqq h,i,j,p,q,r \leqq n)$$

which is derived from (2.2) by differentiation with respect to B^p_{qr}.

Since any L^2_n-equivariant map f of P to Q satisfies

$$f_*(\xi^q_p) = \Xi^q_p \quad \text{and} \quad f_*(\xi^{qr}_p) = \Xi^{qr}_p \quad (1 \leqq p,q,r \leqq n),$$

we get

$$\xi^q_p(Z^\alpha \circ f) = \Xi^q_p(Z^\alpha) \quad \text{and} \quad \xi^{qr}_p(Z^\alpha \circ f) = \Xi^{qr}_p(Z^\alpha), \quad (1 \leqq p,q,r \leqq n),$$

where Z^α denotes any canonical coordinate on Q. We have to use also the conditions $z_h \circ f = w_h$ $(1 \leqq h \leqq n)$. Hence we obtain an explicit system of partial differential equations:

$$(3.1) \quad \sum_{a,b,c} (-\delta^a_p \Gamma^q_{bc} + \delta^q_b \Gamma^a_{pc} + \delta^q_c \Gamma^a_{bp}) \frac{\partial G_{ij}}{\partial \Gamma^a_{bc}} + w_p \frac{\partial G_{ij}}{\partial w_q} = G_{ip}\delta^q_j + G_{jp}\delta^q_i,$$

$$(3.2) \quad \sum_{a,b,c} (-\delta^a_p \Gamma^q_{bc} + \delta^q_b \Gamma^a_{pc} + \delta^q_c \Gamma^a_{bp}) \frac{\partial G^j_i}{\partial \Gamma^a_{bc}} + w_p \frac{\partial G^j_i}{\partial w_q} = - G^q_i \delta^j_p + G^j_p \delta^q_i,$$

$$(3.3) \quad \sum_{a,b,c} (-\delta^a_p \Gamma^q_{bc} + \delta^q_b \Gamma^a_{pc} + \delta^q_c \Gamma^a_{bp}) \frac{\partial G^{ij}}{\partial \Gamma^a_{bc}} + w_p \frac{\partial G^{ij}}{\partial w_q} = -G^{iq}\delta^j_p + G^{jq}\delta^i_p,$$

$$(3.4) \quad \frac{\partial G_{ij}}{\partial \Gamma^p_{qr}} = -\frac{1}{2} w_p (G^q_i \delta^r_j + G^q_j \delta^r_i + G^r_i \delta^q_j + G^r_j \delta^q_i),$$

$$(3.5) \quad \frac{\partial G^j_i}{\partial \Gamma^p_{qr}} = -\frac{1}{2} w_p (G^{jq}\delta^r_i + G^{jr}\delta^q_i),$$

$$(3.6) \quad \frac{\partial G^{ij}}{\partial \Gamma^p_{qr}} = 0, \quad (1 \leqq i,j,p,q,r \leqq n).$$

We shall solve the system (3.1) - (3.6) step by step. To avoid confusion, we <u>do not</u> use the Einstein summation convention up to the end of this section.

(3.6) implies that G^{ij} $(1 \leqq i,j \leqq n)$ do not depend on Γ^p_{qr} $(1 \leqq p,q,r \leqq n)$. Hence $G^{ij} = \lambda^{ij}(w_1, w_2, \ldots, w_n)$ for $1 \leqq i,j \leqq n$. Then we get from (3.3)

$$(3.7) \quad w_p \partial\lambda^{ij}/\partial w_q = -\lambda^{iq}\delta^j_p - \lambda^{jq}\delta^i_p \quad (1 \leqq i,j,p,q \leqq n).$$

Now, if $n \geqq 3$, we can always choose $p \neq i,j$ and we get, at any generic point (for $w_1 w_2 \cdots w_n \neq 0$), $\partial\lambda^{ij}/\partial w_q = 0$ for all i,j,q. Then (3.7) reduces to $\lambda^{iq}\delta_p^j + \lambda^{jq}\delta_p^i = 0$, and contracting with respect to $i=p$, we get $(n+1)\lambda^{jq} = 0$, i.e.,

$$G^{ij} \doteq \lambda^{ij} = 0 \tag{3.8}$$

for all $i,j \in \{1,2,\ldots,n\}$.

Let now $n = 2$. Putting $i=j=p=q$ in (3.7), we get, at a generic point

$$\lambda^{ii} = c^{ii}/(w_i)^2 \quad (i = 1,2),$$

where c^{ii} does not depend on w_i. Further, putting $i=1$, $j=2$, $p=q$ in (3.7), we obtain the equations

$$w_1\partial\lambda^{12}/\partial w_1 = w_2\partial\lambda^{12}/\partial w_2 = -\lambda^{12}.$$

By an elementary calculation we get

$$\lambda^{12} = c^{12}/w_1 w_2,$$

where c^{12} is a constant. Summarizing, we get, at a generic point,

$$G^{ij} = c^{ij}/w_i w_j \quad (i,j = 1,2)$$

where c^{ij} $(i,j = 1,2)$ are constants. But G^{ij} and w_i must satisfy the transformation laws (2.3) and (2.4). This is possible if and only if $c^{ij} = 0$ $(i,j = 1,2)$. Thus we get again

$$G^{ij} = 0 \quad (i,j = 1,2) \tag{3.9}$$

at a generic point, and hence at any point by continuity.

Substitution of (3.8) into (3.5) implies

$$\partial G_i^j/\partial\Gamma_{qr}^p = 0 \quad (1 \leqq i,j,p,q,r \leqq n),$$

hence G_i^j $(1 \leqq i,j \leqq n)$ do not depend on Γ_{qr}^p $(1 \leqq p,q,r \leqq n)$. Put $G_i^j = \mu_i^j(w_1,w_2,\ldots,w_n)$. Then we get from (3.2)

$$w_p\partial\mu_i^j/\partial w_q = -\mu_i^q\delta_p^j + \mu_p^j\delta_i^q \quad (1 \leqq i,j,p,q \leqq n). \tag{3.10}$$

Using (3.10) for $i=j=p=q$ and also for $i=j\neq p=q$, we see that, at a generic point, μ_i^i $(1 \leqq i \leqq n)$ are constants. Putting $i=j=q\neq p$ in (3.10), we obtain $\mu_p^j = 0$. Thus $\mu_i^j = 0$ for $1 \leqq i,j \leqq n$, $i\neq j$. Putting $i=q\neq j=p$ in (3.10), we get immediately $\mu_j^j - \mu_i^i = 0$ $(1 \leqq i,j \leqq n,\ i \neq j)$. This can be written in the form $\mu_i^j = a\delta_i^j$ $(1 \leqq i,j \leqq n)$ for some constant a, i.e., at a generic point,

$$(3.11)\quad G_i^j = a\delta_i^j \quad (1 \leqq i,j \leqq n),$$

where a is a constant. (3.11) then holds at any point.

Substituting (3.11) into (3.4), we get

$$(3.12)\quad \partial G_{ij}/\partial \Gamma_{qr}^p = -aw_p(\delta_j^q\delta_i^r + \delta_i^q\delta_j^r) \quad (1 \leqq i,j,p,q,r \leqq n)$$

from which we get by integration with respect to Γ_{qr}^p,

$$(3.13)\quad G_{ij} = -2a\sum_s w_s\Gamma_{ij}^s + \nu_{ij} \quad (1 \leqq i,j \leqq n),$$

where ν_{ij} $(1 \leqq i,j \leqq n)$ do not depend on Γ_{qr}^p $(1 \leqq p,q,r \leqq n)$. Substituting (3.13) into (3.1) and using (3.12), we obtain, after some calculations,

$$(3.14)\quad w_p\partial\nu_{ij}/\partial w_q = \nu_{ip}\delta_j^q + \nu_{jp}\delta_i^q \quad (1 \leqq i,j,p,q \leqq n).$$

By calculations similar to those for solving (3.7) in the case $n = 2$ we get from (3.14), at a generic point,

$$(3.15)\quad \nu_{ij} = b_{ij}w_iw_j \quad (1 \leqq i,j \leqq n),$$

where b_{ij} $(1 \leqq i,j \leqq n)$ are constants.

Substitute (3.15) into (3.14) for $i=q\neq j$. We get $b_{ij}w_jw_p = b_{jp}w_jw_p$ $(1 \leqq i,j,p \leqq n,\ i \neq j)$ and hence, at a generic point, $b_{ij} = b_{jp}$ whenever $i\neq j$, p arbitrary. Hence <u>all</u> b_{ij} are equal to the same constant b. Thus we get

$$G_{ij} = -2a\sum_s w_s\Gamma_{ij}^s + bw_iw_j \quad (1 \leqq i,j \leqq n)$$

at a generic point, and hence at any point.

This completes the proof of Theorem 2.

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