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SPINGROUPS AND SPHERICAL MEANS II

F. Sommen (*)

Abstract. In this paper we study generalized mean values of functions in R^m over spheres of any codimension, by making use of representations of Spin(m) on spaces of functions in the Clifford algebra over R^m. This leads to several versions, refinements and generalizations of the classical Euler-Poisson-Darboux equation. Furthermore for spheres of codimension 2 we interprete these equations in terms of complex Clifford analysis.

Introduction. The notion of spherical means of a function is known to be useful in partial differential equations as is shown by F. John (see [6]). Especially for operators, which may be expressed in terms of Laplacians (and powers of it), it is applicable, because of the Darboux equation

$$\Delta_x f(\vec{x},r)=(\frac{\partial^2}{\partial r^2}+\frac{m-1}{r}\frac{\partial}{\partial r})f(\vec{x},r),$$

since it transforms the Laplacian into a one-dimensional operator.
In our previous paper [10] we extended spherical means by using the representations of Spin(m) instead of SO(m) and so-called spherical monogenics instead of spherical harmonics. Spherical monogenics are, roughly speaking, hypercomplex generalizations of the classical complex powers $z\to z^k$, $k\in Z$, i.e. they are homogeneous solutions of a Dirac type operator D, with values in a Clifford algebra. These ideas fit completely into the general setting of group representations and integral geometry as is being studied by S. Helgason in [3].
Our previous paper [10] was restricted to spheres of codimension one and so the spherical means have only one extra dimension, the radius of the sphere. Hence the Darboux equations link this radial dimension r to the space variable $\vec{x}\in R^m$.

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In this paper we study mean values of functions over spheres of any dimension. Such spheres are parametrised by their center $\vec{x}$, the radius r and an s-vector ω, which represents the axis so that spherical means depend on coordinates $(\vec{x},r,\omega)$ where r and ω are extra dimensions. Hence there exist Darboux equations which link the radius r with the space variable $\vec{x}$, called radial Darboux equations, and equations which express the "ω-derivatives" in terms of the space derivatives, called angular Darboux equations.
In the first section we recall the main definitions and properties of [10].
The second section is devoted to spherical means of codimension 2. In this section we link the radial and angular Darboux equations together in such a way that we obtain solutions of the complex monogenic system $(D_x+iD_y)f=0$,

$$D_x+iD_y=\sum_{j=1}^{m} e_j\left(\frac{\partial}{\partial x_j}+i\frac{\partial}{\partial y_j}\right)$$

being a complex Dirac type operator in C^m (see [8],[11],[12]).
The study of spherical means of any codimenson is more involved To that end we make use of functions defined in the entire Clifford algebra C_m or in its real part
R_m or in the spaces of s-vectors $R_{m,s}$ (see also [4]). The study of Spin(m)-representations is done in section 3.
In section 4 we study the Darboux equations for spheres of any codimension.

Preliminaries. Let $\{e_1,\ldots,e_m\}$ be an orthonormal basis of R^m. Then by C_m we denote the complex Clifford algebra constructed by means of this basis. Hence a general element $a\in C_m$ is of the form $a=\sum_{A\subseteq N} e_A a_A$, $a_A\in C$, $N=\{1,\ldots,m\}$, where for $A=\{\alpha_1,\ldots,\alpha_h\}$, $\alpha_1<\ldots<\alpha_h$,

$e_A=e_{\alpha_1}\cdots e_{\alpha_h}$.

The product in C_m is determined by the relations

$$e_ie_j+e_je_i=-2\delta_{ij};\ i,j=1,\ldots,m,\ e_\phi=1.$$

By R_m we denote the real Clifford algebra over R^m.
Every $a\in C_m$ may uniquely be written into the form $a=[a]_0+[a]_1+\ldots+[a]_m$, where $[a]_s\in C_{m,s}$; $s=0,\ldots,m$ and where $C_{m,s}$ is the space of complex s-vectors $C_{m,s}=\left\{\sum_{|A|=s} a_Ae_A : a_A\in C\right\}$. The space of real s-vectors will be

denoted by $R_{m,s}$.
An involution on C_m is given by $\overline{a}=\sum_{A\subseteq N}\overline{a}_A\overline{e}_A$, where $\overline{a}_A$ denotes complex conjugation and $\overline{e}_A=\overline{e}_{\alpha_h}\cdots\overline{e}_{\alpha_1}$, $\overline{e}_j=-e_j$; $j=1,\ldots,m$. Notice that on R_m

$$\overline{a}=[a]_0-[a]_1-[a]_2+[a]_3+\ldots\ .$$

An inner product on R_m is given by $\langle a,b\rangle=[\overline{a}b]_0$. This inner product coincides with the one induced from R^{2^n}. The norm of $a\in C_m$ is given by $|a|^2=\sum_A|a_A|^2$ and satisfies $|ab|\leqslant 2^m|a||b|$.

By the identifications $R^{m+1}=R_{m,0}+R_{m,1}$ and $R^m=R_{m,1}$, R^{m+1} and R^m are naturally imbedded in R_m. Hence $(x_0,x_1,\ldots,x_m)\in R^{m+1}$ will be identified with $x_0+\vec{x}$, $\vec{x}=\sum_{j=1}^m x_je_j$. The inner product in R^m will be denoted by $\langle\vec{x},\vec{y}\rangle$.
Let $\Omega\subseteq R^m$ be open; then $f\in C_1(\Omega,C_m)$ will be called left monogenic in Ω if $Df=0$, where $D=\sum_{j=1}^m e_j\frac{\partial}{\partial x_j}$ is a generalized Cauchy-Riemann operator, called Dirac operator or vector derivative.
A function $P_k(\vec{\omega})$ $(Q_k(\vec{\omega}))$, $\vec{\omega}\in S^{m-1}$ is called inner (outer) spherical monogenic of degree k if $r^kP_k(\vec{\omega})$ $(r^{-(k+m-1)}Q_k(\vec{\omega}))$ is left monogenic in R^m (in $R^m\setminus\{0\}$).
Every spherical harmonic admits a unique decomposition $S_k=P_k+Q_{k-1}$ into spherical monogenics.
By ω_m we denote the area of S^{m-1}.

1. Basic representations of Spin(m)

Let $s\in Spin(m)$ and $f\in L_2(S^{m-1};C_m)$. Then we consider the basic representations H_0 and L of Spin(m), given by $H_0(s)f(\vec{x})=f(\overline{s}\vec{x}s)$, $L(s)f(\vec{x})=sf(\overline{s}\vec{x}s)$. H_0 corresponds to the usual representation of SO(m), while L corresponds to spin 1/2-representation.
The Lie algebra of Spin(m) is the space $R_{m,2}$ of real bivectors, the elements of which are of the form $\sum_{i<j}x_{ij}e_{ij}$, $x_{ij}\in R$. Hence the infinitesimal representations of H_0 and L are given by

$$dH_0(e_{ij})=-2L_{ij},\quad dL(e_{ij})=-2L_{ij}+e_{ij},$$

where $L_{ij}=x_i\frac{\partial}{\partial x_j}-x_j\frac{\partial}{\partial x_i}$.

The Casimir operators $C(H_0)$ and $C(L)$ of H_0 and L are hence given by

$$C(H_0)=\Delta_S,\quad C(L)=\Delta_S+\Gamma-\frac{1}{4}\binom{m}{2},$$

where Δ_S is the Laplace-Beltrami operator and $\Gamma=-\sum_{i<j} e_{ij}L_{ij}$, the spherical Dirac operator (see [7], [9], [13]).
The eigenspaces of Δ_S are the classical spaces H_k of spherical harmonics of degree k (eigenvalue $-k(k+m-2)$); the eigenspaces of $C(L)$ are denoted by M_k.
M_k is called the space of spherical monogenics of degree k.
As $\Delta_S=\Gamma(m-2-\Gamma)$, H_k and M_k are of the form

$$H_k=M_{+,k}+M_{-,k}\quad ,M_k=M_{+,k}+M_{-,k},$$

where $M_{\pm,k}$ are the eigenspaces of Γ with eigenvalues $-k$ and $k+m-1$ (see [7], [9], [13]).
The elements of $M_{\pm,k}$ are called inner and outer spherical monogenics of degree k and are denoted by $P_k(\omega)$ and $Q_k(\omega)$, $\omega\in S^{m-1}$.
The projections on H_k, M_k, $M_{+,k}$, $M_{-,k}$ are respectively denoted by S_k, Π_k, P_k, Q_k.
We have that $Q_k(f)=-\vec{\omega}P_k(\vec{\omega}f)$ and

$$P_k(f)(\vec{\omega})=\frac{(-1)^{k+1}}{\omega_m k!}\int_{S^{m-1}}\langle\vec{\omega},\nabla\rangle^k\left(\frac{\vec{u}}{|\vec{u}|^m}\right)\vec{u}f(\vec{u})\,dS_u.$$

Let $D=\sum_{j=1}^{m} e_j\frac{\partial}{\partial x_j}$; then $D=\vec{\omega}\left(\frac{\partial}{\partial r}+\frac{1}{r}\Gamma_\omega\right)$. Hence if P_k,Q_k are spherical monogenic, $r^kP_k(\vec{\omega})$ and $r^{-(k+m-1)}Q_k(\vec{\omega})$ are left monogenic in $R^m\setminus\{0\}$. As D is invariant under the representation L, D commutes with $\Pi_k=P_k+Q_k$. This leads to a refinement of the classical theory of spherical means (see [6], [10]) of which we recall the main definitions and properties.
Let f be a function in a domain of R^m. Then we consider the refined spherical means

$$P(f)(\vec{x},r)=\frac{1}{\omega_m}\int_{S^{m-1}}f(\vec{x}+r\vec{\omega})\,dS_\omega,$$

$$Q(f)(\vec{x},r)=\frac{1}{\omega_m}\int_{S^{m-1}}\vec{\omega}.f(\vec{x}+r\vec{\omega})\,dS_\omega.$$

These refined spherical means satisfy a first order Darboux system of the form

$$D_x P(f)(\vec{x},r)=\left(\frac{\partial}{\partial r}+\frac{m-1}{r}\right)Q(f)(\vec{x},r)$$

$$D_x Q(f)(\vec{x},r)=-\frac{\partial}{\partial r}P(f)(\vec{x},r),$$

which follows straight from $\Pi_0(D_x f(\vec{x}+\vec{y}))=D_y\Pi_0 f(\vec{x}+\vec{y})$, where $\Pi_0(f)(\vec{x}+\vec{y})=P(f)(\vec{x},|\vec{y}|)-\vec{y}/|\vec{y}|\,.Q(f)(\vec{x},|\vec{y}|)$.
Hence we may generalize these spherical means to

$$\Pi_k(f(\vec{x}+\vec{u}))(\vec{y})=P_k(f(\vec{x}+\vec{u}))(\vec{y})-\frac{\vec{y}}{|\vec{y}|}P_k\left(\frac{\vec{u}}{|\vec{u}|}f(\vec{x}+\vec{u})\right)(\vec{y}),$$

leading up to the generalized Darboux system

$$P_{+,k}(Df)=\left(\frac{\partial}{\partial r}+\frac{k+m-1}{r}\right)P_{-,k}(f),$$

$$P_{-,k}(Df)=\left(-\frac{\partial}{\partial r}+\frac{k}{r}\right)P_{+,k}(f),$$

where for $r=|\vec{y}|$,

$$P_{+,k}(f)(\vec{x},r)=P_k(f(\vec{x}+\vec{u})(\vec{y}),$$

$$P_{-,k}(f)(\vec{x},r)=P_k\left(\frac{\vec{u}}{|\vec{u}|}f(\vec{x}+\vec{u})\right)(\vec{y}),$$

and where for fixed $(\vec{x},r)$, $P_{\pm,k}(f)(\vec{x},r)$ have values in $M_{+,k}$.

In terms of the Gegenbauer polynomials $C_\nu^\lambda(\theta)$ (see [5]), we have the following explicit formulae

$$P_{+,k}(f)(\vec{x},r)=\frac{1}{\omega_m}\int_{S^{m-1}}\left(C_k^{\frac{m}{2}}(\theta)+\vec{\omega}\vec{u}C_{k-1}^{\frac{m}{2}}(\theta)\right)f(r\vec{u}+\vec{x})\,dS_u,$$

$$P_{-,k}(f)(\vec{x},r)=\frac{1}{\omega_m}\int_{S^{m-1}}\left(\vec{u}C_k^{\frac{m}{2}}(\theta)-\vec{\omega}C_{k-1}^{\frac{m}{2}}(\theta)\right)f(r\vec{u}+\vec{x})\,dS_u,$$

where $\vec{y}=r\vec{\omega}$, $\vec{\omega}\in S^{m-1}$ and $\theta=\langle\vec{\omega},\vec{u}\rangle$, $\vec{u}\in S^{m-1}$.

2. Spherical means of codimension 2

In view of its importance in complex analysis we treat spherical means of codimension 2 separately.
Let $\Omega\subseteq R^m$ be open and put

$$\hat{\Omega}=\{(\vec{x},\vec{y}):\vec{x}\in\Omega,\vec{x}+S_y\subseteq\Omega\},\quad S_y=\{\vec{u}:|\vec{u}|=|\vec{y}|,\ \langle\vec{u},\vec{y}\rangle=0\}.$$

The component of $\hat{\Omega}$ containing Ω is called the complex harmonic hull of Ω (see e.g. [1]).
First we introduce the 0-th order spherical means by

$$P^1(f)(\vec{x},\vec{y}) = \frac{1}{\omega_{m-1}} \int_{S^{m-1}} \delta(<\vec{u},\vec{\omega}>) f(\vec{x}+r\vec{u})\, dS_u$$

$$Q^1(f)(\vec{x},\vec{y}) = \frac{1}{\omega_{m-1}} \int_{S^{m-1}} \vec{u}\,\delta(<\vec{u},\vec{\omega}>) f(\vec{x}+r\vec{u})\, dS_u,$$

where $\vec{y}=r\vec{\omega}$, $r=|\vec{y}|$ and $(\vec{x},\vec{y})\in\hat{\Omega}$.
From the codimension 1 case we immediately obtain the radial Darboux equations

$$(D_x - \vec{\omega}<\vec{\omega},D_x>)P^1(f) = \left(\frac{\partial}{\partial r}+\frac{m-2}{r}\right)Q^1(f),$$

$$(D_x - \vec{\omega}<\vec{\omega},D_x>)Q^1(f) = -\frac{\partial}{\partial r}P^1(f).$$

However, this only expresses the radial part of the $\vec{y}$-derivatives in terms of $\vec{x}$-derivatives. Of course there will also be an angular version of the Darboux equations. This is obtained in

<u>Theorem 1</u>. $P^1(f)$ and $Q^1(f)$ satisfy the angular Darboux equations

$$r\vec{\omega}<\vec{\omega},D_x>P^1(f) = (1-\Gamma_y)\,Q^1(f)$$

$$r\vec{\omega}<\vec{\omega},D_x>Q^1(f) = \Gamma_y P^1(f),$$

where $r\vec{\omega}=\vec{y}$ and $\Gamma_y = \sum_{i<j} e_{ij}\left(y_j \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial y_j}\right)$.

<u>Proof</u>. As $\delta(<\vec{u},\vec{\omega}>) = |\vec{y}|\,\delta(<\vec{u},\vec{y}>)$, we have that

$$\Gamma_y P^1(f)(\vec{x},\vec{y})$$

$$= \frac{|\vec{y}|}{\omega_{m-1}} \int_{S^{m-1}} \Gamma_y \delta\,(<\vec{u},\vec{y}>) f(\vec{x}+|\vec{y}|\vec{u})\, dS_u$$

$$= \frac{1}{\omega_{m-1}} \int_{S^{m-1}} \delta'(<\vec{u},\vec{\omega}>)(\vec{u}\wedge\vec{\omega}) f(\vec{x}+r\vec{u})\, dS_u$$

$$= -\frac{1}{\omega_{m-1}} \int_{S^{m-1}} \delta(<\vec{u},\vec{\omega}>)<\vec{\omega},D_u>(\vec{u}\wedge\vec{\omega} f(\vec{x}+r\vec{u}))\, dS_u$$

$$= r\vec{\omega}<\vec{\omega},D_x>Q^1(f).$$

Similarly we obtain that

$$\Gamma_y Q\ (f) = -\frac{1}{\omega_{m-1}} \int_{S^{m-1}} \delta(<\vec{u},\vec{\omega}>)<\vec{\omega},D_u>[\ \vec{u}\wedge\vec{\omega}.\ \vec{u}f(\vec{x}+r\vec{u})]\ dS_u$$

$$= -\frac{1}{\omega_{m-1}} \int_{S^{m-1}} \delta(<\vec{u},\vec{\omega}>)\vec{u}\wedge\vec{\omega}[\ \vec{\omega}f(\vec{x}+r\vec{u})+\vec{u}r<\vec{\omega},D_x>f(\vec{x}+r\vec{u})]\ dS_u$$

$$= Q^1(f) - r\vec{\omega}<\vec{\omega},D_x>P^1(f). \blacksquare$$

Notice that the radial Darboux équations follow from the L-invariance of D, together with the commutation relations

$$[\ D_x - \vec{\omega}<\vec{\omega},D_x>, P^1] = [\ D_x - \vec{\omega}<\vec{\omega},D_x>, Q^1] = 0,$$

$$[\ \vec{\omega}<\vec{\omega},D_x>, P^1] = 0,\quad \vec{\omega}<\vec{\omega},D_x>Q^1 = -Q^1\circ\vec{\omega}<\vec{\omega},D_x>.$$

The angular equations were shown independently from this. There is however a nice way to link the radial and angular equations together, which has a meaning in complex analysis.
Indeed, we have that

$$P^1(D_x f) = (\frac{\partial}{\partial r} - \frac{1}{r}\Gamma_y)Q^1(f) + \frac{m-1}{r}Q^1(f)$$

$$= \vec{\omega}(\frac{\partial}{\partial r} + \frac{1}{r}\Gamma_y)(-\vec{\omega}Q^1(f)) = D_y(-\vec{\omega}Q^1(f)).$$

and

$$-\vec{\omega}Q^1(D_x f) = D_y P^1(f).$$

Furthermore, by the above commutation relations, $\vec{\omega}Q^1(D_x f) = -D_x\vec{\omega}Q^1(f)$, so that we arrive at the system

$$(D_x + iD_y)\ [\ P^1(f) - i\vec{\omega}Q^1(f)] = 0.$$

Hence spherical means of codimension 2 provide global solutions of the complex monogenic system $(D_x+iD_y)g=0$, which we already studied partially in [11] (see also [8] ,[12]). It is natural to introduce one single spherical mean of codimension 2 by means of

$$M(f)(\vec{x},\vec{y}) = \frac{1}{\omega_{m-1}} \int_{S^{m-1}} (1+i\vec{u}\wedge\vec{\omega})\,\delta(<\vec{u},\vec{\omega}>)\,f(\vec{x}+r\vec{u})\,dS_u.$$

Then $M(f)$ is a solution of $(D_x+iD_y)g=0$ such that $\lim_{y\to 0} M(f)(x,y) = f(x)$.

Example. Let us take the Dirac measure $\delta(\vec{x}+r\vec{u})$. Then in spherical coordinates, putting $\vec{x}=|\vec{x}|\vec{\xi}$, we have that

$$\delta(\vec{x}+r\vec{u})=\frac{1}{r^{m-1}}\delta(r-|\vec{x}|)\otimes\delta(\vec{u}+\vec{\xi}),\quad \vec{u},\vec{\xi}\in S^{m-1}.$$

Hence the spherical mean of the Dirac measure is given by

$$M(\delta)(\vec{x},\vec{y})=\frac{1}{\omega_{m-1}}\,\frac{1-i\vec{\xi}\wedge\vec{\omega}}{|y|^{m-1}}\delta(|\vec{y}|-|\vec{x}|)\times\delta(<\vec{\xi},\vec{\omega}>),\vec{x}=|\vec{x}|\vec{\xi},\ \vec{y}=|\vec{y}|\vec{\omega}.$$

Notice that $M(\delta)(\vec{x},\vec{y})$ is concentrated on the isotropic sphere in $\mathcal{C}^m$. One can easily show that $M(\delta)(\vec{x},\vec{y})$ is a global distributional solution of $(D_x+iD_y)g=0$.

Next we introduce the k-th spherical means of codimension 2, denoted by $P_{\pm k}(f)(\vec{x},\vec{y})$, $(\vec{x},\vec{y})\in\hat{\Omega}$.

To that end, we first introduce vector bundles over S^{m-1} as follows. For $\vec{\omega}\in S^{m-1}$, $M_{\pm,k}(\vec{\omega})$ are the right $\mathcal{C}_m$-modules of inner and outer spherical monogenics of degree k on $S_\omega=\{\vec{u}\in S^{m-1}:\vec{u}\perp\vec{\omega}\}$ and $P_{k,\omega}$ is the projection onto $M_{+,k}(\vec{\omega})$. Furthermore, we put $M_k(\vec{\omega})=M_{+,k}(\vec{\omega})+M_{-,k}(\vec{\omega})$ and $H_k(\vec{\omega})=M_{+,k}(\vec{\omega})+M_{-,k-1}(\vec{\omega})$ and denote by $\Pi_{k,\omega}$ and $S_{k,\omega}$ the corresponding projection operators. Notice that $\Pi_{k,\omega}=P_{k,\omega}-\vec{\nu}P_{k,\omega}\vec{\nu}$, where $\vec{\nu}$ is the unit normal vectorfield on S_ω.

<u>Definition 1</u>. The k-th inner and outer spherical means of codim 2 are given by

$$P^1_{+,k}f(\vec{x},r\vec{\omega})=P_{k,\omega}(f(\vec{x}+r\vec{u})),$$

$$P^1_{-,k}f(\vec{x},r\vec{\omega})=P_{k,\omega}(\vec{u}f(\vec{x}+r\vec{u})),$$

and are considered as sections of $M_{+,k}(\vec{\omega})$ (for fixed $\vec{x}$).

Putting $\theta=<\vec{u},\vec{\nu}>$, we have that in terms of the Gegenbauer polynomials,

$$P_{+,k}(f)(\vec{x},r\vec{\omega})(\vec{\nu})$$
$$=\frac{1}{\omega_{m-1}}\int_{S^{m-1}}\delta(<\vec{\omega},\vec{u}>)\Big(C_k^{\frac{m-1}{2}}(\theta)+\vec{\nu}\vec{u}C_{k-1}^{\frac{m-1}{2}}(\theta)\Big)f(r\vec{u}+\vec{x})\,dS_u$$

and

$P^1_{-,k}(f)(x,r\vec{\omega})(\vec{\nu})$

$$=\frac{1}{\omega_{m-1}}\int_{S^{m-1}}\delta(\langle\vec{\omega},\vec{u}\rangle)[\vec{u}C_k^{\frac{m-1}{2}}(\theta)-\vec{\nu}C_{k-1}^{\frac{m-1}{2}}(\theta)]\,f(r\vec{u}+\vec{x})\,dS_x.$$

Of course $P_{\pm,k}(f)(\vec{x},r\vec{\omega})\in M_{+,k}(\vec{\omega})$ only for $\vec{\nu}\perp\vec{\omega}$.

The radical Darboux equations are now of the form

$$P^1_{+,k}((D_x-\vec{\omega}\langle\vec{\omega},D_x\rangle)f)=(\frac{\partial}{\partial r}+\frac{k+m-2}{r})P^1_{-,k}(f)$$

$$P^1_{-,k}((D_x-\vec{\omega}\langle\vec{\omega},D_x\rangle)f)=(-\frac{\partial}{\partial r}+\frac{k}{r})P^1_{+,k}(f).$$

The angular Darboux equations are not expressed nicely in terms of $P^1_{\pm,k}$. In order to obtain them , we first write $P^1_{\pm,k}$ into the form

$$P^1_{+,k}(f)=A_{+,k}(f)+\vec{\nu}A_{-,k-1}(f)$$

$$P^1_{-,k}(f)=A_{-,k}(f)-\vec{\nu}A_{+,k-1}(f),$$

where

$$A_{+,k}(f)=\frac{1}{\omega_{m-1}}\int_{S^{m-1}}\delta(\langle\vec{\omega},\vec{u}\rangle)C_k^{\frac{m-1}{2}}(\theta)f(r\vec{u}+\vec{x})\,dS_n$$

and $A_{-,k}(f)=A_{+,k}(\vec{u}f)$. Similar to Theorem 1, we obtain that for

$\langle\vec{\omega},\vec{\nu}\rangle=0$,

$$r\vec{\omega}\langle\vec{\omega},D_x\rangle\ A_{+,k}(f)=(1-\Gamma_\omega)A_{-,k}(f),$$

$$r\vec{\omega}\langle\vec{\omega},D_x\rangle\ A_{-,k}(f)=\Gamma_\omega A_{+,k}(f).$$

Next we introduce

Definition 2. The k-th spherical harmonic means of f are given by

$$S^1_{+,k}(f(\vec{x}+r\vec{u}))=A_{+,k}(f)-A_{+,k-2}(f),$$

$$S^1_{-,k}(f(\vec{x}+r\vec{u}))=A_{-,k}(f)-A_{-,k-2}(f).$$

Notice that formally $S^1_{+,k}=P^1_{+,k}-\vec{\nu}P^1_{-k}$ and

$$S^1_{-,k}(f)=S^1_{+,k}(\vec{u}f)=P^1_{-,k}(f)+\vec{\nu}P^1_{+,k}(f).$$

Next we prove the generalized Darboux system for the k-th spherical harmonic means.

Theorem 2. Let $\vec{y}=r\vec{\omega}$, $\vec{\nu}\in S^{m-1}$ such that $<\vec{\nu},\vec{\omega}>=0$ and let Γ_ν be the spherical Dirac operator on S_ω. Then $S^1_{+,k}(f)$ and $S^1_{-,k}(f)$ satisfy the system

$$(D_x+iD_y-\frac{i\vec{\omega}\Gamma_\nu}{r})(S^1_{+,k}(f)-i\vec{\omega}S^1_{-,k}(f))=0.$$

Proof. First notice that $S^1_{\pm,k}(f)$ satisfy the same angular Darboux system from Theorem 1. Next, the radial Darboux system for $P_{\pm,k}$ leads to

$$S^1_{+,k}((D_x-\vec{\omega}<\vec{\omega},D_x>)f)=(\frac{\partial}{\partial r}+\frac{m-2-\Gamma\nu}{r})S^1_{-,k}(f),$$

$$S^1_{-,k}((D_x-\vec{\omega}<\vec{\omega},D_x>)f)=-(\frac{\partial}{\partial r}+\frac{\Gamma\nu}{r})S^1_{+,k}(f).$$

Hence, by combining both systems, we obtain that for $<\vec{\nu},\vec{\omega}>=0$, $\vec{y}=r\vec{\omega}$,

$$S^1_{+,k}(D_xf)=D_y(-\vec{\omega}S^1_{-,k}(f))-\frac{\Gamma\nu}{r}S^1_{-,k}(f),$$

$$-\vec{\omega}S^1_{-,k}(D_xf)=D_yS^1_{+,k}(f)+\frac{\Gamma\nu\vec{\omega}}{r}S_{+,k}(f).$$

It is now clear that $S^1_{+,k}(D_xf)=D_xS^1_{+,k}(f)$ while straightforward compu- leads to

$$S^1_{-,k}((D_x-\vec{\omega}<\vec{\omega},D_x>)f)$$

$$=-2\frac{\Gamma\nu}{r}S^1_{+,k}(f)+(D_x-\vec{\omega}<\vec{\omega},D_x>)S^1_{-,k}(f).$$

Hence, as $S^1_{-,k}(\vec{\omega}<\vec{\omega},D_x>f)=-\vec{\omega}<\vec{\omega},D_x>S^1_{-,k}(f)$, we obtain that for $<\vec{\nu},\vec{\omega}>=0$,

$$D_xS^1_{+,k}(f)=D_y(-\vec{\omega}S^1_{-,k}(f))-\frac{\Gamma\nu}{r}S^1_{-,k}(f),$$

$$D_x(\vec{\omega}S^1_{-,k}(f))=D_y(S^1_{+,k}(f))-\frac{\Gamma\nu\vec{\omega}}{r}S^1_{+,k}(f),$$

which may be simplified to the stated system. ∎

Notice that the above equation should be considered as an equation for sections of the bundle $S_k(\omega)$, on which Γ_ν acts as a finite dimensional linear operator.

3. Extended representations of Spin(m)

Let $R_{m,s}$ be the space of real s-vectors and let $\tilde{R}_{m,s}$ be the cone of elements of the form $\vec{y}=\vec{y}_1.\vec{y}_2\ldots\vec{y}_1$ with $\vec{y}_1\perp\ldots\perp\vec{y}_s$. Notice that $\tilde{R}_{m,s}\setminus\{0\}=\tilde{G}_{m,s}(R)\times R_+$, $\tilde{G}_{m,s}(R)$ being the Grassmann manifold of oriented s-dimensional subspaces of R^m.

First of all we introduce extended representations H and L of Spin(m) as follows. Let $\Omega\subseteq R_m$, f a function in Ω and $t\in\mathrm{Spin}(m)$. Then we put $H(t)f(y)=f(\bar{t}yt)$, $L(t)f(y)=tf(\bar{t}yt)$, $y\in\Omega$.

Furthermore, $y\in R_m$ may be written as

$$y=[y]_0+[y]_1+\ldots+[y]_m,\quad [y]_s\in R_{m,s},\quad s=0,\ldots,m,$$

and $\bar{t}[y]_s t=[\bar{t}yt]_s$, $t\in\mathrm{Spin}(m)$. Hence the representations H and L are well defined for functions in $\Omega\subseteq R_{m,s}$.

Furthermore, if y is of the form $y=\vec{y}_1\ldots\vec{y}_s\in\tilde{R}_{m,s}$ then $\bar{t}yt=\bar{t}\vec{y}_1 t\bar{t}\vec{y}_2 t\ldots\bar{t}\vec{y}_s t\in\tilde{R}_{m,s}$. Hence H and L may even act on functions defined on $\tilde{R}_{m,s}$.

The Casimir operator of H is of the form

$$C(H)=\frac{1}{4}\sum_{i<j}(dH(e_{ij}))^2,$$

where $dH(e_{ij})$ are the infinitesimal representations of e_{ij}. Let $\Delta_{\tilde{G}_{m,s}}$ be the Laplace-Beltrami operator on $\tilde{G}_{m,s}(R)$, then $\Delta_{\tilde{G}_{m,s}}$ equals the restriction of G(H) to $\tilde{R}_{m,s}$.

The infinitesimal representations of e_{ij} corresponding to L are given by $dL(e_{ij})=dH(e_j)+e_{ij}$. Hence the Casimir operator of L is given by

$$C(L)=C(H)+\Gamma-\frac{1}{4}\binom{m}{2},$$

where $\Gamma=\frac{1}{2}\sum_{i<j}e_{ij}dH(e_{ij})$.

Notice that $\Gamma^2=[\Gamma^2]_0+[\Gamma^2]_2+[\Gamma^2]_4$, where

$$[\Gamma^2]_0=C(H),\quad [\Gamma^2]_2=(m-2)\Gamma$$

and

$$[\Gamma^2]_4=\frac{1}{4}\sum_{i<j<k<l}e_{ijkl}(dH(e_{ij})dH(e_{kl})-dH(e_{ik})dH(e_{jl})$$

$$+dH(e_{il})dH(e_{jk})).$$

Next, consider the Clifford derivative on R_m, introduced by D. Hestenes and G. Sobczyk in [4] and given by $\mathcal{D}=\sum_A e_A \frac{\partial}{\partial y_A}$. Then on R_m we have that

$$dH(e_{ij})f(y)=\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(f((1-\varepsilon e_{ij})y(1+\varepsilon e_{ij}))-f(y))$$

$$=<[y,e_{ij}],\mathcal{D}>f=<e_{ij},\bar{y}\mathcal{D}+y\bar{\mathcal{D}}>f,$$

where $<y,u>=[\bar{y}u]_0=[y\bar{u}]_0$, $u,y\in R_m$.

Hence on R_m we obtain that

$$\Gamma=\frac{1}{2}[\bar{y}\mathcal{D}+y\bar{\mathcal{D}}]_2 .$$

Furthermore, let $D_{m,s}$ be the s-vector derivative, given by $\sum_{|A|=s} e_A\frac{\partial}{\partial y_A}$, then the restrictions of Γ to $R_{m,s}$ and $\tilde{R}_{m,s}$ are both of the form

$$\Gamma|R_{m,s}=\frac{1}{2}[\bar{y}D_{m,s}+y\bar{D}_{m,s}]_2,$$

and will be denoted by $\Gamma_{y,s}$.

Examples. (i) For s=1 we have that $[\Gamma^2_{y,s}]_4=0$ so that $\Delta_S=\Gamma(m-2-\Gamma)$.

(ii) For s=2 we put $y=\sum_{k<l} y_{kl}e_{kl}$ and $y_{kl}=-y_{lk}$ and we have that

$$dH(e_{ij})=2\sum_{k\neq i,j}(y_{kj}\frac{\partial}{\partial y_{ki}}-y_{ki}\frac{\partial}{\partial y_{kj}}).$$

Hence $\Gamma_{y,2}$ is given by

$$\Gamma_{y,2}=\sum_{i<j}\sum_{k\neq i,j} e_{ij}(y_{kj}\frac{\partial}{\partial y_{ki}}-y_{ki}\frac{\partial}{\partial y_{kj}}).$$

Notice that in this case $[\Gamma^2_{y,2}]_4\neq 0$, which makes $\Gamma_{y,2}$ quite independent from $\Delta_{\tilde{G}_{m,2}}$. $\Gamma_{y,2}$ is even not an elliptic operator.

4. Spherical means of higher codimension

Let $s<m-1$ and $\Omega\subseteq R^m$ open. Than by $\hat{\Omega}_s$ we denote the set of all spheres

of codimension s+1 inside Ω. We parametrise $\hat{\Omega}_s$ as follows.
Let $\vec{\omega}_1,\dots,\vec{\omega}_s$ be an orthonormal s-frame; then $\omega=\vec{\omega}_1\dots\vec{\omega}_s$ represents the oriented s-space spanned by $\vec{\omega}_1,\dots,\vec{\omega}_s$. Hence $\omega\in\tilde{G}_{m,s}(\mathbb{R})=\tilde{\mathbb{R}}_{m,s}\cap S^{2^m-1}$.
A sphere of codimension s+1 is determined by its center $\vec{x}$, its radius r and the s-vector ω which represents the axis.
Hence $\hat{\Omega}_s=\{(\vec{x},r\omega):\vec{x}+\vec{y}\in\Omega,|y|=r,\vec{y}\perp\omega\}$.
The normal vectors to span$\{\vec{\omega}_1,\dots,\vec{\omega}_s\}$ are given by the equations $\langle\vec{\omega}_j,\vec{u}\rangle=0$, $j=1,\dots,s$, and the Dirac measure on the space $N(\omega)$ of normal vectors is given by

$$\delta(\langle\vec{u},\vec{\omega}_1\rangle)\dots\delta(\langle\vec{u},\vec{\omega}_s\rangle)=\delta(\langle\vec{u},\omega\rangle).$$

Definition 3. The 0-th spherical means of $f\in C_0(\Omega)$ of codimension s+1 are given by

$$P^s(f)(\vec{x},r\omega)=\frac{1}{\omega_{m-s}}\int_{S^{m-1}}\prod_{j=1}^{s}\delta(\langle\vec{u},\vec{\omega}_j\rangle)f(\vec{x}+r\vec{u})dS_u,$$

$$Q^s(f)(\vec{x},r\omega)=\frac{1}{\omega_{m-s}}\int_{S^{m-1}}\prod_{j=1}^{s}\delta(\langle\vec{u},\vec{\omega}_j\rangle)\vec{u}f(\vec{x}+r\vec{u})dS_u,$$

where $(\vec{x},r\omega)\in\hat{\Omega}_s$.

Notice that, when s is odd,

$$\vec{u}\omega+\omega\vec{u}=2\sum_{j=1}^{s}(-1)^j\langle\vec{u},\vec{\omega}_j\rangle\hat{\omega}_j\ ,$$

whereas for s even,

$$\vec{u}\omega-\omega\vec{u}=2\sum_{j=1}^{s}(-1)^j\langle\vec{u},\vec{\omega}_j\rangle\ \hat{\omega}_j,$$

where $\hat{\omega}_j=\vec{\omega}_1\dots\not{\omega}_j\dots\vec{\omega}_s$.

For s odd we put $-\langle\vec{u},\omega\rangle=\frac{1}{2}(\vec{u}\omega+\omega\vec{u})$, whereas for s even, $-\langle\vec{u},\omega\rangle=\frac{1}{2}(\vec{u}\omega-\omega\vec{u})$.
Hence $\langle\vec{u},\omega\rangle$ is an (s-1)-vector in the Clifford algebra spanned by $\vec{\omega}_1,\dots,\vec{\omega}_s$, which we denote by $A(\omega)$.
Hence $\langle\vec{u},\omega\rangle$ behaves like an s-dimensional vector in $A(\omega)$. This justifies the notation $\delta(\langle\vec{u},\omega\rangle)$ for the Dirac measure on $N(\omega)$. We now have

Lemma 1. The Dirac operator may be decomposed as $D=D_+(\omega)+D_-(\omega)$ where

$$D_+(\omega)=\frac{1}{2}\sum_{j=1}^{m}\overline{\omega}\{\omega,e_j\}\frac{\partial}{\partial x_j},\quad D_-(\omega)=\frac{1}{2}\sum_{j=1}^{m}\overline{\omega}[\omega,e_j]\frac{\partial}{\partial x_j}.$$

Furthermore for s even (resp. s odd),

$$D_{\mp}(\omega)=\sum_{j=1}^{s}\vec{\omega}_j\langle\vec{\omega}_j,D\rangle$$

Hence we obtain the radial Darboux equations

Theorem 3. For s even (resp. s odd), we have that

$$D_{\pm}(\omega)P^s(f)=\left(\frac{\partial}{\partial r}+\frac{m-s+1}{r}\right)Q^s(f)$$

$$D_{\pm}(\omega)Q^s(f)=-\frac{\partial}{\partial r}P^s(f).$$

In order to establish the angular Darboux equations, we first study the action of the operator $\Gamma_{y,s}$, introduced in the previous section, section, on $\delta(\langle u,\omega\rangle)$.

Lemma 2. For s odd (resp. s even), we have that

$$\Gamma_{y,s}\delta(\langle\vec{u},\omega\rangle)=\vec{u}\wedge D_{\pm}(\omega)\delta(\langle\vec{u},\omega\rangle).$$

Proof. First consider any smooth function $f(\vec{y}_1,\ldots,\vec{y}_s)$, defined in a neighbourhood of the cone

$$K=\{(\vec{y}_1,\ldots,\vec{y}_s)\in(R^m\setminus\{0\}) : \vec{y}_1\perp\ldots\perp\vec{y}_s\},$$

such that $f|K$ depends only on the s-vector $\vec{y}_1\ldots\vec{y}_s$. Then $f|K$ determines a function on $\tilde{R}_{m,s}$, which we denote by $f|\tilde{R}_{m,s}$. Of course this is no restriction in the classical sense, since K is a bundle over $\tilde{R}_{m,s}$ in which $\tilde{R}_{m,s}$ is not inbedded as a classical surface. In any case, we may define a representation H' of Spin(m) on f by $H'(t)f(\vec{y}_1,\ldots,\vec{y}_s)=f(\overline{t}\vec{y}_1t,\ldots,\overline{t}\vec{y}_st)$ and $H'(t)f(\vec{y}_1,\ldots,\vec{y}_s)$ may still be"restricted" to $\tilde{R}_{m,s}$.

Furthermore $(H'(t)f)|_{\tilde{R}_{m,s}}=f|_{\tilde{R}_{m,s}}(\overline{t}\vec{y}_1\ldots\vec{y}_s\overline{t})=H(t)(f|\tilde{R}_{m,s})$,

so that also

$$dH(e_{ij})(f|\tilde{R}_{m,s})=(dH'(e_{ij})f)|\tilde{R}_{m,s}$$

$$=-2\sum_{k=1}^{s}(L_{ij}^{k}f)|\tilde{R}_{m,s},$$

where $L_{ij}^{k}=y_{ki}\frac{\partial}{\partial y_{kj}}-y_{kj}\frac{\partial}{\partial y_{ki}}$. Hence we arrive at

$$\Gamma_{y,s}(f|\tilde{R}_{m,s})=-(\sum_{i<j}e_{ij}\sum_{k=1}^{s}L_{ij}^{k}f)|\tilde{R}_{m,s}.$$

We now apply this to the function

$$f(\vec{y}_1,\ldots,\vec{y}_s)=|\vec{y}_1|\ldots|\vec{y}_s|\delta(<\vec{u},\vec{y}_1>)\ldots\delta(<\vec{u},\vec{y}_s>),$$

which, after action on a testfunction $\varphi(u)$ behaves like a C_∞-function. Notice that $f|\tilde{R}_{m,s}=\delta(<\vec{u},\omega>)$. Hence, putting $\vec{y}_j=|\vec{y}_j|\vec{\omega}_j$ and $\Gamma_{yk}=-\sum_{i<j}e_{ij}L_{ij}^{k}$, we arrive at

$$\Gamma_{y,s}\delta(<\vec{u},\omega>)=|\vec{y}_1|\ldots|\vec{y}_s|\Gamma_{y,s}(\delta(<\vec{u},\vec{y}_1>)\ldots\delta(<\vec{u},\vec{y}_s>))$$

$$=(\vec{u}\wedge\sum_{k=1}^{s}\vec{\omega}_k\ \delta'(<\vec{u},\vec{\omega}_k>)\prod_{j\neq k}\delta(<\vec{u},\vec{\omega}_j>))|\tilde{R}_{m,s}.$$

On the other hand, for $<\vec{\omega}_h,\vec{\omega}_j>=\delta_{kj}$, i.e. on K,

$$\sum_{k=1}^{s}\vec{\omega}_k<\vec{\omega}_k,D_u>\delta(<\vec{u},\omega>)=\sum_{k=1}^{s}\vec{\omega}_k\delta'(<\vec{u},\vec{\omega}_k>)\prod_{j\neq k}\delta(<\vec{u},\vec{\omega}_j>),$$

which, in view of Lemma 1, leads to the stated identity. ∎
This leads to the angular Darboux equations.

<u>Theorem 4.</u> For s odd (resp. s even), we have that

$$D_\pm(\omega)P^s(f)=\frac{1}{r}(s-\Gamma_{y,s})Q^s(f),$$

$$D_\pm(\omega)Q^s(f)=\frac{1}{r}\Gamma_{y,s}P^s(f).$$

<u>Proof</u>. We have that $\vec{u}\wedge D_\pm(\omega)=\sum_{k=1}^{s}<\vec{\omega}_k,D_u>\vec{u}\wedge\vec{\omega}_k=0$, so that, in view of Lemma 2,

$$\Gamma_{y,s}P^s(f)(\vec{x},r\omega)$$

$$=\frac{1}{\omega_{m-s}}\int_{S^{m-1}}\vec{u}\wedge(D_\pm(\omega)\delta(<\vec{u},\omega>))f(\vec{x}+r\vec{u})dS_u$$

$$=-\frac{1}{\omega_{m-s}}\int_{S^{m-1}}\delta(<\vec{u},\omega>)\vec{u}\wedge(r\sum_{k=1}^{s}\vec{\omega}_k<\vec{\omega}_k,D_x>)f(\vec{x}+r\vec{u})dS_u$$

$$=rD_\pm(\omega)Q^s(f)(\vec{x},r\omega),$$

since for $\vec{u}\perp\vec{\omega}_k$, $\vec{u}\wedge\vec{\omega}_k=-\vec{\omega}_k\wedge\vec{u}=-\vec{\omega}_k\vec{u}$.

Similarly, as $D_\pm(\omega)\vec{u}=\sum_{k=1}^{s}\vec{\omega}_k<\vec{\omega}_k,P'_u>\vec{u}=-s$,

$$\Gamma_{y,s}Q^s(f)(\vec{x},r\omega)$$
$$=-\frac{1}{\omega_{m-s}}\int_{S^{m-1}}\delta(<\vec{u},\omega>)\vec{u}\wedge D_\pm(\omega)(\vec{u}f(x+r\vec{u}))dS_u$$
$$=sQ^s(f)-rD_\pm(\omega)P^s(f). \blacksquare$$

Notice that for s odd (resp. s even), $D_-(\omega)$ commutes with both P^s and Q^s, while $D_\pm(\omega)$ commutes with P^s and anticommutes with Q^s. Hence Theorems 3 and 4 lead to the system

$$P^s(D_xf)=(\frac{\partial}{\partial r}-\frac{1}{r}\ \Gamma_{y,s})Q^s(f)+\frac{m-1}{r}Q^s(f),$$

$$Q^s(D_xf)=-(\frac{\partial}{\partial r}+\frac{1}{r}\ \Gamma_{y,s})P^s(f).$$

Furthermore, for s even $D_+(\omega)$ anticommutes with ω, while for s odd. $D_-(\omega)$ commutes with ω. This means that for s even (resp. s odd) D_x commutes (resp. anticommutes) with Q^s. Hence the second Darboux equations may be written as

$$D_x\omega Q^s(f)=(-1)^{s+1}\omega(\frac{\partial}{\partial r}+\frac{1}{r}\Gamma_{y,s})P^s(f).$$

Next, put $y=r\omega$. Then we shall establish an expression for $\Gamma_{y,s}(yf(y))$ in terms of $y\Gamma_{y,s}(f(y))$ and $yf(y)$. This corresponds to the hypercomplex refinement of the Kelvin inversion, given by $\Gamma(\vec{y}f(\vec{y}))=-\vec{y}\Gamma_yf(\vec{y})+m\vec{y}f(\vec{y})$, so that the map $f(\vec{y})\rightarrow\frac{\vec{y}}{|\vec{y}|^m}f(\frac{\vec{y}}{|\vec{y}|^2})$ preserves monogenicity and changes inner spherical monogenics into outer spherical monogenics and vice versa (see [7], [9], [13]). First we prove

<u>Lemma 3</u>. Let $\omega=\vec{\omega}_1\ldots\vec{\omega}_s\in\tilde{G}_{m,s}(R)$ and let $(\vec{u}_1,\ldots,\vec{u}_{m-s})$ be a local orthonormal frame, orthogonal to ω. Then $\Gamma_{y,s}$ is locally given by

$$\Gamma_{y,s}=r\sum_{j,k}(-1)^k\vec{\omega}_k\vec{u}_j<\vec{u}_j\hat{\omega}_k\ ,D_{m,s}>,$$

where $r\omega=y$ and $\hat{\omega}_k=\vec{\omega}_1\ \ldots\ \vec{\omega}_{k-1}\ \vec{\omega}_{k+1}\ \ldots\ \vec{\omega}_s$.

<u>Proof</u>. Let us recall that $\Gamma_{y,s}$ is given by

$$\Gamma_{y,s}=\frac{1}{2}[\bar{y}D_{m,s}+y\bar{D}_{m,s}]_2.$$

Next, consider local orthonormal frames $(\vec{\omega}_1,\ldots,\vec{\omega}_s)$ and $(\vec{u}_1,\ldots,\vec{u}_{m-s})$ such that $\vec{\omega}=\vec{\omega}_1 \ldots \vec{\omega}_s$ and $(\vec{u}_1,\ldots,\vec{u}_{m-s})$ is orthogonal to ω. Then it is easy to see that

$$D_{m,s}=\omega\langle\omega,D_{m,s}\rangle+\sum_{j,k}\vec{u}_j\hat{\omega}_k\langle\vec{u}_j\hat{\omega}_k,D_{m,s}\rangle+L_{m,s},$$

where $L_{m,s}$ is normal to $\tilde{R}_{m,s}$. Hence, as $y=r\omega$ and $\bar{y}=r\bar{\omega}$, we obtain that

$$[\bar{y}D_{m,s}]_2=r\sum_{j,k}[\bar{\omega}\vec{u}_j\hat{\omega}_k]_2\langle\vec{u}_j\hat{\omega}_k,D_{m,s}\rangle,$$

$$[y\bar{D}_{m,s}]_2=r\sum_{j,k}[\overline{\omega\vec{u}_j\hat{\omega}_k}]_2\langle\vec{u}_j\hat{\omega}_k,D_{m,s}\rangle,$$

since $[\bar{\omega}L_{m,s}]_2=[\omega\bar{L}_{m,s}]_2=0$.

Now $\vec{u}_j\hat{\omega}_k=(-1)^{s-1}\hat{\vec{\omega}}_k\vec{u}_j$ and $\bar{\omega}=(-1)^{s-k}\bar{\vec{\omega}}_k\bar{\hat{\omega}}_k$, so that $\bar{\omega}\vec{u}_j\hat{\omega}_k=(-1)^k\vec{\omega}_k\vec{u}_j$.

On the other hand, $\omega=(-1)^{k-1}\vec{\omega}_k\hat{\omega}_k$ so that $\overline{\omega\vec{u}_j\hat{\omega}_k}=(-1)^{k-1}\vec{\omega}_k\bar{\vec{u}}_j$ $=(-1)^k\vec{\omega}_k\vec{u}_j$. This leads to the stated lemma. ∎

Theorem 5. Let $f(y)$ be a function on $\tilde{R}_{m,s}$. Then we have that

$$\Gamma_{y,s}yf(y)=-y\Gamma_{y,s}f(y)+s(m-s)yf(y).$$

Proof. Putting $y=\sum_A y_A e_A$, we have that

$$\Gamma_{y,s}yf(y)=\sum_{|A|=s}y_A\Gamma_{y,s}e_Af(y)+\Gamma_{y,s}(y)f(y).$$

For s odd, ω commutes with $\vec{\omega}_k$ and anticommutes with $\vec{u}_j$, whereas for s even, ω commutes with $\vec{u}_j$ and anticommutes with $\vec{\omega}_k$. Hence we obtain that

$$\sum_{|A|=s}y_A\Gamma_{y,s}e_Af(y)=r\sum_{j,k}(-1)^k\vec{\omega}_k\vec{u}_j(r\omega)\langle\vec{u}_j\hat{\omega}_k,D_{m,s}\rangle f(y)$$
$$=-y\Gamma_{y,s}\,f(y).$$

Furthermore we have that

$$\Gamma_{y,s}y=r\sum_{j,k}(-1)^k\vec{\omega}_k\vec{u}_j\sum_{|A|=s}\langle\vec{u}_j\hat{\omega}_k,e_A\rangle e_A$$
$$=r\sum_{j,k}\omega=s(m-s)y. \blacksquare$$

In order to establish the complete system of Darboux equations, we introduce a new differential operator.

Definition 4. The operator D_y on $\tilde{R}_{m,s}$ is given by $D_y=\omega(\frac{\partial}{\partial r}+\frac{1}{r}\Gamma_{y,s})$.

Proposition 1. Let $\vec{\omega}_1 \ldots \vec{\omega}_s=\omega$ and let $(\vec{u}_1,\ldots,\vec{u}_{m-s})$ be an orthonormal basis, orthogonal to ω. Then we have that

$$D_y=\omega<\omega,D_{m,s}>+\sum_{j,k}\vec{u}_j\hat{\omega}_k<\vec{u}_j\hat{\omega}_k,D_{m,s}>,$$

or, in other words, D_y is the projection of $D_{m,s}$, tangent to $\tilde{R}_{m,s}$.

Proof. This follows easily from the fact that
$\frac{\partial}{\partial r}=<\omega,D_{m,s}>$ and $(-1)^k\omega\vec{\omega}_k\vec{u}_j=(-1)^k\vec{u}_j\vec{\omega}_k\omega=\vec{u}_j\hat{\omega}_k$

and the fact that an orthonormal basis for the tangent space of $\tilde{R}_{m,s}$ in $R_{m,s}$ is given by $\{\omega,\vec{u}_j\hat{\omega}_k:j,k\}$. ∎

Notice that if f is a C_1-function in a neighbourhood Ω of a point of $\tilde{R}_{m,s}$ such that in $\Omega\cap\tilde{R}_{m,s}$ all normal derivations to $\tilde{R}_{m,s}$ of f vanish, then $D_y(f|\tilde{R}_{m,s})=(D_{m,k}f)|\tilde{R}_{m,s}$. We now have the Darboux system.

Theorem 5. The spherical means of codim s+1 satisfy the system

$$D_xP^s(f)=(-1)^{\frac{s(s+1)}{2}}\left(D_y+\frac{(s-1)(s+1-m)\omega}{r}\right)\omega Q^s(f),$$

$$D_x\omega Q^s(f)=(-1)^{s+1}D_yP^s(f).$$

Proof. As $\omega^2=(-1)^{\frac{s(s+1)}{2}}$, we have that

$$\begin{aligned}&(\tfrac{\partial}{\partial r}-\tfrac{1}{r}\Gamma_{y,s})Q^s(f)\\ &=(-1)^{\frac{s(s+1)}{2}}(\tfrac{\partial}{\partial r}-\tfrac{1}{r}\Gamma_{y,s})\omega.\omega Q^s(f)\\ &=(-1)^{\frac{s(s+1)}{2}}[D_y\omega Q^s(f)-\tfrac{s(m-s)}{r}\omega^2Q^s(f)],\end{aligned}$$

while clearly
$$D_x\omega Q^s(f)=(-1)^{s+1}D_yP^s(f).$$ ∎

General spherical means of codimension s+1 are introduced as follows. First, denote for $\omega\in\tilde{G}_{m,s}(R)$, $M_{\pm,k}(\omega)$ the right-module of inner (outer)

spherical monogenics of degree k on $S_\omega=\{\vec{u}\in S^{m-1};\ \langle\vec{u},\omega\rangle=0\}$.
Let $P_{k,\omega}$ be the projection on $M'_{+,k}(\omega)$ and put

$$M_k(\omega)=M_{+,k}(\omega)+M_{-,k}(\omega),\ H_k(\omega)=M_{+,k}(\omega)+M_{-,k-1}(\omega);$$

then the projections on $M_k(\omega)$ and $S_k(\omega)$ are denoted by $\Pi_{k,\omega}$ and $S_{k,\omega}$.

Definition 5. Let f be a continuous function in $\Omega\subseteq R^m$. Then the k-th inner and outer spherical means of codim s+1 of f are defined by

$$P^s_{+,k}f(\vec{x},r\omega)=P_{k,\omega}(f(\vec{x}+r\vec{u})),$$

$$P^s_{-,k}f(\vec{x},r\omega)=P_{k,\omega}(\vec{u}f(\vec{x}+r\vec{u})),$$

and are considered as sections of $M_{+,k}(\omega)$ such that $(\vec{x},r\omega)\in\hat{\Omega}_s$.

Notice that, if $\vec{\nu}$ is the unit normal on S_ω, $\theta=\langle\vec{u},\vec{\nu}\rangle$, then $P^s_{+,k}$ is given by

$$P^s_{+,k}(f)(\vec{x},r\omega)(\vec{\nu})$$
$$=\frac{1}{\omega_{m-s}}\int_{S^{m-1}}\prod_1^s\delta(\langle\vec{u},\vec{\omega}_j\rangle)\left(C_k^{\frac{m-s}{2}}(\theta)+\vec{\nu}\vec{u}C_{k-1}^{\frac{m-s}{2}}(\theta)\right)f(r\vec{u}+\vec{x})\,dS_u.$$

Furthermore, the radial Darboux equations are given by (s being even and odd respectively)

$$P^s_{+,k}(D_\pm(\omega)f)=\left(\frac{\partial}{\partial r}+\frac{k+m-s-1}{r}\right)P^s_{-,k}(f),$$

$$P^s_{-,k}(D_\pm(\omega)f)=\left(-\frac{\partial}{\partial r}+\frac{k}{r}\right)P^s_{+,k}(f).$$

The construction of angular Darboux equations is similar to the one in section 2 and uses the operator $\Gamma_{y,s}$. To that end, let

$$S^s_{+,k}(f)=P^s_{+,k}(f)-\vec{\nu}P^s_{-,k-1}(f),\quad S^s_{-,k}(f)=P^s_{-,k}(f)+\vec{\nu}P^s_{+,k-1}(f).$$

We then obtain

Proposition 2. For s even (resp. s odd), $S^s_{+,k}$ and $S^s_{-,k}$ satisfy the angular Darboux system

$$D_{\mp}(\omega)S^s_{+,k}(f)=\frac{1}{r}(s-\Gamma_{y,s})S^s_{-,k}(f),$$

$$D_{\mp}(\omega)S^s_{-,k}(f)=\frac{1}{r}\Gamma_{y,s}S^s_{+,k}(f).$$

This finally leads to the complete Darboux system.

Theorem 7. The k-th spherical harmonic means of codimension s+1 satisfy the system

$$D_x S^s_{+,k}(f)=(-1)^{\frac{s(s+1)}{2}}\left(D_y+\frac{(s-1)(s+1-m)\omega}{r}-\frac{\omega\Gamma_\nu}{r}\right)\omega S^s_{-,k}(f),$$

$$D_x\omega S^s_{-,k}(f)=(-1)^{s+1}\left(D_y-\frac{\omega\Gamma_\nu}{r}\right)S^s_{+,k}(f).$$

Proof. The radial and angular Darboux equations already lead to the system

$$S^s_{+,k}(D_x f)=\left(\frac{\partial}{\partial r}-\frac{1}{r}\Gamma_{y,s}+\frac{m-1}{r}-\frac{\Gamma_\nu}{r}\right)S^s_{-,k}(f),$$

$$S^s_{-,k}(D_x f)=-\left(\frac{\partial}{\partial r}+\frac{1}{r}\Gamma_{y,s}+\frac{1}{r}\Gamma_\nu\right)S^s_{+,k}(f).$$

The rest follows easily from the fact that D_x commutes with $S^s_{+,k}$ while

$$S^s_{-,k}(D_{\mp}(\omega)f)=-D_{\mp}(\omega)S^s_{-,k}(f),$$

$$S^s_{-,k}(D_{\pm}(\omega)f)=D_{\pm}(\omega)S^s_{-,k}(f)-\frac{2\Gamma_\nu}{r}S^s_{+,k}(f),$$

so that

$$\omega S^s_{+,k}(D_x f)=(-1)^s D_x\omega S^s_{-,k}(f)-2\frac{\omega\Gamma_\nu}{r}S^s_{+,k}(f). \blacksquare$$

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